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Boltzmann-Langevin Equation and Hydrodynamic Fluctuations*

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The one-particle distribution function which satisfies the Boltzmann equation is interpreted as the average of the phase-space function $\sum_{i=1}^{N}\delta(\vec{r}-\vec{r}_i)\delta(\vec{v}-\vec{v}_i)$. The equation of motion for this function is a generalized Langevin equation. This equation is the linear Boltzmann equation to which a fluctuating force term is added. An expression for the second moment of this force, in terms of the Boltzmann kernel and the equilibrium second moment of the distribution function, is derived in analogy with the known procedure involving the Langevin equation. The second moments of the fluctuating pressure tensor and the heat-flow vector are evaluated by using the first Chapman-Enskog approximation. They are equal to the expressions derived by Landau and Lifshitz, using thermodynamic fluctuation theory in relation to the linearized hydrodynamic equations.

I. INTRODUCTION

In this paper, we present a generalization of the linearized Boltzmann equation that includes fluctuation effects. The resulting equation is related to the familiar Boltzmann equation, in the same way that the Langevin equation of motion for the velocity of a Brownian particle is related to the equation of motion for the average velocity of that particle. For this reason, we refer to our generalization as the Boltzmann-Langevin equation.

In a later paper, we will present a derivation of the Boltzmann-Langevin equation from first principles. Here, we obtain the result by arguments based on physical intuition and analogy.

As an illustration of the utility of the Boltzmann-Langevin equation, we obtain from it a theory of hydrodynamic fluctuations in a low-density gas. The results are identical with expressions obtained by Landau and Lifshitz, using entirely different arguments.

Our discussion is motivated by the following observation. As is well known, the single-particle distribution function $\bar{f}(\vec{r},\vec{v};t)$ that satisfies the Boltzmann equation can be interpreted in two different ways.[1-5] For example, it is proportional to the probability density of finding a particle at a position $\vec{r}$ with a velocity $\vec{v}$ at time t. In this interpretation, $\bar{f}$ is found from the N-particle probability density $f_N(\vec{R}^N,\vec{v}^N;t)$ in the phase space of the entire gas by integrating over the positions and velocities of all but one particle

$$\bar{f}(\vec{R}_1,\vec{v}_1;t)=N\int d\vec{R}^{N-1}\int d\vec{v}^{N-1}f_N(\vec{R}^N,\vec{v}^N;t). \quad (1)$$

The single-particle distribution function $\bar{f}$ can also be viewed as the ensemble average of a phase function $f(\vec{r},\vec{v};t)$, defined as the physical density in a six-dimensional position-velocity space (the "μ space")

$$f(\vec{r},\vec{v};t)=\sum_{j=1}^{N}\delta(\vec{R}_j(t)-\vec{r})\delta(\vec{v}_j(t)-\vec{v}), \quad (2)$$

$$\bar{f}(\vec{r},\vec{v};t)=\langle f(\vec{r},\vec{v};t)\rangle^{(0)}. \quad (3)$$

The ensemble average (denoted by $\langle\ \rangle^{(0)}$) is taken over some given initial distribution in phase space, and the time-dependent positions and velocities $\vec{R}_j(t)$ and $\vec{v}_j(t)$ are parametric functions of the initial positions and velocities. It is evident that in a purely mathematical sense both interpretations are equivalent.

But the second interpretation has an interesting consequence that is not so apparent in the first interpretation. In any individual member of an ensemble (i.e., for any specified initial state of the N-particle system), the actual density $f(\vec{r},\vec{v};t)$ is not identical with its average value. There are fluctuations about the average. The Boltzmann equation describes the time dependence of the average density $\bar{f}$; so we expect that there is some equation of motion for the actual density f, such that its average is the Boltzmann equation. This equation, in the linear approximation of small deviations of the actual density f from its equilibrium value $f_B(\vec{r},\vec{v})$, is the Boltzmann-Langevin equation.

It is possible to derive a linear equation of motion for the deviation $f-f_B$ by means of a technique due to Mori.[6] When this is done, and then an expansion in powers of density is performed, the Boltzmann-Langevin equation is obtained in the lowest order in density. Higher-order corrections in density can also be obtained. In a later paper, we will describe this derivation in detail.

Here, we follow a more intuitive procedure. The difference between the Boltzmann-Langevin equation and its average is a random-force term. This is expected to be a very complicated function of time and of the initial state of the N-particle system. But for many applications one does not have to know its structure in detail. Often it is sufficient to know its mean value (which is zero) and its second moment (or correlation function).

The correlation function of the random force can be obtained from the Boltzmann-Langevin equation itself, using standard techniques, if we assume that the random force is a Markov process. In other words, we suppose that the random force at any particular instant of time is not correlated with its value at any other time. This assumption allows us to evaluate the correlation function by a method analogous to that used in the derivation of the fluctuation-dissipation theorem from the ordinary Langevin equation.[7]

The Markovian assumption is consistent with treatment of the ordinary Boltzmann equation as a Markovian equation, in which the collision integral does not contain any memory of the past behavior of the system. It is expected to be a reasonable approximation when one is dealing with slowly varying, or low-frequency, processes. In the limit of low density, it will fail only when the time scale is of the order of the duration of a collision.

As an application of the Boltzmann-Langevin equation, we discuss the role of fluctuations in the hydrodynamic equations of a low-density gas. We show that the random-force term in the Boltzmann-Langevin equation gives rise to a fluctuating stress tensor and a fluctuating heat current in the hydrodynamic equations, and we calculate the correlation functions of these hydrodynamic fluctuations. This is done in the framework of the first Chapman-Enskog approximation.

Our results are equivalent to the hydrodynamic fluctuation theory presented by Landau and Lifshitz (in the low-density limit).[8] Their derivation was based on use of thermodynamic fluctuation theory in connection with the linearized hydrodynamic equation directly, while we proceed to the same results more indirectly by means of the Boltzmann-Langevin equation.

II. RANDOM-FORCE TERM IN THE BOLTZMANN-LANGEVIN EQUATION

In this section, we introduce the Boltzmann-Langevin equation, together with its random-force term, and we show how the correlation function of the random force is related to the collision operator.

First we write the ordinary linearized Boltzmann equation in a form suitable for generalization. It is convenient to use the relative deviation from equilibrium, in place of the average single-particle distribution function. The relative deviation $\overline{\phi}(\vec{r}, \vec{v}; t)$ is defined by

$$\overline{\phi}(\vec{r}, \vec{v}; t) = [f_B(\vec{r}, \vec{v})]^{-1} [\overline{f}(\vec{r}, \vec{v}; t) - f_B(\vec{r}, \vec{v})]. \quad (4)$$

In this form, the linearized Boltzmann equation is

$$\frac{\partial \overline{\phi}}{\partial t} + \vec{v} \cdot \vec{\nabla}_r \overline{\phi} = J\overline{\phi}, \quad (5)$$

where J denotes the linearized collision operator. (Since the explicit structure of this operator is not used in the following treatment, we do not need to write it out in detail.)

It is important to note, however, that the operator J is Markovian; that is, it converts the function $\overline{\phi}(\vec{r}, \vec{v}; t)$ to a new function at the same time t, and does not contain any memory effects.

As was indicated in the Introduction, $\overline{f}$ is interpreted as the ensemble average of the μ-space density $f(\vec{r}, \vec{v}; t)$. