D. Ti_8O_{15}

The conductivity of Ti_8O_{15} is activated over the whole temperature range investigated. The activation energy of 0.087 eV could be involved in exciting the d electrons into a band from a localized state, or it could be associated with an activated mobility with the electrons hopping from one localized state to another.

V. SUMMARY

Several transitions have been observed in single crystals of the titanium oxides by electrical conductivity measurements and differential thermal analysis. Most of these transitions correspond closely with the transitions observed in previously reported magnetic susceptibility data obtained on powders or sintered pellets of these compounds. The one notable exception is the lower-temperature transition in Ti_4O_7 which has been shown to involve some structural rearrangement without affecting the magnetic susceptibility.

The behavior of Ti_4O_7 is similar to that of V_2O_3 except that Ti_4O_7 undergoes two separate transitions. The first involves a change in magnetic susceptibility, while the second involves structural rearrangement with no change in susceptibility. It was pointed out that existing theories apparently do not provide a satisfactory explanation of the semiconductor-to-metal transition in Ti_4O_7 or account for the properties of Ti_5O_9 and Ti_6O_{11}. The striking changes brought about by relatively small structural alterations in this series of materials suggest that highly detailed theoretical treatments will be needed for any satisfactory degree of understanding.

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Quantum Corrections in the Theory of the Anomalous Skin Effect

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A comparison is made between the semiclassical and the quantum-mechanical treatment of the anomalous skin effect. The latter theory is based on the density matrix. The author calculates the value of the surface impedance with both theories. The difference depends on a parameter λ, which is the ratio of the skin depth to a typical de Broglie wavelength. This wavelength corresponds to the velocity component, normal to the surface, of an effective electron. The order of magnitude of the correction depends on the relaxation time, and can amount to 10% for sufficiently pure metals.

1. INTRODUCTION

BECAUSE much information about the structure of solids is obtained by means of electric conductivity measurements in the frequency region of the anomalous skin effect (ase), it is obvious that a proper description of these experiments is needed. In the literature two different approaches have been followed for treating the ase: (i) the semiclassical approach, which is based on the concept of the distribution function,[1] and (ii) the quantum-mechanical approach founded on the density matrix.[2,3] It has been reported by several authors[4] that no significant discrepancies exist between the treatments. In this paper it is shown that this conclusion is not correct. The reason for this is the fact that the motion of electrons, close to the boundaries, was not properly accounted for in Ref. 2. This is because use has been made of the "speculum assumption," defined in Sec. 2, which is not correct for a quantum treatment, as is shown in Secs. 6 and 7. Although the subject of the ase has tended to become almost "classical," it should be emphasized at this point that the treatments given in the textbooks are fundamentally incorrect for this reason. In order to make a quantitative comparison between the two approaches, let us confine our attention to the value of the surface impedance of an electron system with a metallic density. In order to make it clear at which point the present analysis differs with the existing ones, a formal description is given of the response of the electron system to an applied electromagnetic field. In Sec. 3 the relation between this response and the surface impedance is established. In Secs. 4 and 5 a brief

[1] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).
[2] D. C. Mattis and G. Dresselhaus, Phys. Rev. **111**, 403 (1958).
[3] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).
[4] A. B. Pippard, *Documents on Modern Physics, The Dynamics of Conduction Electrons* (Gordon and Breach, Science Publishers, Inc., New York, 1964), Chap. 5.

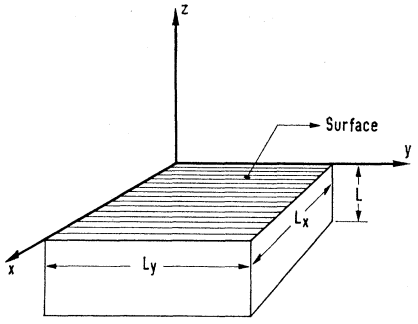

FIG. 1. Geometry of the system.

indication is made of the contents of the semiclassical theory of the ase, which is in agreement with the existing literature.[1] In Secs. 6 and 7 a quantum treatment of the ase is given, which differs from that of Ref. 3. The results of linear response theory which are needed in the text are given in the Appendix. Preliminary results of this investigation have been given elsewhere.[5]

A reason why it has not yet been possible to identify the quantum corrections is that all observed deviations of the surface impedance can be ascribed to surface roughness within the improper semiclassical framework. It should be possible to distinguish experimentally between surface roughness and our quantum corrections, however. For this purpose, one needs a pure metal ($\omega\tau>1$) with a sufficiently smooth surface (80–90% specular reflection at glancing angles is attainable today). The frequency range between 1–100 Gc/sec seems most promising. The rather rapid decay of the corrections for $\omega\tau<1$ may be useful aid.

2. FORMAL DESCRIPTION OF RESPONSE

In order to evaluate the value of the surface impedance, it is necessary to make some specifications about the surface of the system. For convenience, it is considered to be smooth and rigid, which implies that it is represented by an infinite potential wall. Let these walls be located at $z=0$ and $z=-L$, as shown in Fig. 1. The other boundary conditions are considered to be periodic with periodicities L_x and L_y along the x and y directions. Ultimately, the limits L_x, L_y, $L\to\infty$ are taken so that the system becomes semi-infinite. For convenience, the system is assumed to be reflection symmetric about the $x-z$ plane. This is realized for a free-electron system with the geometry of Fig. 1. The simplification is that the current-density components along the y direction do not couple to electric field components along the x direction and vice versa. In order to apply the response theory of the Appendix, the system is considered to have been in thermal equilibrium until $t=0$, the instant at which the experiment is started. The Fourier transforms of currents and fields are accordingly one-sided in time and analytical in the upper half of the complex ω plane. The current-density and electric field components, J_x and E_x, are related to each other in two different ways: (i) by means of the response theory and (ii) by means of Maxwell's equations. If the average values of J_x and E_x in a plane z are denoted by $\bar{J}_x(z,\omega)$ and $\bar{E}_x(z,\omega)$, respectively, the linear response theory gives a relation which is of the following type:

$$\bar{J}_x(z,\omega)=\int_{-L}^{0} dz'\, K(z,z')\bar{E}_x(z',\omega), \tag{2.1}$$

if the system has the just mentioned symmetries. The function $K(z,z')$ depends on ω. Physically, $K(z,z')$ describes the transport of electrons from the plane z', where electrons are accelerated by the electric field $\bar{E}_x(z',\omega)$, towards the plane z. The function K may hence be looked upon as a propagator of signals which propagate through the electron system. If we neglect the influence of the boundaries for the moment, then, if the system were reflection-symmetric about the x-y plane, the propagator $K(z,z')$ would only depend on the value of $|z-z'|$. Let us denote this propagator by $C(|z-z'|)$, so that we would have

$$K(z,z')=C(|z-z'|). \tag{2.2}$$

The crudest and most simple effect of the presence of a boundary at $z=0$ is the following: a signal which is emitted at z' by means of the electric field propagates towards the wall and will be reemitted back into the bulk. If the signal were not distorted by this reemission process, it would arrive at z as though it had traveled a total distance $(z+z')$ through the bulk without being disturbed. If this reasoning were correct, we would have

$$K(z,z')=C(|z-z'|)+C|z-z'|). \tag{2.3}$$

It is not at all obvious that this result is correct, so that one generally expects the following trivial expression for $K(z,z')$:

$$K(z,z')=C(|z-z'|)+C(|z+z'|)+Q(z,z'), \tag{2.4}$$

where all corrections which are not incorporated in the crude assumptions leading to Eq. (2.3) are contained in Q. It should be mentioned at this point that the assumptions leading to Eq. (2.3), which imply that $Q=0$, are equivalent to the frequently used *speculum assumption*.[6] This is readily seen if one substitutes Eq. (2.3) into (2.1), which gives

$$\bar{J}_x(z,\omega)=\int_{-\infty}^{0} dz'\, C(|z-z'|)\bar{E}_x(z',\omega) + \int_{0}^{\infty} dz'\, C(|z-z'|)\bar{E}_x(-z',\omega). \tag{2.5}$$

[5] A. P. van Gelder, Phys. Letters **21**, 18 (1966).

[6] C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, 1964), Chap. 16.

Physically, this assumption implies that the presence of the surface can be neglected altogether, provided one extends the system across its boundary in such a way that the electric field would be continued evenly with respect to the original boundary, and the electrons are allowed to penetrate unhindered. It should be emphasized that this assumption *has not been and cannot be* justified in general. It is shown to be valid for the semiclassical case in Secs. 4 and 5. However, in Secs. 6 and 7 the assumption is proved to be invalid for the quantum treatment, where Q does not vanish. The resulting corrections are due to this nonvanishing $Q(z,z')$ term.

3. EXPRESSIONS FOR SURFACE IMPEDANCE

In the preceding section the electric field was considered to be a "given" function of space and time. In order to know *how* the electromagnetic field is given, one has to solve Maxwell's equations, subject to the boundary conditions which correspond to the experimental conditions. Let us first define the following transform:

$$\tilde{g}(z,\omega)=\int_0^\infty dt\; e^{i\omega t}(L_xL_y)^{-1}\int dx\int dy\; g(\mathbf{r},t), \quad (3.1)$$

where g represents a current density component J_x or a component of the electric field $\mathbf{E}$ or one of the magnetic field $\mathbf{B}$. For reasons which become apparent later, it is convenient to consider ω as a complex quantity with a positive imaginary part ϵ. Let us furthermore define

$$\tilde{g}(k,\omega)=2\int_{-L}^{0} dz\;\cos kz\;\tilde{g}(z,\omega). \quad (3.2)$$

Maxwell's equations imply

$$\text{curl}\;\mathbf{E}(\mathbf{r},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \quad (3.3)$$

and

$$\text{curl}\;\mathbf{B}(\mathbf{r},t)=\mu_0\mathbf{J}(\mathbf{r},t)+\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t), \quad (3.4)$$

where mks units are used and c is the velocity of light. These equations can be transformed according to (3.1). Since the system is assumed to have been in thermal equilibrium until $t=0$, one has $\mathbf{E}(\mathbf{r},0)=0$ and $\mathbf{B}(\mathbf{r},0)=0$. This gives

$$(d/dz)\tilde{E}_x(z,\omega)=i\omega\tilde{B}_y(z,\omega) \quad (3.5)$$

and

$$\frac{d}{dz}\tilde{B}_y(z,\omega)=-\mu_0\tilde{J}_x(z,\omega)+\frac{i\omega}{c^2}\tilde{E}_x(z,\omega). \quad (3.6)$$

Equations (3.5) and (3.6) are, of course, not the only contractions which can be made out of (3.3) and (3.4), but other possibilities will not be needed for what follows. In deriving Eqs. (3.5) and (3.6), explicit use has been made of the periodic boundary conditions along the x and y directions. Elimination of $\tilde{B}_y(z,\omega)$ from (3.5) and (3.6) gives

$$\left(\frac{d^2}{dz^2}+\frac{\omega^2}{c^2}\right)\tilde{E}_x(z,\omega)=-i\omega\mu_0\tilde{J}_x(z,\omega). \quad (3.7)$$

From Eq. (3.7) one can determine the value of the surface impedance, which is defined as the ratio of the electric field $\tilde{E}_x(z,\omega)$ and the magnetic field $\mu_0^{-1}\tilde{B}_y(z,\omega)$ at the surface $z=0$. This quantity equals $i\omega\mu_0 Z(0,\omega)$, if the function $Z(z,\omega)$ is defined as

$$Z(z,\omega)=\tilde{E}_x(z,\omega)/a, \quad (3.8)$$

where a is defined as

$$a=(d/dz)\tilde{E}_x(z,\omega)\,|_{z=0}. \quad (3.9)$$

For the calculation of $Z(z,\omega)$, it is necessary to specify the boundary condition at $z=-L$. Let us require the fields to vanish at $z=-L$, which implies that no radiation is transmitted through the system. This is realized experimentally for an absorbing medium with $L\to\infty$. Application of the transform of Eq. (3.2) to Eq. (3.7) and integration by parts gives

$$[k^2-(\omega/c)^2]\tilde{E}_x(k,\omega)=i\omega\mu_0\tilde{J}_x(k,\omega)+2a. \quad (3.10)$$

It should be noted that the last term of this equation is due to partial integration. In the conventional treatments of the ase, one introduces the artificial device of a current sheet which gives rise to a similar term.

It follows from Eq. (2.1) that $\tilde{J}_x(k,\omega)$ is a functional of $\tilde{E}_x(k,\omega)$, explicitly:

$$\tilde{J}_x(k,\omega)=\sigma(k,\omega)\tilde{E}_x(k,\omega)+\tilde{J}_E(k,\omega), \quad (3.11)$$

where σ and J_E are abbreviations of

$$\sigma(k,\omega)=2\int_0^\infty dz\;\cos kz\;C(|z|) \quad (3.12)$$

and

$$\tilde{J}_E(k,\omega)=\int_0^\infty dk'\;G(k,k')\tilde{E}_x(k',\omega), \quad (3.13)$$

where

$$G(k,k')=\frac{2}{\pi}\int_{-\infty}^{0} dz\int_{-\infty}^{0} dz'\;\cos kz\;\cos k'z'\;Q(z,z'). \quad (3.14)$$

The physical interpretation of Eq. (3.11) is the following: The first term on the right corresponds to the bulk propagation of signals, whereas the second one accounts for the corrections which are due to the inadequacy of the speculum assumption of Eq. (2.5). Substitution of Eq.(3.11) into (3.10) gives

$$[k^2-(\omega/c)^2-i\omega\mu_0\sigma(k,\omega)]\tilde{E}_x(k,\omega)=2a+i\omega\mu_0\tilde{J}_E(k,\omega). \quad (3.15)$$

If one denotes the value of $\tilde{E}_x$ by $\tilde{E}_x{}^0$ when the speculum assumption is used, it follows from Eq. (3.15) that

$$[k^2-(\omega/c)^2-i\omega\mu_0\sigma(k,\omega)]\tilde{E}_x{}^0(k,\omega)=2a. \quad (3.16)$$

