Liquid ^{3}He. V. Calculations of Landau Parameters by Brueckner Theory*

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Using the reaction matrix or G matrix obtained by Brueckner theory for liquid ^{3}He, rearrangement terms are evaluated, and the Landau f function is estimated from microscopic theory. Taking the second variational derivative with respect to the particle occupation number of the total energy, written in terms of Brueckner theory, an approximative expression for the f function is obtained. It includes the G matrix, and the first and second derivatives of the G-matrix elements. The density dependence of the G matrix, i.e., the derivatives with respect to the Fermi momentum, is evaluated numerically, and the spin-independent and spin-dependent parts of the Landau f function are calculated. The effective interaction changes completely, from an average attractive G matrix to an average repulsive f function. Also, the coefficients of the expansion of the Landau f function in terms of Legendre polynomials are estimated, and the calculated values are in fair agreement with experimental results. In lowest order, the calculations give 5.1 to 8.7 for F_0, 2.8 to 3.7 for F_1, -0.8 to -0.4 for Z_0, and -1.2 to -0.4 for Z_1. The experimental values are, respectively, 10.77, 6.25, -0.665, and -0.72. The value -0.72 for Z_1 is, however, obtained from the exclusion-principle sum rule for the scattering amplitude in a way which is, at best, very uncertain. According to our calculated coefficients for $L>1$, the experimental value for Z_1 is underestimated. The sign is correct, but the absolute value should be larger.

I. INTRODUCTION

A complete theory of liquid ^{3}He should give the properties of the liquid as a function of temperature, starting from an empirically derived interaction between the atoms. The most important properties which should then be understood are energy properties, density, specific heat, compressibility, magnetic susceptibility, thermal expansion, transport properties, and sound propagation.

Landau[1,2] has introduced a phenomenological theory of a system of strongly interacting fermions at very low temperatures. The theory is based on the validity of perturbation theory, starting from a gas of noninteracting atoms or particles, and is semiphenomenological, giving the macroscopic behavior of normal Fermi liquids at low temperatures. The theory has been applied to liquid ^{3}He, and has been used to correlate experimental properties, assuming an effective interaction given by experimental studies of the properties of the liquid. The Landau theory is valid when the system is highly degenerate, and only macroscopic phenomena are considered. Then, the number of excited quasiparticles involved remains very small compared to the total density, and the theory is useful if the lifetime of a quasiparticle is much longer than the time between successive collisions. The Landau theory corresponds, in fact, to an expansion in powers of the fraction of excited particles.

The basic function $f(\vec{k}, \vec{k}'; \vec{\sigma},\vec{\sigma}')$ in the Landau theory is the second variational derivative of the total energy with respect to the particle occupation number. The f function is the effective interaction between a pair of quasiparticles, and depends on all the particles in the many-body system. It is in principle a function of the complete state of the system, and not only of the particle momenta and spins. For states near the Fermi surface, the momenta $\vec{k}$ and $\vec{k}'$ in f can be replaced by vectors equal to k_F in magnitude. Then f is a function of only the angle between $\vec{k}$ and $\vec{k}'$, and of the spins.

After the assumption that all the spin dependence of f is of exchange origin, the function is separated into a spin-independent or "direct" part, and a spin-dependent or "exchange" part, which are each expanded in terms of Legendre polynomials. These functions are considered to be phenomenological parameters, and the coefficients of lowest order in the series are determined from the experimental values of the compressibility or the hydrodynamic sound velocity, the specific heat, and the magnetic susceptibility. Neither these quantities nor the total energy, the pressure, the equilibrium density, or the coefficient of thermal expansion, can then be predicted from the theory. But it is possible to make qualitative predictions of the transport coefficients, i.e., the viscosity, the thermal conductivity, and the spin diffusion. The transport coefficients depend on averages over the angles of the final momenta as well as over the angles of the initial momenta, but this information cannot be obtained from the

f function itself. There is, however, a close relationship between the stationary properties of the liquid and the transport coefficients.

Formal quantum field theory has been applied to the quantum statistical mechanics of the many-body problem by, for example, Galitskii and Migdal,[3] Martin and Schwinger,[4] and Luttinger and Ward.[5] Their work shows that the Landau theory can be derived from a microscopic theory where the true system is related to the system of non-interacting particles by a convergent perturbation theory. The validity of perturbation theory is fairly well supported by experimental results, and theoretical proofs are given by Klein[6] and by Noziéres and Luttinger.[7] This gives a definition of a quasiparticle, but we are not yet able to calculate exactly its properties from the properties of the free ^{3}He atoms. There are, however, several approximate methods or microscopic theories.

The first serious attempt to calculate properties of liquid ^{3}He at zero temperature from first principles was made by Brueckner and Gammel.[8] The basic quantity in the Brueckner theory is the reaction matrix or G matrix, which is obtained by solution of a two-body problem. If we know an accurate method for determining the G matrix, those of the liquid properties mentioned above, which the Landau theory cannot provide, can then be calculated or estimated, and thus predicted from a microscopic theory. Also the transport coefficients can be calculated or at least estimated, which provides a comparison with the Landau theory. Calculations of G-matrix elements, and then of most of these liquid properties, have already been reported by the author in some earlier papers.[9-12] The Brueckner theory can, however, be formulated in such a way that it can be compared directly with the Landau theory of a Fermi liquid, and then used to determine the phenomenological function $f(\vec{k}, \vec{k}'; \vec{\sigma}, \vec{\sigma}')$ from a microscopic point of view. We can calculate the coefficients in the series expansion of f to any order, and most of the properties of liquid ^{3}He can then be estimated.

Two particles in a many-body system interact not only directly via the two-body potential, but also indirectly through all the other particles in the medium. Scattering phase shifts indicate that the interaction of two free ^{3}He atoms with the momenta and spins of quasiparticles in liquid ^{3}He is on the average attractive. The experimental spin-averaged effective quasiparticle interaction f defined in the Landau theory is, however, repulsive in liquid ^{3}He. Thus, the direct interaction and even the G-matrix elements are attractive, and give, for instance, a negative binding energy, while the net effective Landau interaction or f function is repulsive, giving, for instance, a low compressibility. Somehow the many-body medium transforms the attractive interaction into an effective repulsive f interaction. This is due to the rearrangement effects. If a particle is added to a many-body system at constant volume, the binding energy per particle decreases as the system moves from its equilibrium density. The particles originally present are redistributed, and this rearrangement contribution is large enough to make the angular average of the spin-averaged effective quasiparticle f function repulsive.

II. LANDAU THEORY

The Landau theory[1,2] of a Fermi liquid assumes that there is a one-particle distribution function $n(\vec{k}, \vec{\sigma})$ for the quasiparticles, where $\vec{k}$ is the momentum or wave vector, and $\vec{\sigma}$ is the spin orientation or vector. Then the entropy S is

$$S = -\sum_{k,\sigma} [n \ln n + (1-n)\ln(1-n)], \tag{2.1}$$

and the total number of particles is

$$N = \sum_{k,\sigma} n(\vec{k}, \vec{\sigma}). \tag{2.2}$$

We consider a very large number N of atoms with equal mass in a very large volume Ω.

If E is the total energy, then one can define the energy $\epsilon(\vec{k}, \vec{\sigma})$ of a quasiparticle by the variation in the total energy when one quasiparticle is added to the system, i.e.,

$$\delta E = \sum_{k,\sigma} \epsilon\, \delta n = (2\pi)^{-3} \sum_{\sigma} \int \epsilon(\vec{k}, \vec{\sigma}) \delta n(\vec{k}, \vec{\sigma}) d^3k. \tag{2.3}$$

The quasiparticle distribution function is found by optimizing S with N and E constant, which gives

$$n(\vec{k}, \vec{\sigma}) = \{1 + \exp[(\epsilon - \mu)/KT]\}^{-1}, \tag{2.4}$$

where K is Boltzmann's constant, T is the temperature, and μ is the chemical potential. The total number of particles in the Fermi sea is

$$N = (2\pi)^{-3} 2\Omega \int_{k<k_F} d^3k = \tfrac{1}{3}\Omega k_F^{\,3}/\pi^2, \tag{2.5}$$

where a factor of 2 comes from two spin states per momentum state. The Fermi momentum k_F is then related to the density by

$$\rho = N/\Omega = \tfrac{1}{3} k_F^{\,3}/\pi^2. \tag{2.6}$$

We define an effective mass at the Fermi surface by

$$\nabla_k \epsilon(\vec{k},\vec{\sigma})|_{k=k_F} = (\hbar^2/M^*)k_F , \tag{2.7}$$

$$\text{or} \quad M^* = \hbar^2 k_F / [\partial\epsilon/\partial k]_{k=k_F} . \tag{2.8}$$

The effective mass is significant because, at low temperatures only the region near the Fermi surface is of importance. In fact, it is only in this region that the quasiparticles are well defined.

The interparticle forces will enter into the quasiparticle energy $\epsilon(\vec{k},\vec{\sigma})$, and the influence of one quasiparticle on the energy of another is taken into account in defining the basic function $f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')$ of the Landau theory by the relation

$$\delta\epsilon(\vec{k},\vec{\sigma}) = \sum_{k',\sigma'} f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')\delta n(\vec{k}',\vec{\sigma}')$$

$$= (2\pi)^{-3} \sum_{\sigma'} \int f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')\delta n(\vec{k}',\vec{\sigma}')d^3k' . \tag{2.9}$$

For an ideal gas, the f function is zero, but it is important in the theory of a Fermi liquid. This quantity must be determined phenomenologically in the Landau theory, but it is the aim of microscopic theories to derive it from two-body forces. If we assume that all the spin dependence of f is of exchange origin, we may write f in the form

$$f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}') = f(\vec{k},\vec{k}') + \vec{\sigma}\cdot\vec{\sigma}'\cdot z(\vec{k},\vec{k}') . \tag{2.10}$$

If the temperature T is small compared to the degeneracy temperature T_F of the system, then $n(\vec{k},\vec{\sigma})$ may be expanded in powers of T/T_F, and, to lowest order, the sound velocity v_S, the specific heat C, and the magnetic susceptibility χ can be expressed in terms of averages of $f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')$ with

$$|\vec{k}| = |\vec{k}'| = k_F . \tag{2.11}$$

The function (2.10) is now needed for values of its arguments near the Fermi surface, so we write

$$f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')|_{k=k'=k_F} = f(\theta,\vec{\sigma},\vec{\sigma}')$$

$$= (\pi^2\hbar^2/k_F M^*)\sum_L (F_L + \vec{\sigma}\cdot\vec{\sigma}' Z_L)\mathcal{P}_L(\cos\theta), \tag{2.12}$$

where the spin dependence is fixed by invariance arguments, and θ is the angle between $\vec{k}$ and $\vec{k}'$. The function $\mathcal{P}_L(\cos\theta)$ is the Legendre polynomial of order L, and the coefficients F_L and Z_L can be determined by experiment for low L. The coefficient in front of the sum in Eq. (2.12) is the inverse of the density ν of quasiparticle states per unit volume at the Fermi surface, i.e.,

$$\nu = 8\pi\hbar k_F M^*/(2\pi\hbar)^3 = k_F M^*/\pi^2\hbar^2 . \tag{2.13}$$

The first term in the Legendre polynomial series in Eq. (2.12) is finally given by

$$F_0 = 3MM^* v_s^{\,2}/\hbar^2 k_F^{\,2} - 1,$$
$$Z_0 = \gamma^2 k_F M^*/\pi^2\hbar^2\chi - 1, \tag{2.14}$$

where γ is the magnetic moment of a ^{3}He atom. Our definition of Z_L then differs by a factor of 4 from what is often used by others, because we use Dirac's definition of σ instead of Landau's definition. With his formalism, Landau also derived from Galilean relativity or invariance a relation between the effective mass M^* and the correlation function f. It is given through the coefficient F_1 by

$$F_1 = 3(M^*/M - 1) . \tag{2.15}$$

In the limit of low temperature, the specific heat C is related to the corresponding value for an ideal gas by

$$C/C_F = M^*/M = m^* . \tag{2.16}$$

The velocity of sound is determined by

$$v_s^{\,2} = \frac{\partial\mathcal{P}}{\partial(M\rho)} = \frac{N/M}{\partial\mu/\partial N} = \frac{1}{3}\frac{k_F}{M}\frac{\partial\rho}{\partial k_F} , \tag{2.17}$$

where $\mathcal{P}$ is the pressure, and the magnetic susceptibility χ is related to that of an ideal gas χ_F by

$$\chi/\chi_F = (1+\tfrac{1}{3}F_1)/(1+Z_0), \tag{2.18}$$

where $\chi_F = \gamma^2 k_F M/\pi^2\hbar^2$. (2.19)