This equation would also have been obtained if Eq. (3.15) would be iterated with respect to J_E, a procedure which is only meaningful if the correction due to J_E is small. Equations (3.15) and (3.16) can be combined to give

$$\tilde{E}_x(k,\omega)-\tilde{E}_x^0(k,\omega)=\tfrac{1}{2}i\omega\mu_0 a^{-1}\tilde{J}_E(k,\omega)\tilde{E}_x^0(k,\omega). \quad (3.17)$$

Integration with respect to k gives

$$Z(0,\omega)-Z^0(0,\omega)=\frac{i\omega\mu_0}{2\pi}a^{-2}\int_0^\infty dk\,\tilde{J}_E(k,\omega)\tilde{E}_x^0(k,\omega). \quad (3.18)$$

For this derivation we have used the inverse transform of Eq. (3.2), valid for $L\to\infty$:

$$\tilde{g}(z,\omega)=\frac{1}{\pi}\int_0^\infty dk\,\cos kz\,\tilde{g}(k,\omega). \quad (3.19)$$

Note that Eq. (3.18) is an exact expression for the correction to $Z(0,\omega)$, because of the inadequacy of the speculum assumption. Another expression for this difference follows directly from Eq. (3.18):

$$Z(0,\omega)-Z^0(0,\omega)=i\omega\mu_0 a^{-2}\int_{-\infty}^0 dz\,\tilde{J}_E(z,\omega)\tilde{E}_x^0(z,\omega). \quad (3.20)$$

Although these results are mathematically exact, their usefulness depends on whether or not the iteration of Eq. (3.15) with respect to J_E converges rapidly. In order to calculate the corrections to lowest order of the iteration procedure, one has to evaluate the functional J_E with respect to E_x^0, rather than E_x, in Eq. (3.13).

It should be mentioned that Eqs. (3.15) and (3.16) determine the *ratio* of the field $\tilde{E}_x(k,\omega)$ and a. The latter quantity may very well be considered as fixed and for instance be made equal to 1, in which case $\tilde{E}_x(z,\omega)$ reduces to $Z(z,\omega)$. It should finally be mentioned that the transforms of Eqs. (3.2) and (3.19) should be applied with care. For instance, to interchange differentiations with integrations, in general, is not allowed, because of nonuniform convergence.

4. SEMICLASSICAL TREATMENT OF ANOMALOUS SKIN EFFECT

Except for the restrictions of Sec. 2, our treatment has been quite general thus far. For instance, it applies to systems of interacting electrons. Let us, from now on restrict ourselves to the model of a free-electron system. This does not imply that the particles do not scatter at the boundaries, however. The motion of an electron is hence only determined by the self-consistent electromagnetic field and by the walls of the system. In principle, there is a difficulty connected with the longitudinal field which, on one hand, is a result of two-particle interaction and on the other, a part of the electromagnetic field. However, on account of symmetry, this field does not contribute to the value of the surface impedance. The symmetries responsible are reflections about the $x-z$ and the $y-z$ planes, symmetries which vanish in the presence of an applied magnetic field. If the frequency is sufficiently high, as is the case for the extreme anomalous-skin-effect region, this model is well known to be adequate for conduction in metals at low temperatures. This is due to the fact that the "bottleneck" of the energy transfer from the electromagnetic field to the system is not the transfer of momentum and energy from the electron system to the lattice, but rather the transfer of energy from the electromagnetic field to the electron system.

In order to derive a semiclassical expression for the propagator $K(z,z')$ of Eq. (2.1) or for the equivalent expression in Eq. (3.11), it is necessary to solve the Boltzmann equation for the distribution function $f(\mathbf{r},\mathbf{v},t)$:

$$\frac{\partial f}{\partial t}=-\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\dot{\mathbf{v}}\cdot\frac{\partial f}{\partial \mathbf{v}}. \quad (4.1)$$

Since the boundaries at $z=0$ and $z=-L$ are smooth and rigid, Eq. (4.1) must be solved with the boundary conditions

$$f(x,y,0,v_x,v_y,v_z,t)=f(x,y,0,v_x,v_y,-v_z,t)$$
and
$$f(x,y,-L,v_x,v_y,v_z,t)=f(x,y,-L,v_x,v_y,-v_z,t). \quad (4.2)$$

These conditions, used by Reuter and Sondheimer,[1] are referred to as the speculum conditions for the Boltzmann equation. In order to avoid confusion, we would like to mention that this boundary condition is not *a priori* equivalent to the "speculum assumption" which was defined in Sec. 2. The acceleration $\dot{\mathbf{v}}$ in Eq. (4.1) equals $(e/M)[\mathbf{E}(\mathbf{r},t)+\mathbf{v}\times\mathbf{B}(\mathbf{r},t)]$, where e is the charge of an electron and M is its mass.

In order to solve Eq. (4.1) for linear response theory with respect to the electromagnetic field, f is replaced by $n^0 f^0(\mathbf{v})+f^1(\mathbf{r},\mathbf{v},t)$, where f^0 is the semiclassical equilibrium distribution function and f^1 is the first-order correction. Here, n^0 is the unperturbed electron density, so that

$$\int d^3v\, f^0(\mathbf{v})=1.$$

In order to solve Eq. (4.1), we define the functions

$$F(v_z)=\int dv_x\int dv_y\, f^0(\mathbf{v}), \quad (4.3)$$

$$f_x(z,v_z,\omega)=\int_0^\infty dt\, e^{i\omega t}(L_xL_y)^{-1}\int dx\int dy\int dv_x\int dv_y \times v_x f^1(\mathbf{r},\mathbf{v},t),$$

$$f^+(z,v_z,\omega)=f_x(z,v_z,\omega)+f_x(z,-v_z,\omega), \quad (4.4)$$

and

$$f^-(z,v_z,\omega)=f_x(z,v_z,\omega)-f_x(z,-v_z,\omega). \quad (4.5)$$

It is straightforward to show that f^+ and f^- satisfy the following set of linearized equations:

$$i\omega f^+(z,v_z,\omega)=v_z\frac{\partial}{\partial z}f^-(z,v_z,\omega)-2\frac{n^0e}{M}F(v_z)\bar{E}_x(z,\omega) \quad (4.6)$$

and

$$i\omega f^-(z,v_z,\omega)=v_z(\partial/\partial z)f^+(z,v_z,\omega). \quad (4.7)$$

Partial derivatives with respect to x and y have disappeared owing to the periodic boundary conditions. Equation (4.2) requires f^- to vanish at $z=0$ and $z=-L$, so that $v_z(\partial/\partial z)f^+(z,v_z,\omega)$ also vanishes at these boundaries. Elimination of f^- out of Eqs. (4.6) and (4.7) gives rise to

$$\left(\omega^2+v_z^2\frac{\partial^2}{\partial z^2}\right)f^+(z,v_z,\omega)=2i\omega\frac{n^0e}{M}F(v_z)\bar{E}_x(z,\omega). \quad (4.8)$$

Application of (3.2) gives

$$(\omega^2-v_z^2k^2)\tilde{f}^+(k,v_z,\omega)=2i\omega(n^0e/M)F(v_z)\tilde{E}_x(k,\omega).$$

The current density

$$\tilde{J}_x(k,\omega)=e\int_0^\infty dv_z\,\tilde{f}^+(k,v_z,\omega)$$

is hence given by

$$\tilde{J}_x(k,\omega)=\sigma(k,\omega)\tilde{E}_x(k,\omega), \quad (4.9)$$

with

$$\sigma(k,\omega)=\frac{in^0e^2}{M}\int_{-\infty}^{+\infty}dv_z\frac{F(v_z)}{\omega-v_zk}. \quad (4.10)$$

Comparison of this result with Eq. (3.11) shows the validity of the speculum assumption for a semiclassical free-electron system.