The function $f(\vec{k},\vec{k}';\ \vec{\sigma},\vec{\sigma}')$ is in general temperature-dependent. For normal systems at very low temperatures, it becomes independent of T, and C is proportional to T, i.e.,

$$C = \tfrac{1}{3}(M/\hbar^2)k_F(1+\tfrac{1}{3}F_1)K^2T, \tag{2.20}$$

and v_S and χ should be constant. As we see from Eqs. (2.14) and (2.15),

$$v_s = \frac{\hbar k_F[\frac{1}{3}(1+F_0)/(1+\frac{1}{3}F_1)]^{1/2}}{M},$$
$$\chi = \gamma^2 k_F M(1+\tfrac{1}{3}F_1)/[\pi^2\hbar^2(1+Z_0)] . \tag{2.21}$$

The transport properties of the Fermi liquid have been considered by Landau.[1,13] Neglecting spin dependence and including a collision term, he assumed that for nonequilibrium problems, the distribution function $n(\vec{k},\vec{r},t)$ can be obtained from a Boltzmann equation. Abrikosov and Khalatnikov[14] have calculated the viscosity η and the thermal conductivity κ at very low temperatures, and their method has been used by Hone[15] to calculate the spin-diffusion coefficient D. η and κ may be expressed in terms of the mean-free times τ_η and τ_κ, and D by a corresponding relaxation time τ_D.

At very low temperatures, τ_η, τ_κ, and τ_D are given mainly by scattering of particles on the Fermi surface, and the calculations involve a scattering function between quasiparticles, $\omega(\theta,\varphi)$, which is the transition probability for the scattering of quasiparticles around the Fermi surface. Here, θ is the angle between the initial momenta of the colliding quasiparticles, and φ is the angle between the planes formed by the initial and final pairs of momentum vectors. In the Landau theory, the scattering function $\omega(\theta,\varphi)$ is obtained from the forward-scattering amplitude $a(\theta,\vec{\sigma},\vec{\sigma}')$ or $a(\theta,\varphi)$ for two quasiparticles colliding at initial angle θ. This can be written

$$\omega(\theta,\varphi)\approx(2\pi/\hbar)[a^2(\theta,\vec{\sigma},\vec{\sigma}')]_{\mathrm{av}} \quad . \tag{2.22}$$

The Landau theory is formulated so that the spin of a quasiparticle is approximately a good quantum number. For calculations of the viscosity η and the thermal conductivity κ, collisions involving parallel and antiparallel spins must be weighted equally in the average of a^2 over spins, because transport of momentum and energy is involved, and there are no explicitly spin-dependent quantities. But in the calculation of the diffusion coefficient D, we must consider only antiparallel spin collisions.

The quantity $a(\theta,\vec{\sigma},\vec{\sigma}')$ may be related to $f(\theta,\vec{\sigma},\vec{\sigma}')$, but it is not equal to f because in a, one considers a scattering with no energy change, but with some small momentum transfer. In f, the quasiparticles travel forward, but scatter with some small energy transfer. The functions f and a are both proportional to the vertex part of a two-particle Green's function in the limit where both the momentum transfer q and the energy transfer w vanish. But in the calculation of f the limit is taken so that $(w/q)\to\infty$, while in the calculation of a the limit is such that $(w/q)\to 0$.

Landau has shown how to relate $f(\theta,\vec{\sigma},\vec{\sigma}')$ and $\omega(\theta,0)$ at zero temperature, or how to relate f and a, and then get numerical estimates for η,κ, and D. The function $a(\theta,\vec{\sigma},\vec{\sigma}')$ is given by a relation analogous to Eq. (2.12), i.e.,

$$a(\theta,\vec{\sigma},\vec{\sigma}')=(\pi^2\hbar^2/k_F M^*) \times \sum_L (B_L+\vec{\sigma}\cdot\vec{\sigma}' C_L)P_L(\cos\theta), \tag{2.23}$$

where B_L and C_L are related to F_L and Z_L in a simple way by the connection

$$B_L=F_L/[1+F_L/(2L+1)] \quad , \qquad C_L=Z_L/[1+Z_L/(2L+1)] \quad . \tag{2.24}$$

At very low temperatures, f and ω become approximately independent of temperature, and Landau's relationship should then be a fairly good approximation. We have found earlier,[12] however, that the φ dependence of the scattering amplitude is important in calculations of the transport coefficients directly from the G matrix.

For an uncharged Fermi liquid, we have an exact relationship between the Legendre polynomial expansion coefficients of the Landau f function in the form of a sum rule. It depends on the fact that the forward scattering amplitude for two fermions of the same spin vanishes. Then we can write

$$\sum_L (B_L+C_L)=\sum_L F_L/\left(1+\frac{F_L}{2L+1}\right) +\sum_L Z_L/\left(1+\frac{Z_L}{2L+1}\right)=0 \quad , \tag{2.25}$$

which is the exclusion-principle sum rule for the system.

A comparison of the Landau theory with experiments has been given by Hone,[16] and later experimental results are given by Abel *et al.*[17] and by Anderson *et al.*[18] The hydrodynamic velocity of sound, the low-temperature specific heat, and the magnetic susceptibility all have approximately the appropriate temperature dependence, and imply the following values for the Landau coefficients:

$$F_0=10.77, \quad F_1=6.25, \qquad Z_0=-0.665, \quad Z_1\approx -0.72 \quad . \tag{2.26}$$

We see that F_0, or the angular average of the spin-averaged effective quasiparticle interaction, is positive, indicating a repulsive interaction which gives the low compressibility of liquid ^{3}He. F_1 indicates a surprisingly large value for the effective mass, and the value for Z_0 shows that the system is close to being ferromagnetic. The value for Z_1 is obtained from the three other coefficients in Eq. (2.26), by assuming that all F_L and Z_L are equal to zero for $L>1$, and then using the sum rule (2.25).

III. BRUECKNER THEORY

By application of Brueckner theory, various properties of liquid ^{3}He have already been calculated by the author,[11,12] and the coefficients (2.26) can be obtained from these estimates. The coefficient F_0 can be obtained from the compressibility, and estimates of the effective mass and the magnetic susceptibility give values for F_1 and Z_0. A rough value for Z_1 can be obtained from the exclusion-principle sum rule (2.25), if we assume that all the coefficients of higher order of L are equal to zero.