Note that the imaginary part of ω is positive, so that the integral of (4.10) is well defined. In order to find the conductivity for a physical frequency ω, one has to take the limit $\mathrm{Im}(\omega)\to 0$. If the Boltzmann equation had contained a collision term with a relaxation time τ, one would have had to identify $\mathrm{Im}(\omega)$ with τ^{-1}.

5. SURFACE IMPEDANCE FOR SEMICLASSICAL CASE

The value of the surface impedance is obtained if one substitutes the result of Eq. (4.10) into (3.16). Let us consider the low-temperature limit, in which case $F(v_z)$, defined in Eq. (4.3), can be approximated by

$$F(v_z)=\tfrac{3}{4}v_F^{-3}(v_F^2-v_z^2) \quad \text{if} \quad |v_z|\leqslant v_F,$$

and by zero elsewhere. Here, v_F is the velocity of an electron at the Fermi surface. For frequencies which are not too large, it is sufficient to consider only those values of k for which $|\omega/k|\ll v_F$. For metallic densities this approximation is only valid if $\omega\ll 10^{14}$ sec^{-1}. In this limit, Eq. (4.10) reduces to

$$\sigma(k,\omega)=\tfrac{3}{4}\pi n^0e^2/(Mv_F|k|).$$

Consequently, one obtains

$$\bar{E}_x{}^0(k,\omega)/a=2(k^2-i\alpha^3/|k|)^{-1}, \quad (5.1)$$

where the characteristic wave number α is defined as

$$\alpha=(\tfrac{3}{4}\pi\omega\omega_p{}^2)^{1/3}(c^2v_F)^{-1/3}, \quad (5.2)$$

and ω_p is the plasma frequency, $\omega_p=(n^0e^2)^{1/2}(\epsilon_0M)^{-1/2}$. The reciprocal value of α is referred to as the skin depth; it is the characteristic distance over which the electromagnetic field decays into a metal. It is sometimes convenient to represent $Z(z,\omega)$ of Eq. (3.8) as $(2/\pi\alpha)\mathcal{E}(\alpha z)$, if the function $\mathcal{E}$ is defined as

$$\mathcal{E}(x)=\int_0^\infty dk\,\cos kx\frac{k}{k^3-i}. \quad (5.3)$$

The surface impedance follows from $Z(0,\omega)=\frac{2}{3}\alpha^{-1}\times(1+\frac{1}{3}i\sqrt{3})$ in this approximation. The approximation implies that $|\omega/\alpha|\ll v_F$, so that $\omega\ll 10^{14}$ sec^{-1} for metallic densities. The real part of the function $\mathcal{E}$ is just $\mathrm{Re}[\mathcal{E}(x)]=\frac{1}{6}\pi e^x+\frac{1}{3}\pi e^{(1/2)x}\sin(\frac{1}{2}x\sqrt{3}+\frac{1}{6}\pi)$.

6. QUANTUM TREATMENT OF ANOMALOUS SKIN EFFECT

In order to derive a quantum-mechanical expression for the response of the free-electron system, it would not be correct to make use of the procedure of the previous section because the *notion* of a distribution function is not compatible with the principles of quantum mechanics, as is well known. In order to make a comparison with the semiclassical theory, an electron system will be considered which is dynamically identical to that of the preceding section. It is again our purpose to derive an expression like Eq. (3.11) for the response. For the details of the linear response theory, we refer to the Appendix.

The linearized current-density component along the x direction, $J_x(\mathbf{r},t)$, is given by the following expectation value:

$$J_x(\mathbf{r},t)=\tfrac{1}{2}e\sum_j\langle[v_{xj},\,\delta(\mathbf{r}-\mathbf{q}_j)]_+\rangle_t - (e^2/M)A_x(\mathbf{r},t)\sum_j\langle\delta(\mathbf{r}-\mathbf{q}_j)\rangle_{\mathrm{eq}}. \quad (6.1)$$

Here, e is the charge of the electron and M is its mass. The vector potential of the electromagnetic field is $\mathbf{A}(\mathbf{r},t)$. The symbol $\mathbf{v}$ is an abbreviation of $M^{-1}\hbar\mathbf{p}$, where $\hbar\mathbf{p}$ is the momentum operator of the electron. The operator $\mathbf{v}$ corresponds to the velocity if the electromagnetic field were absent, which is the case for the unperturbed system, where $\mathbf{A}=0$. The symbols $\langle\ \rangle_t$ and $\langle\ \rangle_{\mathrm{eq}}$ refer to the nonequilibrium expectation value at time t and to the equilibrium value, respectively. Because the system is translation-invariant

along the x and y directions, the expectation value $\sum_j \langle\delta(\mathbf{r}-\mathbf{q}_j)\rangle_{\rm eq}$ can be replaced by $(L_xL_y)^{-1}\sum_j\langle\delta(z-q_{zj})\rangle_{\rm eq}$. According to Eq. (3.1), one has

$$\bar{J}_x(z,\omega)=e(2L_xL_y)^{-1}\int_0^\infty dt\, e^{i\omega t}\sum_j\langle[v_{xj},\,\delta(z-q_{zj})]_+\rangle_t$$
$$-e^2(ML_xL_y)^{-1}\bar{A}_x(z,\omega)\sum_j\langle\delta(z-q_{zj})\rangle_{\rm eq}. \quad (6.2)$$

The nonequilibrium expectation value of this equation can be evaluated with the aid of Eq. (A4) of the Appendix. In order to do this, one has to replace the single-particle operator j by the corresponding operator $[v_x,\delta(z-q_z)]_+$ in the text. For the evaluation one has to know the complete set of eigenfunctions of the single-particle Hamiltonian of the unperturbed system, and the interaction Hamiltonian H^1.

If, as in Secs. 4 and 5, the boundary at $z=0$ is represented by an infinite potential well, the eigenfunctions of the unperturbed Hamiltonian are given by $\langle\mathbf{r}|\mathbf{k}\rangle=\langle\mathbf{r}|k_x,k_y,k_z\rangle=2^{1/2}\{L_xL_yL\}^{-1/2}\exp[i(k_xx+k_yy)]\sin(k_zz)$. The energy which corresponds to this wave function is $E(\mathbf{k})=(\hbar^2/2M)(k_x^2+k_y^2+k_z^2)$. A complete set of wave functions corresponds to the assembly of quantum numbers: $k_x=2\pi(L_x)^{-1}n_x$, $k_y=2\pi(L_y)^{-1}n_y$, and $k_z=\pi(L)^{-1}n_z$, if n_x and n_y are integers and if n_z is a *positive* integer. It should be emphasized that the corrections which will be found are due to the fact that we use the proper set of eigenfunctions rather than a plane-wave representation.