Experimentally, a value for the compressibility β is generally obtained from the velocity v_S of hydrodynamic sound in the Fermi liquid, by the relation

$$v_s^2 = (\beta\rho M)^{-1} \quad . \tag{3.1}$$

The isothermal compressibility then is

$$\beta = -\frac{\partial\Omega}{\partial\mathcal{P}}\Big/\Omega = \rho^{-1}\Big/\frac{\partial\mathcal{P}}{\partial\rho}$$

$$= \frac{3}{k_F}\Big/\frac{\partial}{\partial k_F}\left(\frac{\rho^2\partial\mu}{\partial\rho}\right)^{-1} = (\rho M v_s^{\,2})^{-1} \quad . \tag{3.2}$$

From Eq. (2.6), we see that

$$\frac{\partial k_F}{\partial\rho} = \tfrac{1}{3}k_F/\rho \tag{3.3}$$

and $$(\beta\rho)^{-1} = \frac{\rho\partial\mu}{\partial\rho} = \tfrac{1}{3}(\hbar^2/M^*)k_F^{\,2}(1+F_0) \tag{3.4}$$

finally give

$$\beta^{-1} = \tfrac{1}{9}(\hbar^2/M)(k_F^{\,5}/\pi^2)(1+F_0)/(1+\tfrac{1}{3}F_1) \tag{3.5}$$

or $$F_0 = 9\pi^2 M^*/\beta\hbar^2 k_F^{\,5} - 1 \quad , \tag{3.6}$$

which can also be obtained directly from Eqs. (2.14), (2.21), and (3.1). The coefficients F_1 and Z_0 we get directly from Eqs. (2.14), (2.15), and (2.18).

Using the results of earlier calculations[11,12] of the compressibility, the effective mass, and the magnetic susceptibility, we get the values given in Table I for the low-L-order Legendre polynomial expansion coefficients. These results can be compared with experimental values, and also with the results obtained later in Sec. IV, and shown in Table II. As defined earlier,[11,12] the YS potential is the Yntema-Schneider potential[8]

$$V_{YS}(r) = 7250[1200\exp(-4.82r) - 1.24/r^6 - 1.89/r^8] \quad , \tag{3.7}$$

TABLE I. Landau parameters or expansion coefficients for liquid ^{3}He. Calculated by Brueckner theory.

Property	YS potential	FM potential	Experimental value
β[% atm^{-1}]	4.3	4.3	3.5
m^*	2.5	2.5	3.08
χ/χ_F	11	8	9.2
F_0	7.85	7.03	10.77
F_1	4.50	4.50	6.25
Z_0	−0.77	−0.69	−0.665
Z_1	0.93	−0.41	−0.72
B_0	0.89	0.88	0.915
B_1	1.80	1.80	2.03
C_0	−3.39	−2.20	−1.99
C_1	0.71	−0.48	−0.95

and the FM potential is the Frost-Musulin potential[19]

$$V_{FM}(r) = -12.54[1+8.01(1-2.98/r)] \times \exp[8.01(1-r/2.98)] \quad \text{for } r<3.5\text{ Å},$$

$$= -7250[1.41/r^6 + 3.82/r^8] \quad \text{for } r>3.5\text{ Å}, \tag{3.8}$$

in [°K], where r is measured in Å.

IV. CALCULATION OF LANDAU PARAMETERS BY BRUECKNER THEORY

The basic quantity in the Brueckner theory is the G matrix, which can be written

$$G = G_D - \tfrac{1}{2}(1+\vec{\sigma}\cdot\vec{\sigma}')G_E \quad . \tag{4.1}$$

Here G_D is the direct part, and G_E is the exchange part which includes only exchange of momenta. The spin exchange is obtained by multiplying G_E by the spin-exchange operator $\frac{1}{2}(1+\vec{\sigma}\cdot\vec{\sigma}')$. Then

$$G_D = a_e + a_O,$$
$$G_E = a_e - a_O, \tag{4.2}$$

where $$a_e = \sum_{\text{even } L}(2L+1)\langle k|G_L|k\rangle \quad ,$$

$$a_O = \sum_{\text{odd } L}(2L+1)\langle k|G_L|k\rangle \quad , \tag{4.3}$$

and the G_L-matrix elements are calculated as

TABLE II. Landau parameters or expansion coefficients for liquid ^{3}He. Calculated by Brueckner theory.

Potential	YS potential		FM potential		Experimental
k_F[Å^{-1}]	0.75	0.78	0.75	0.78	value
Coefficient					
F_0	5.09	7.22	5.79	8.69	10.77
F_1	2.80	3.27	3.14	3.73	6.25
F_2	1.64	1.72	1.91	1.98	
F_3	0.97	1.10	1.13	1.28	
F_4	0.56	0.68	0.64	0.78	
F_5	0.45	0.52	0.51	0.59	
Z_0	−0.56	−0.44	−0.75	−0.66	−0.665
Z_1	−0.74	−1.17	−0.42	−0.84	−0.72
Z_2	1.46	1.49	1.83	1.87	
Z_3	0.80	0.87	1.00	1.13	
Z_4	0.34	0.43	0.45	0.52	
Z_5	0.19	0.19	0.25	0.28	
B_0	0.84	0.88	0.85	0.90	0.915
B_1	1.45	1.56	1.54	1.66	2.03
B_2	1.23	1.28	1.38	1.42	
B_3	0.85	0.95	0.97	1.08	
B_4	0.53	0.63	0.60	0.72	
B_5	0.43	0.49	0.49	0.56	
C_0	−1.29	−0.80	−2.97	−1.95	−1.99
C_1	−0.98	−1.91	−0.49	−1.18	−0.95
C_2	1.13	1.15	1.34	1.36	
C_3	0.72	0.78	0.88	0.97	
C_4	0.33	0.41	0.43	0.50	
C_5	0.19	0.19	0.24	0.27	

$$\langle k|G_L|k\rangle = (4\pi/k^2)\int_0^\infty j_L(kr)v(r)u_L(k,r)\,dr\ . \quad (4.4)$$

Here, k is the relative momentum, $v(r)$ is the two-body potential, and $j_L(kr)$ and $u_L(k,r)$ are the unperturbed and perturbed partial wave functions, as explained in detail earlier.[9]

To make a connection between the Brueckner theory and the fundamental equation of the Landau theory, we start from the assumption that the liquid properties can be obtained from an expression for the total energy. The function $f(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\vec{\sigma}_j)$ can be related to the G matrix through the relation

$$f(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\sigma_j) = \delta^2E/\delta n(\vec{k}_i,\vec{\sigma}_i)\delta n(\vec{k}_j,\vec{\sigma}_j)\ , \quad (4.5)$$

with the Brueckner expression for the total energy E. Since this energy is approximate, the final expression for f will also be an approximation. Identification with Landau's phenomenological method is only possible if the approximations made in the calculations of the G-matrix elements do not destroy certain invariance properties which hold for the full Brueckner-Goldstone perturbation series.[20] The terms in the expression for the total energy should satisfy Galilean invariance, and the values of the effective mass calculated from the f function by Landau's formulas should be equal to the values calculated directly from the energy spectrum. This cannot be satisfied by a nonquadratic form of the single-particle energy spectrum. A reference energy spectrum should, however, satisfy this requirement. But we still encounter, at least in principle, problems arising from G-matrix elements calculated off the energy shell, and from the gap or the discontinuity in the single-particle energy spectrum at the Fermi surface.

We could, however, as in the calculations of the transport coefficients,[12,21,22] rather identify the G matrix with the scattering amplitude (2.23) and calculate $\omega(\theta,\varphi)$ from a microscopic theory. The relation with f would then be given through Eq. (2.24). On the other hand, it seems not to be possible to obtain the Landau f function through the scattering amplitude, since the usual relation between them depends on diagrams which in principle are not included in the G matrix. The scattering amplitude calculated by Brueckner theory is not very reliable, because rearrangement terms are not included; it gives poor theoretical agreement with experimental results for the transport coefficients.[12] And the Landau theory

gives no information about nonforward scattering. The angle between the relative momenta before and after a scattering process should be taken into account, but the Landau theory cannot provide this angular dependence of the scattering amplitude.

So we write the total energy as

$$E=\tfrac{1}{2}\sum_i n_i\frac{\hbar^2}{M}k_i^2+\tfrac{1}{2}\sum_i\sum_j n_i n_j(\langle ij|G|ij\rangle-\langle ij|G|ji\rangle)=\tfrac{1}{2}\sum_{k_i,\sigma_i} n(\vec{k}_i,\vec{\sigma}_i)\frac{\hbar^2}{M}k_i^2$$

$$+\tfrac{1}{2}\sum_{k_i,\sigma_i}\sum_{k_j,\sigma_j} n(\vec{k}_i,\vec{\sigma}_i)n(\vec{k}_j,\vec{\sigma}_j)[G_D(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\vec{\sigma}_j)-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_j)G_E(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\vec{\sigma}_j)]\,. \quad (4.6)$$

The first variational derivative of Eq. (4.6) is

$$\delta E/\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)=\tfrac{1}{2}(\hbar^2/M)k_\alpha^2+\sum_{k_i,\sigma_i} n(\vec{k}_i,\vec{\sigma}_i)[G_D(\vec{k}_i,\vec{k}_\alpha;\vec{\sigma}_i,\vec{\sigma}_\alpha)-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_\alpha)G_E(\vec{k}_i,\vec{k}_\alpha;\vec{\sigma}_i,\vec{\sigma}_\alpha)]$$

$$+\tfrac{1}{2}\sum_{k_i,\sigma_i}\sum_{k_j,\sigma_j} n(\vec{k}_i,\vec{\sigma}_i)n(\vec{k}_j,\vec{\sigma}_j)[\delta G_D(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\sigma_j)/\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)$$

$$-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_j)\delta G_E(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\vec{\sigma}_j)/\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)]\,, \quad (4.7)$$

and the second variational derivative is

$$\frac{\delta^2 E}{\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)\delta n(\vec{k}_\beta,\vec{\sigma}_\beta)}=G_D(\vec{k}_\alpha,\vec{k}_\beta;\vec{\sigma}_\alpha,\vec{\sigma}_\beta)-\tfrac{1}{2}(1+\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta)G_E(\vec{k}_\alpha,\vec{k}_\beta;\vec{\sigma}_\alpha,\vec{\sigma}_\beta)+\sum_{k_i,\sigma_i} n(\vec{k}_i,\vec{\sigma}_i)$$

$$\times\frac{\delta[G_D(\vec{k}_i,\vec{k}_\beta,\vec{\sigma}_i,\vec{\sigma}_\beta)-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_\beta)G_E(\vec{k}_i,\vec{k}_\beta;\vec{\sigma}_i,\vec{\sigma}_\beta)]}{\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)}+\sum_{k_i,\sigma_i} n(\vec{k}_i,\vec{\sigma}_i)$$

$$\times\frac{\delta[G_D(\vec{k}_i,\vec{k}_\alpha;\vec{\sigma}_i,\vec{\sigma}_\alpha)-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_\alpha)G_E(\vec{k}_i,\vec{k}_\alpha;\vec{\sigma}_i,\vec{\sigma}_\alpha)]}{\delta n(\vec{k}_\beta,\vec{\sigma}_\beta)}+\tfrac{1}{2}\sum_{k_i,\sigma_i}\sum_{k_j,\sigma_j} n(\vec{k}_i,\vec{\sigma}_i)n(\vec{k}_j,\vec{\sigma}_j)$$

$$\times\frac{\delta^2[G_D(\vec{k}_i,\vec{k}_j,\vec{\sigma}_i,\vec{\sigma}_j)-\tfrac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_j)G_E(\vec{k}_i,\vec{k}_j;\vec{\sigma}_i,\vec{\sigma}_j)]}{\delta n(\vec{k}_\alpha,\vec{\sigma}_\alpha)\delta n(\vec{k}_\beta,\vec{\sigma}_\beta)}\,. \quad (4.8)$$