The single-particle interaction Hamiltonian H_t^1 is given by

$$H_t^1=-(e\hbar/M)\mathbf{p}\cdot\mathbf{A}(\mathbf{q},t)+e\Phi(\mathbf{q},t). \quad (6.3)$$

Here, $\mathbf{A}$ is the vector potential of the electromagnetic field and Φ is the scalar potential which corresponds to the self-consistent longitudinal field. It should be mentioned with respect to Eq. (6.3) that the term which is proportional to $\mathbf{A}^2$ has been left out, because we are only interested in the *linear* response with respect to the electromagnetic field. Furthermore, we have chosen Coulomb's gauge, div $\mathbf{A}(\mathbf{r},t)=0$.

In the Appendix we define the operator

$$H_\omega^1=\int_0^\infty dt\, e^{i\omega t}H_t^1.$$

Because the system is translation-invariant along x and y directions and reflection-symmetric about the x-z and y-z planes, it is obvious that only the following part of H_ω^1 will contribute to the value of $\bar{J}_x(z,\omega)$:

$$-(e\hbar/M)p_x\bar{A}_x(q_z,\omega). \quad (6.4)$$

This statement can also be verified by inspection of Eq. (A4). Application of (A4) to the expression for the current density, Eq. (6.2), gives

$$\bar{J}_x(z,\omega)=\frac{4e^2\hbar}{M^2VL}\sum_{k_x,k_y}\sum_{k_{1z},k_{2z}}k_x^2\sin(k_{1z}z)\sin(k_{2z}z)$$
$$\times\frac{f^0(k_x,k_y,k_{2z})-f^0(k_x,k_y,k_{1z})}{\omega-(\hbar/2M)(k_{2z}^2-k_{1z}^2)}\int_{-L}^0 dz'\sin(k_{1z}z')$$
$$\times\sin(k_{2z}z')\bar{A}_x(z',\omega)-\frac{2e^2}{MV}\bar{A}_x(z,\omega)$$
$$\times\sum_{k_x,k_y}\sum_{k_z}f^0(k_x,k_y,k_z)\sin^2(k_zz). \quad (6.5)$$

Here, V is an abbreviation of L_xL_yL, the volume of the system. The convention for the summation is to sum over all quantum numbers, where it is understood that k_x and k_y can be positive and negative, whereas k_{1z} and k_{2z} can only be positive. Because the summand is even with respect to k_{1z} and k_{2z}, one may extend the summations with respect to these variables to negative values as well, provided a factor $\frac{1}{2}$ is taken into account. In the limit $L\to\infty$ we replace the summations by integrations. Let us also define the following variables: $k_{1z}=p-\frac{1}{2}k$ and $k_{2z}=p+\frac{1}{2}k$. Then Eq. (6.5) reduces to

$$\bar{J}_x(z,\omega)=\frac{e^2\hbar}{16M^2\pi^4}\int_{-\infty}^\infty dk_x\int_{-\infty}^\infty dk_y\int_{-\infty}^\infty dp\int_{-\infty}^\infty dk\,k_x^2$$
$$\times(\cos kz-\cos 2pz)\frac{f^0(k_x,k_y,p+\frac{1}{2}k)-f^0(k_x,k_y,p-\frac{1}{2}k)}{\omega-\hbar pk/M}$$
$$\times\int_{-\infty}^0 dz'(\cos kz'-\cos 2pz')\bar{A}_x(z',\omega)+\frac{e^2}{8\pi^3M}\bar{A}_x(z,\omega)$$
$$\times\int_{-\infty}^\infty dk_xdk_ydk_z\,f^0(k_x,k_y,k_z)(\cos 2k_zz-1). \quad (6.6)$$

Since $f^0(\mathbf{k})$ is even with respect to the components of $\mathbf{k}$, Eq. (6.6) reduces to

$$\bar{J}_x(z,\omega)=\frac{e^2\hbar}{4M^2\pi^4}\int_0^\infty dk(\cos kz)\int_{-\infty}^\infty dp\int_{-\infty}^0 dz'$$
$$\times(\cos kz'-\cos 2pz')\bar{A}_x(z',\omega)\int_{-\infty}^\infty dk_x\int_{-\infty}^\infty dk_y\,k_x^2$$
$$\times\frac{f^0(k_x,k_y,p+\frac{1}{2}k)-f^0(k_x,k_y,p-\frac{1}{2}k)}{\omega-\hbar pk/M}+\frac{e^2}{8M\pi^3}\bar{A}_x(z,\omega)$$
$$\times\int_{-\infty}^\infty dk_xdk_ydk_z\,f^0(k_x,k_y,k_z)(\cos 2k_zz-1). \quad (6.7)$$

The transformations $p\to\frac{1}{2}k$ and $k\to 2p$ are used to derive Eq. (6.7). Let us define the normalized distri-

bution function $f^n(\mathbf{k})$ according to

$$f^n(\mathbf{k})=f^0(\mathbf{k})\left[\int \mathbf{d}^3\mathbf{k}\, f^0(\mathbf{k})\right]^{-1}=(2\pi)^{-3}(n^0)^{-1}f^0(\mathbf{k}),$$

where $n^0=N/V$ is the average electron density. The function $F(k_z)$ is defined as

$$F(k_z)=\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\, f^n(\mathbf{k}). \tag{6.8}$$

Note that

$$\int_{-\infty}^{\infty}F(k)dk=1.$$

Application of the transform of Eq. (3.2) to (6.7) gives

$$\tilde{J}_x(k,\omega)=\frac{n^0e^2\hbar}{M^2}\int_{-\infty}^{\infty}dp[\tilde{A}_x(k,\omega)-\tilde{A}_x(2p,\omega)]\int_{-\infty}^{\infty}dk_x$$
$$\times\int_{-\infty}^{\infty}dk_y\, k_x{}^2\frac{f^n(k_x,k_y,p+\frac{1}{2}k)-f^n(k_x,k_y,p-\frac{1}{2}k)}{\omega-\hbar pk/M}$$
$$+\frac{n^0e^2}{M}\int_{-\infty}^{\infty}dk_z F(k_z)\tilde{A}_x(k-2k_z,\omega)-\frac{n^0e^2}{M}\tilde{A}_x(k,\omega). \tag{6.9}$$

In deriving Eq. (6.9), no other approximations have been made except for the replacement of discrete sums over $\mathbf{k}$ by integrations, and linearization with respect to the electromagnetic field. Let us again consider the low-temperature limit for a free-electron system with metallic density. The function F reduces to $\frac{3}{4}k_F{}^{-3}\times(k_F{}^2-k_z{}^2)$ if $|k_z|<k_F$, and zero elsewhere. For the evaluation of Eq. (6.9) in this limit, the following identities are used:

$$\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\, k_x{}^2f^n(k_x,k_y,k_z)=\frac{3}{16}\frac{(k_F{}^2-k_z{}^2)^2}{k_F{}^3},$$
$$\text{if}\quad |k_z|<k_F$$
$$=0,\qquad \text{elsewhere}$$

and

$$\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\, k_x{}^2[f^n(k_x,k_y,p+\tfrac{1}{2}k)$$
$$-f^n(k_x,k_y,p-\tfrac{1}{2}k)]=-pkF((p^2+\tfrac{1}{4}k^2)^{1/2}),$$
$$\text{if}\quad |p+\tfrac{1}{2}k|<k_F\quad\text{and}\quad |p-\tfrac{1}{2}k|<k_F.$$