To facilitate the identification with the Landau function $f(\vec{k}_\alpha,\vec{k}_\beta;\vec{\sigma}_\alpha,\vec{\sigma}_\beta)$, we want to separate or project out the spin dependence on $\vec{\sigma}_\alpha$ and $\vec{\sigma}_\beta$, so we transform Eq. (4.8) by summing over the spins $\vec{\sigma}_i$ and $\vec{\sigma}_j$. Using Eq. (4.2) the final result is written

$$f(\vec{k}_\alpha,\vec{k}_\beta;\vec{\sigma}_\alpha,\vec{\sigma}_\beta)=f(\vec{k}_\alpha,\vec{k}_\beta)+\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta\, z(\vec{k}_\alpha,\vec{k}_\beta)=\tfrac{1}{2}[a_e(k_\alpha,k_\beta)+3a_O(k_\alpha,k_\beta)]+\tfrac{1}{2}\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta$$

$$\times[a_O(k_\alpha,k_\beta)-a_e(k_\alpha,k_\beta)]+2\sum_{k_i} n_i\frac{\partial}{\partial n_\alpha}[a_e(k_i,k_\beta)+3a_O(k_i,k_\beta)+2\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta a_O(k_i,k_\beta)] \quad (4.9)$$

$$+\sum_{k_i}\sum_{k_j} n_i n_j\frac{\partial^2}{\partial n_\alpha\partial n_\beta}\{\tfrac{3}{4}a_e(k_i,k_j)+\tfrac{13}{4}a_O(k_i,k_j)+\vec{\sigma}_\alpha\cdot\vec{\sigma}_\beta[\tfrac{1}{4}a_e(k_i,k_j)+\tfrac{7}{4}a_O(k_i,k_j)]\}\,.$$

The definition of the function $z(\vec{k}_\alpha,\vec{k}_\beta)$ and the coefficients Z_L then differ by a factor of 4 from what is often used by others.

Within the Landau theory, we use the assumption (2.11), i.e.,

$$|\vec{k}_\alpha| = |\vec{k}_\beta| = k_F. \tag{4.10}$$

The first or direct term on the right-hand side of Eq. (4.9) is then a function of the angle θ between $\vec{k}_\alpha$ and $\vec{k}_\beta$, or of the relative momentum k according to the relation

$$k = k_F \sin\tfrac{1}{2}\theta \ . \tag{4.11}$$

The second and third terms, i.e., the rearrangement terms, are, however, given by summation over k_i and k_j for each k_α and k_β.

The total energy and the G matrix depend on all the occupation numbers. And only when all the occupation numbers are equal to unity below the Fermi surface and zero otherwise, can this dependence be reduced to a dependence on the total number of particles or the Fermi momentum, from the density relation (2.6). Analogous to Eq. (3.3), we get

$$\frac{\partial k_F}{\partial N} = \tfrac{1}{3} k_F / N \ , \tag{4.12}$$

and using $\int d\vec{k}_i = 16\pi \int_0^{k_F} k^2(1 - k/k_F)\,dk$,

$$\int d\vec{k}_i \int d\vec{k}_j = \tfrac{128}{3}\pi^2 k_F{}^3 \int_0^{k_F} k^2(1 - \tfrac{3}{2}k/k_F + \tfrac{1}{2}k^3/k_F{}^3)dk, \tag{4.13}$$

we get, at the top of the Fermi sea,

$$\begin{aligned} f(k_F, k_F) &= \tfrac{1}{2}[a_e(k_F, k_F) + 3a_O(k_F, k_F)] + 4\int_0^1 x^2(1-x)k_F \left(\frac{\partial a_e}{\partial k_F} + 3\frac{\partial a_O}{\partial k_F}\right)dx \\ &\quad + \int_0^1 x^2(1 - \tfrac{3}{2}x + \tfrac{1}{2}x^3)\left[k_F\left(k_F \frac{\partial^2}{\partial k_F{}^2} - 2\frac{\partial}{\partial k_F}\right)(a_e + \tfrac{13}{3}a_O)\right]dx \ , \\ z(k_F, k_F) &= \tfrac{1}{2}[a_O(k_F, k_F) - a_e(k_F, k_F)] + 8\int_0^1 x^2(1-x)k_F \frac{\partial a_O}{\partial k_F}\,dx \\ &\quad + \int_0^1 x^2(1 - \tfrac{3}{2}x + \tfrac{1}{2}x^3)\left[k_F\left(k_F \frac{\partial^2}{\partial k_F{}^2} - 2\frac{\partial}{\partial k_F}\right)(\tfrac{1}{3}a_e + \tfrac{7}{3}a_O)\right]dx \ , \end{aligned} \tag{4.14}$$

where $x = k/k_F$. (4.15)

To get functions of the angle θ or the relative momentum k, Eq. (4.14) is rewritten

$$\begin{aligned} f(k) &= \tfrac{1}{2}[a_e(k) + 3a_O(k)] + \int_0^1 x^2 \left((2 - x - x^3)k_F \frac{\partial a_e}{\partial k_F} + (\tfrac{10}{3} + \tfrac{2}{3}x - \tfrac{13}{3}x^3)k_F \frac{\partial a_O}{\partial k_F}\right. \\ &\quad \left. + (1 - \tfrac{3}{2}x + \tfrac{1}{2}x^3)k_F{}^2 \frac{\partial^2 a_e}{\partial k_F{}^2} + (\tfrac{13}{3} - \tfrac{13}{2}x + \tfrac{13}{6}x^3)k_F{}^2 \frac{\partial^2 a_O}{\partial k_F{}^2}\right)dx \ , \\ z(k) &= \tfrac{1}{2}[a_O(k) - a_e(k)] + \int_0^1 x^2 \left((-\tfrac{2}{3} + x - \tfrac{1}{3}x^3)k_F \frac{\partial a_e}{\partial k_F} + (\tfrac{10}{3} - x - \tfrac{7}{3}x^3)k_F \frac{\partial a_O}{\partial k_F}\right. \\ &\quad \left. + (\tfrac{1}{3} - \tfrac{1}{2}x + \tfrac{1}{6}x^3)k_F{}^2 \frac{\partial^2 a_e}{\partial k_F{}^2} + (\tfrac{7}{3} - \tfrac{7}{2}x + \tfrac{7}{6}x^3)k_F{}^2 \frac{\partial^2 a_O}{\partial k_F{}^2}\right)dx \ . \end{aligned} \tag{4.16}$$

Our main problem is to improve the estimate of the dependence of the rearrangement term on the angle θ. The G-matrix elements which give the first term on the right-hand side of Eq. (4.16) are calculated as functions of k, i.e., θ. The detailed angular dependence of the rearrangement terms, or the dependence of the derivatives of the G-matrix elements on the relative momentum k is, however, uncertain and very complicated to calculate or estimate. This is because the variational derivatives with respect to the occupation numbers are not really given by only the simple dependence on the Fermi momentum. The differentiation of the G matrix is carried out in the numerical solution of the Bethe-Goldstone equation by making a variation or finite shift in the population of the Fermi gas at the chosen momentum. Then the derivatives are determined by the finite shift in the G matrix. Investigations similar to the calculations by Brueckner *et al.*[23] of the rearrangement energy in nuclear matter indicate that the rearrangement terms are approximately twice as large for $k=0$ as for $k=k_F$, and that

$$V_R(k) \approx (\tfrac{3}{2} + \tfrac{1}{2}\cos\theta) V_R(k_F) \tag{4.17}$$

is a fair approximation for the momentum dependence of the rearrangement potential energy. This is dominated by the momentum dependence of the derivatives of a_e, because

$$\frac{\partial a_e}{\partial k_F} \approx 20 \frac{\partial a_O}{\partial k_F} , \tag{4.18}$$

with a similar relation for the second derivatives. The derivatives of a_O show little variation as functions of k, and we can use just the same value for the corresponding rearrangement term for all θ.

The momentum dependence estimated above for the derivatives is also supported by our values for F_0 and F_1 calculated in Sec. III. These values should be roughly reproduced if we manage to include properly in our G-matrix elements the contribution from three-body correlations. We could, in fact, use the values for F_0 and F_1 in Sec. III to estimate the rearrangement term if we were unhappy about the approximation (4.17).

The G-matrix elements and the derivatives are now calculated as explained earlier,[12] with the same choice of input parameter values. The functions $f(\theta)$ and $z(\theta)$ are obtained from $G(\theta)$, starting from the functions

$$g(\theta) = \tfrac{1}{2}(a_e + 3a_O) , \tag{4.19}$$

and $$h(\theta) = \tfrac{1}{2}(a_O - a_e) , \tag{4.20}$$

and the resulting functions are given in Figs. 1 and 2. Similar curves are obtained by Tan and Feenberg,[24] and for $f(\theta)$ by Burkhardt[25] by just raising curves for $g(\theta)$ to fit values for the compressibility.

Assuming that the total effective interaction can be expanded in a series of Legendre polynomials, we write

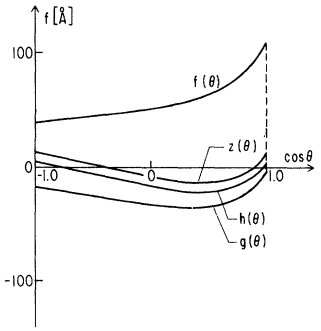

FIG. 1. Functions $f(\theta)$, $g(\theta) = \frac{1}{2}(a_e + 3a_O)$, $z(\theta)$, and $h(\theta) = \frac{1}{2}(a_O - a_e)$ in (Å), calculated by Brueckner theory. YS potential. $k_F = 0.78$ Å^{-1}.

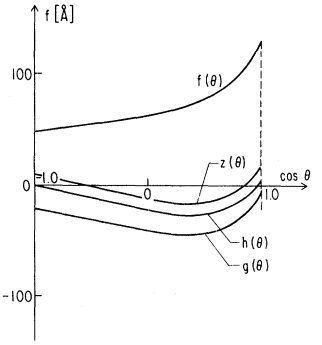

FIG. 2. Functions $f(\theta)$, $g(\theta) = \frac{1}{2}(a_e + 3a_O)$, $z(\theta)$, and $h(\theta) = \frac{1}{2}(a_O - a_e)$ in (Å), calculated by Brueckner theory. FM potential. $k_F = 0.78$ Å^{-1}.

$$F_L = \tfrac{1}{2}(2L+1)(k_F M^*/\pi^2\hbar^2)\int_0^\pi f(\theta)\mathcal{P}_L(\cos\theta)\sin\theta\, d\theta \ ,$$
$$Z_L = \tfrac{1}{2}(2L+1)(k_F M^*/\pi^2\hbar^2)\int_0^\pi z(\theta)\mathcal{P}_L(\cos\theta)\sin\theta\, d\theta \ , \tag{4.21}$$

because $$\int_{-1}^{1}\mathcal{P}_L(x)\mathcal{P}_{L'}(x)dx = 2\delta_{LL'}/(2L+1) \ . \tag{4.22}$$

Afterwards, the coefficients B_L and C_L are calculated according to Eq. (2.24).

Results for $L<6$ are given in Table II. We have used G-matrix elements calculated for the two different potentials defined in Eqs. (3.7) and (3.8), and for two different densities.

We see that the rearrangement contribution is large enough to make the angular average of the spin-averaged effective interaction repulsive, while the G matrix is on the average attractive. The calculated rearrangement effects are too small, however, to reproduce the large experimental values for F_0 and F_1. This means that we have not succeeded in including properly the contribution from three-body correlations, because otherwise a value for F_0 corresponding to the calculated compressibility[12] should occur. And the slope of the interaction function is not sufficiently steep to produce a good value for the effective mass or F_1, which indicates the same trouble. The results can otherwise be compared with the calculations by Tan and Feenberg.[24]

It has earlier been assumed, or rather hoped, that the coefficients F_L, Z_L, B_L, and C_L would be small for $L>1$. This assumption would correspond to an effective interaction function of approximately the form

$$f(\theta) \approx A + B\cos\theta \ , \tag{4.23}$$

and a similar angular dependence for $z(\theta)$. This is, however, not very likely according to our calculations. The G matrix, or the functions $g(\theta)$ and $h(\theta)$ plotted as functions of θ, have a completely different angular dependence, and show that the rearrangement terms must have a rather strange dependence on k or θ to possibly correct for this. All our calculations and estimates of derivatives and rearrangement terms show this to be unlikely, which is also indicated by the fact that the calculated coefficients for $L>1$ in Table II are all positive and not very small. The present use by others of the exclusion-principle sum rule to calculate the coefficients Z_1 and C_1 is therefore a very uncertain way to estimate their values.