Because the electric field $E_x(\mathbf{r},t)$ is given by $-(\partial/\partial t)A_x(\mathbf{r},t)-(\partial/\partial x)\Phi(\mathbf{r},t)$, it follows that $\tilde{E}_x(k,\omega)=i\omega\tilde{A}_x(k,\omega)$. The integrations over k_z and p in Eq. (6.9) are limited to values smaller than $\frac{1}{2}k_F$. This cutoff introduces an error of order α/k_F for the value of the surface impedance, which can be neglected for what follows. Consequently, we obtain the following expression for the current density:

$$\tilde{J}_x(k,\omega)=\sigma(k,\omega)\tilde{E}_x(k,\omega)+\tilde{J}_E(k,\omega), \tag{6.10}$$

where $\sigma(k,\omega)$ and $\tilde{J}_E(k,\omega)$ are defined as follows:

$$\sigma(k,\omega)=\frac{in^0e^2}{2M}\int_{-k_F}^{k_F}dp\frac{F(\frac{1}{2}(p^2+k^2)^{1/2})}{\omega-\hbar pk/2M} \tag{6.11}$$

and

$$\tilde{J}_E(k,\omega)=-\frac{in^0e^2}{2M}\int_{-k_F}^{k_F}dp\,\tilde{E}_x(p,\omega)\frac{F(\frac{1}{2}(p^2+k^2)^{1/2})}{\omega-\hbar pk/2M}. \tag{6.12}$$

Comparison of this result with Eq. (3.11) shows that the speculum assumption is not correct for the quantum-mechanical case, because, if it were correct, the J_E term would have to vanish. It is interesting to note, at this point, that the J_E term would be absent if we had used the improper set of eigenfunctions for the Hamiltonian $e^{ik\cdot r}$. In this case no proper consideraton would be given to the boundaries of the system.

7. SURFACE IMPEDANCE FOR QUANTUM-MECHANICAL CASE

Equation (6.11) for $\sigma(k,\omega)$ is readily identified with the semiclassical expression Eq. (4.10), apart from a small correction of order $(k/k_F)^2$. This correction corresponds to the Landau diamagnetism, and can be neglected in what follows. The extra contribution J_E hence contains the quantum corrections to the semiclassical theory.

If it is assumed that the correction to the surface impedance due to J_E will be small, the integral equation for the quantity $\tilde{Z}(k,\omega)=\tilde{E}_x(k,\omega)/a$ can be solved by iteration with respect to J_E. The obtained results are in agreement with this assumption. As mentioned in Sec. 3, the lowest-order correction of the surface impedance is obtained by evaluating the functional J_E with respect to $E_x{}^0$, rather than E_x. Consequently, $\tilde{E}_x(p,\omega)$ on the right of Eq. (6.12) has to be replaced by $\tilde{E}_x{}^0(p,\omega)$ in lowest order. The function $\tilde{E}_x{}^0(p,\omega)$ is given by Eq. (5.1). As a result, one obtains

$$\frac{\tilde{J}_E(k,\omega)}{a}=-\frac{2in^0e^2}{\alpha^3\hbar}\int_{-g}^{+g}dp\left(p^2-\frac{i}{|p|}\right)^{-1}$$
$$\times\frac{F(\frac{1}{2}(\alpha^2p^2+k^2)^{1/2})}{\lambda-pk/\alpha+i\epsilon}, \tag{7.1}$$

where $g=k_F/\alpha$, and the parameter λ is defined as

$$\lambda=2M\omega/(\alpha^2\hbar). \tag{7.2}$$

It will be shown that the quantum correction in the theory of the ase is determined by this parameter λ. If the Fermi surface is not spherical, the quantum correction is likewise determined by a parameter λ which satisfies Eq. (7.2), where M represents the effec-

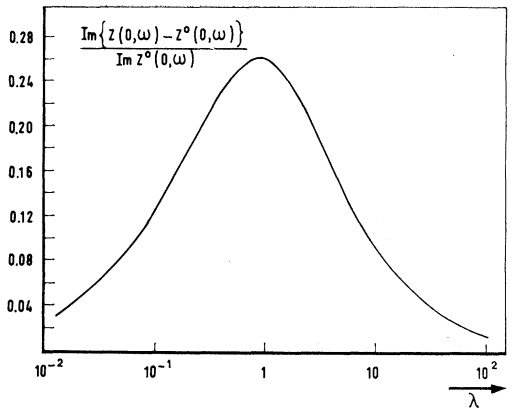

FIG. 2. Relative quantum correction of the real part of the surface impedance for a free-electron system ($\tau=\infty$).

tive mass for the Z direction; explicitly,

$$\lambda=2\hbar\omega\alpha^{-2}(\partial^2E/\partial k_z^2)^{-1}.$$

The orders of magnitude of λ and g for an isotropic free-electron system with metallic density are $\lambda\sim10^{-2}\ \omega^{1/3}$ and $g\sim10^7\ \omega^{-1/3}$. More precisely, in mks units, $\lambda=0.023\ \omega^{1/3}(e/E_F)^{2/3}$, if E_F is the Fermi energy of an isotropic free-electron system.

The just-mentioned assumption that J_E is a small correction to the semiclassical treatment implies that the value of $|i\omega\mu_0\tilde{J}_E(k,\omega)|$ is small in comparison to $|2a|$, in view of Eq. (3.15), This implies that

$$\left|\frac{1}{\pi}\int_{-g}^{+g}\frac{dp}{p^2-i/|p|}\,\frac{1}{\lambda-pk/\alpha+i\epsilon}\right|<1,$$

if use is made of the definition of α, and if $F(\frac{1}{2}(\alpha^2p^2+k^2)^{1/2})$ has been replaced by $3/4k_F$, which is legimate apart from a correction of the order of g^{-2}. According to this estimation, the extra current-density term in Eq. (3.15) is small if $\lambda>1$. For this reason we expect the lowest-order result of the iteration procedure to be a good estimation for $\lambda>1$ or for $\omega>10^6$ sec^{-1}. For the proof we have made use of the following identity:

$$\lim_{\epsilon\to0+}\int_{-\infty}^{+\infty}dp\frac{1}{p^2-i/|p|}\,\frac{1}{p-a-i\epsilon}=-\tfrac{2}{3}\pi\frac{a}{a^2+1}$$
$$-2i\frac{a\ln(a)}{a^6+1}-\frac{2\pi i\sqrt{3}}{9}\frac{a^3(a^2+1)}{a^6+1}+i\pi\frac{a^4}{a^6+1},\quad(7.3)$$