As a test of the sum rule (2.25) we get from our calculations

$$\sum_{L<2}(B_L + C_L) = 0.02 \quad \text{for} \quad k_F = 0.75 \text{ Å}^{-1}$$
$$= -0.27 \quad \text{for} \quad k_F = 0.78 \text{ Å}^{-1} \ , \tag{4.24}$$
$$\sum_{L<6}(B_L + C_L) = 5.42 \quad \text{for} \quad k_F = 0.75 \text{ Å}^{-1}$$
$$= 5.60 \quad \text{for} \quad k_F = 0.78 \text{ Å}^{-1} \ ,$$

for the YS potential, and

$$\sum_{L<2}(B_L + C_L) = -1.08 \quad \text{for} \quad k_F = 0.75 \text{ Å}^{-1}$$
$$= -0.57 \quad \text{for} \quad k_F = 0.78 \text{ Å}^{-1} \ ,$$
$$\sum_{L<6}(B_L + C_L) = 5.25 \quad \text{for} \quad k_F = 0.75 \text{ Å}^{-1} \tag{4.25}$$
$$= 6.32 \quad \text{for} \quad k_F = 0.78 \text{ Å}^{-1} \ ,$$

for the FM potential. This indicates that we always have

$$\sum_{L>1}(B_L + C_L) > 0 \ , \tag{4.26}$$

which means that the value for Z_1 is generally underestimated. The sign is correct [it is negative, as in

Eq. (2.26)], but the absolute value should be larger. However, the exclusion-principle sum rule should be violated if the effective interaction in the Landau theory is calculated by functional differentiation of the total energy obtained by the Brueckner method, because of the unsymmetrical treatment of particles and holes in the Brueckner theory.[26]

V. DISCUSSION

In the Landau theory, there is a one-to-one correspondence between the low excited states of an ideal noninteracting Fermi gas with particles at the Fermi surface, and the low excited states of a real Fermi liquid with corresponding quasiparticles. We have assumed that the Landau f function can be obtained by taking the second variational derivative with respect to the particle occupation number of the total energy written in terms of Brueckner theory. This gives the reaction matrix or G matrix, plus rearrangement terms containing the first and second variational derivatives of the G matrix. The approximation that the Landau f function in this way can be related to the G matrix and its derivatives, is based on the assumption that our energy expression is a reasonable approximation for the total energy. This assumption is supported by calculations of the binding energy of liquid ^{3}He, with contributions from both two-body and three-body correlations.[11] Here, the three-body contribution is included as explained earlier,[12] i.e., partly included in the two-body calculations and partly added separately.

The Goldstone perturbation series gives a linked-cluster expansion for the total energy, which gives a theoretical expression for the ground-state energy of the system. It is, however, difficult to make any statements about the convergence of this series, or rather about the resulting compact-cluster expansion.[27] Liquid ^{3}He is a denser system than (for instance) nuclear matter, but the Pauli exclusion principle helps to suppress the energy contribution from higher n-body terms. The increasing antisymmetry of the n-body wave functions with increasing n helps to suppress the energy contribution from higher clusters, i.e., n-body contributions for $n > 3$.

The G-matrix elements and the derivatives are calculated as explained earlier.[12] The values for the input parameters are already determined by calculations[11,12] of, for instance, binding energy, effective mass, compressibility, and magnetic susceptibility. The same calculations give a check on the values of the Landau parameters F_0, F_1, and Z_0, as indicated in Sec. III. Both the G matrix and its derivatives are, however, not very sensitive to the detailed shape of the input single-particle energy spectrum on the energy shell for particles in the Fermi sea and to the corresponding input parameters in the calculations.

The effective interaction functions and the Landau parameters would be more sensitive to the energy spectrum for excited particles outside the Fermi sea and the corresponding effective mass. But as explained earlier,[9] we have chosen to set the intermediate-state potential energies off the energy shell equal to zero (i.e., the corresponding effective mass equal to one), and calculate the three-body energy contribution separately. However, the energy gap between the on-energy-shell spectrum and the off-energy-shell spectrum will probably damp the virtual excitations at low energy. The use of the gap is justified for calculations of the ground-state energy and other properties, because the effect or possible error is averaged out when using a reference energy spectrum. And other terms can be added afterwards by calculating separately, for instance, the contribution from three-body correlations. But the derivatives of the three-body cluster energy with respect to the particle occupation number may differ quite a lot from the derivatives evaluated directly from the G-matrix elements, and this problem cannot be solved in an accurate way at the moment.

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Momentum and Energy Exchange between Beams of Particles Interacting by Yukawa-Type Potentials

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A statistical theory of the momentum and energy exchange between beams of particles interacting by Yukawa-type potentials is presented. A theoretical description is given for the slowing down of the beams, the thermalization and heating of the beams, and the establishment of the isothermal state in the beams. The associated collision integrals are evaluated, with consideration of the velocity dependence of various Coulomb logarithms. The kinetic processes are shown to depend crucially on whether the relative velocity of the beams is supersonic or subsonic. The theory is applicable to (i) beams of charged and neutral particles interacting by a Coulomb potential screened by bound electrons, and (ii) beams of charged particles interacting by a Coulomb potential screened by free charges. In the region of low energies, the results are applicable to the components of collision-dominated ionized gases and plasmas, and represent an essential contribution to the transport theory of collision-dominated media with nondivergent Coulomb interactions.

I. INTRODUCTION

In many cases, the interaction of a pair of particles r and s can be characterized by constants e_r and e_s that determine the strength, and by a length l_{rs} that determines the range of the interaction. An analytically simple potential that considers these elementary properties of an interaction is the so-called Yukawa potential in the quasistatic approximation[1]

$$U = e_r e_s e^{-r/l_{rs}}/r. \tag{1}$$

In the region of high energies, Eq. (1) describes the nuclear interactions of the range of the Compton wavelength of a pion $\lambda_\pi = \hbar/m_\pi c \cong 10^{-13}$ cm. In the region of low energies, Eq. (1) describes various weak interactions of relatively long ranges.

This investigation is concerned with the statistical mechanics of the momentum and energy exchange between various types of quasihomogeneous, collision-dominated particle beams, interacting by a Yukawa potential of the form of Eq. (1):

(i) *Beams of Charged and Neutral Particles.* According to the Thomas-Fermi statistical model, the interaction potential for the scattering of electrons or ions of charge e_r by neutral atoms of nuclear charge e_s is approximately represented by Eq. (1). The range of this interaction is determined by the screening radius of the atomic electrons[2]

$$l_{rs} \to R_{rs} = \frac{\hbar^2}{me^2}\left(\left|\frac{e_r}{e}\right|^{2/3} + \left|\frac{e_s}{e}\right|^{2/3}\right)^{1/2} \gtrsim 10^{-8}\ \text{cm}. \tag{2}$$