if $\mathrm{Im}(a)\geqslant0$. The quantum corrections are extremal for k values in the vicinity of $k\lesssim\alpha\lambda$. These values of k are small compared to k_F if $\lambda<g$, which corresponds to $\omega<10^{14}$ sec^{-1}. For this reason we restrict our analysis, based on iteration of Eq. (3.15), to the frequency region of $10^6<\omega<10^{14}$ sec^{-1}. The quantum correction to the surface impedance follows from Eq. (3.18):

$$Z(0,\omega)-Z^0(0,\omega)$$
$$=2\alpha^{-1}\pi^{-2}\lim_{\epsilon\to0}\int_0^\infty dk\left(k^2-\frac{i}{|k|}\right)^{-1}$$
$$\times\int_{-g}^{g}dp\left(p^2-\frac{i}{|p|}\right)^{-1}(\lambda-pk+i\epsilon)^{-1}$$
$$=-4(\alpha\lambda\pi)^{-1}\int_0^\infty dk\left(k^2-\frac{i}{|k|}\right)^{-1}\left[-\frac{1}{3}\frac{1}{1+\xi^2}\right.$$
$$\left.+i\left(\frac{1}{\pi}\frac{\xi^4\ln(\xi)}{1+\xi^6}-\frac{\sqrt{3}}{9}\frac{1+\xi^2}{1+\xi^6}+\frac{1}{2}\frac{\xi}{1+\xi^6}\right)\right],\quad(7.4)$$

where ξ is an abbreviation of $\lambda^{-1}k$. Here, the second identity has been derived with the aid of Eq. (7.3). Equation (7.4) can be evaluated analytically:

$$Z(0,\omega)-Z^0(0,\omega)$$
$$=\frac{2}{\pi\alpha}\frac{\lambda}{1-\lambda^6}\left[\frac{1}{\pi}\ln^2\lambda+\frac{\pi}{54}(-8\lambda^4+9\lambda^3-8\lambda^2+7)\right]$$
$$+\frac{2}{\pi\alpha}\frac{i\lambda}{1-\lambda^6}\left[(\lambda^3-\tfrac{1}{3})\ln\lambda+\frac{4\pi\sqrt{3}}{27}(\lambda^2-\lambda^4)\right].\quad(7.5)$$

The relative increment of the imaginary part of $Z(0,\omega)$ with respect to the semiclassical treatment is shown in Fig. 2.

For the derivation of our results it was explicitly assumed that the electrons are free. The question arises, to what extent our calculations apply to a realistic electron system with interactions. To answer it, we consider a relaxation-time model in the usual sense, i.e., by replacing the frequency ω with $\omega+i/\tau$. It follows from a comparison between Eqs. (6.12) and (7.1) that the parameter λ has to be replaced by$\lambda\ \{1+i/\omega\tau\}$. Here, τ is referred to as the *relaxation time*. Careful inspection of the steps leading to Eqs. (7.4) and (7.5) shows that Eq. (7.5) is also valid for complex values of λ. This extension of the theory is only valid if the electric field is given by Eq. (5.1), i.e., for the extreme anomalous-skin-effect region. It appears that the effect of relaxation is to reduce the quantum correction which we have calculated for the free-electron model. The effect of this reduction is shown in Fig. 3.

A physical explanation for the quantum correction which we calculated in this section is the following. Although *all* electrons contribute to the value of the surface impedance in the semiclassical theory, it is obvious, after inspection of Eqs. (4.10) and (5.1), that the contribution of certain electrons is more important than that of the others. In particular, it appears that the electrons for which $v_z\sim\omega/k\sim\omega/\alpha$, i.e., those which contribute to the pole of Eq. (4.10), are

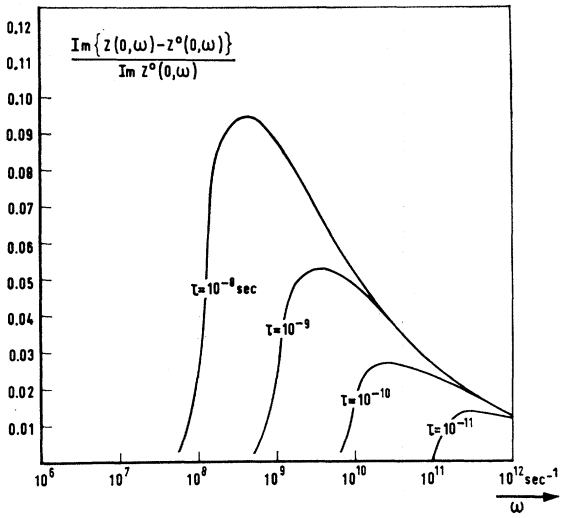

FIG. 3. Relative quantum correction of the real part of the surface impedance for different values of the relaxation time τ. For convenience we have taken $\lambda=0.01\omega^{1/3}$.

dominant. Corresponding to Pippard's terminology,[7] such electrons are referred to as "effective." If the electrons are free ($\tau\rightarrow\infty$), an effective electron spends a time comparable with the period of the electromagnetic field inside the skin layer and is hence capable of absorbing an optimum amount of energy out of the radiation field. The classical boundary condition, the speculum condition, implies that the velocity and position coordinates at $z=0$ of these effective electrons have to be determined simultaneously. From a quantum-mechanical point of view such a statement can be made provided the de Broglie wavelength, which corresponds to the velocity component normal to the surface, is sufficiently small—in particular, if it is small compared to the skin depth. For an effective electron this implies that $2M\omega/\alpha^2\hbar=\lambda\gg1$, which is in agreement with Eq. (7.5) for the free-electron case. If $\omega\tau\gg1$, Pippard's effectiveness criterion implies that those electrons are effective which spend a time comparable with the relaxation time inside the skin region. Formally, this corresponds to replacement of ω by $1/\tau$, which implies again that the semiclassical treatment is valid for $|\lambda|\gg1$.

Summarizing, the quantum corrections for the semiclassical theory of the anomalous skin effect depend on the ratio of the skin depth to a typical de Broglie wavelength. This wavelength corresponds to the velocity component, normal to the surface, of an effective electron. The magnitude of the correction is given by Eq. (7.5), where λ is given by Eq. (7.2). The quantity ω is the frequency if the electrons are free, and it is the frequency multiplied with $1+i/\omega\tau$ if a relaxation time is introduced. The results of this treatment are assumed to be valid for the frequency region $10^6<\omega<10^{14}$ sec^{-1} on behalf of the approximations which have been made.

[7] A. B. Pippard, Proc. Roy. Soc. (London) **A224**, 273 (1954).

It may be that in practice the quantum corrections are larger than calculated here, owing to the effective mass of the electrons. It should finally be mentioned that the quantum corrections, which have been discussed in this paper, may very well have been overlooked by experimentalists. The reason is that diffuse scattering also gives rise to an increment of the surface resistivity which can be comparable with the quantum correction.

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APPENDIX: LINEAR RESPONSE THEORY

The expectation value of an operator F at time t is given by $\langle F\rangle_t=\mathrm{Tr}[\rho(t)F]$, where $\rho(t)$ is the density operator which satisfies $d\rho(t)/dt=(i\hbar)^{-1}[\mathcal{H},\rho(t)]_-$.[8–10] The Hamiltonian $\mathcal{H}$ may depend on t. Obviously, $\langle F\rangle_t$ satisfies $d/dt\langle F\rangle_t=i\hbar^{-1}\langle[\mathcal{H},F]_-\rangle_t$.

The single-particle density operator $f(t)$ is defined by the requirement that $\langle a^\dagger(k_1)a(k_2)\rangle_t=\langle k_2|f(t)|k_1\rangle$, where $a^\dagger(k)$ and $a(k)$ are creation and destruction operators of a particle in a single-particle state $|k\rangle$. The expectation value of an operator $J=\sum_n j_n$ (n is the particle index) is given by $\langle J\rangle_t=\mathrm{Tr}[f(t)j]$, where j is the single-particle operator corresponding to J. The trace in this expression must be evaluated for a complete set of single-particle states. In case $\mathcal{H}=\sum_n H_n$, where H is the single-particle Hamiltonian, it follows that $(d/dt)f(t)=(i\hbar)^{-1}[H,f(t)]_-$. In order to prove this, one has to investigate the equation of motion of the operator $F=a^\dagger(k_1)a(k_2)$, using the proper (anti)commutation relations. If the system is in thermal equilibrium, the operator f is diagonal in the representation of H, so that $f=f^0=(e^{\beta(H-\mu)}\pm1)^{-1}$ (+ sign for Fermi, − sign for Bose statistics).

If $\mathcal{H}$ contains a time-independent part $\mathcal{H}^0$ and a time-dependent interaction $\mathcal{H}_t^1$, the equation of motion for $\rho(t)$ can be linearized with respect to the interaction. In this approximation one obtains for $\langle F\rangle_t$

$$\langle F\rangle_t=(i\hbar)^{-1}\int_0^t dt_1\,\mathrm{Tr}\{\rho^0[F(t-t_1),\mathcal{H}_{t_1}^1]_-\}. \quad \text{(A1)}$$

Here, the operator $F(t)$ is defined as $F(t)=e^{i\hbar^{-1}\mathcal{H}^0t}\times Fe^{-i\hbar^{-1}\mathcal{H}^0t}$. Equation (A1) can also be written as

$$\int_0^\infty dt\,e^{i\omega t}\langle F\rangle_t=(i\hbar)^{-1}\langle[F_\omega,\mathcal{H}_\omega^1]_-\rangle_{\mathrm{eq}}, \quad \text{(A2)}$$

[8] W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957).
[9] H. Ehrenreich and M. H. Cohen, Phys. Rev. **115**, 786 (1959).
[10] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

if

$$F_\omega=\int_0^\infty dt\, e^{i\omega t}F(t) \quad \text{and} \quad \mathcal{H}_\omega{}^1=\int_0^\infty dt\, e^{i\omega t}\mathcal{H}_t{}^1.$$

The symbol $\langle\ \rangle_{\rm eq}$ refers to the equilibrium expectation value.

If the particles are noninteracting, it is obvious that the expectation value of an operator $J=\sum_n j_n$ satisfies the analogous equation

$$\int_0^\infty dt\, e^{i\omega t}\langle J\rangle_t=(i\hbar)^{-1}\langle[j_\omega,H_\omega{}^1]_-\rangle_{\rm eq}. \tag{A3}$$

The proof of Eq. (A3) is analogous to that of (A2) on account of the analogy of the equations of motion for $\rho(t)$ and $f(t)$. Equation (A3) can be made explicit for the complete set of single particle eigenstates of H, denoted by $|n\rangle$:

$$\int_0^\infty dt\, e^{i\omega t}\langle J\rangle_t=\sum_n\sum_m \langle n|j|m\rangle\langle m|H_\omega{}^1|n\rangle \times\frac{f^0(E_n)-f^0(E_m)}{\hbar\omega+E_n-E_m}. \tag{A4}$$

Classically, the expectation value of an observable $F(q_k,p_k)$ is defined as

$$\langle F\rangle_t=\int dq_k dp_k\, \rho(t)F(q_kp_k),$$

if $\rho(t)$ is the density function and (q_kp_k) is a symbolic notation for all coordinates and momenta of the particles. By convention, $q_k(t)$ and $p_k(t)$ refer to the values of these quantities if the system were not disturbed, i.e., if $\mathcal{H}=\mathcal{H}^0$. Furthermore, $q_k(0)=q_k$ and $p_k(0)=p_k$. The density function satisfies $(d/dt)\rho(t)=\{\mathcal{H},\rho(t)\}_{\rm PB}$, where $\{A,B\}_{\rm PB}$ is the Poisson bracket, defined as

$$\sum_k\left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k}-\frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}\right).$$

Linearization with respect to the interaction $\mathcal{H}_t{}^1$ leads to the classical analog of (A1):

$$\int_0^\infty dt\, e^{i\omega t}\langle F\rangle_t=\langle\{F_\omega,\mathcal{H}_\omega{}^1\}_{\rm PB}\rangle_{\rm eq}.$$

Here

$$F_\omega=\int_0^\infty dt\, e^{i\omega t}F[q_k(t)p_k(t)], \tag{A5}$$

and

$$\mathcal{H}_\omega{}^1=\int_0^\infty dt\, e^{i\omega t}\mathcal{H}_t{}^1(q_kp_k).$$

If the particles are noninteracting, it is convenient to make use of the single-particle distribution function $f(\mathbf{r},\mathbf{v},t)$, which satisfies $(d/dt)f(t)=\{H,f(t)\}_{\rm PB}$, where the PB refers to the coordinates and momentum components of a single particle. Hence classical analog of (A3) is

$$\int_0^\infty dt\, e^{i\omega t}\langle J\rangle_t=\langle\{j_\omega,H_\omega{}^1\}_{\rm PB}\rangle_{\rm eq}. \tag{A6}$$

Here, the Poisson bracket $\{\ \}_{\rm PB}$ refers to the position and momentum coordinates of a single particle. Equation (A6) is also used for the semiclassical case, where the unperturbed distribution function is identified with the quantum-statistical one.

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Quantum Corrections in the Theory of the Anomalous Skin Effect in the Presence of a Magnetic Field

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A comparison is made between the semiclassical and the quantum treatment of the anomalous skin effect for an electron system which is placed in a static magnetic field. It is shown that the surface impedance can generally be expressed in terms of a Green's function. For weak applied magnetic fields, there exist two types of quantum corrections, one of which has been treated by Prange; and the other is discussed in this paper. Both corrections imply that the surface resistance oscillates as a function of the applied field B. The frequency dependence of the values of B for which the oscillations occur, $\omega^{3/2}$, is in agreement with experiments by Koch, Kuo, Khaikin, *et al.*

1. INTRODUCTION

IN a previous paper,[1] referred to as I, we have investigated the inequivalence of the semiclassical and the quantum treatments of the anomalous skin effect for a free-electron system in the absence of an external magnetic field. It was shown that both theories lead to equivalent results, provided the value of the skin depth is large compared to a characteristic de Broglie wavelength. This wavelength was shown to correspond to the velocity component of an effective electron in a direction perpendicular to the surface. If this condition is not satisfied, the uncertainty relation

[1] A. P. van Gelder, preceding paper, Phys. Rev. 184, 833 (1969).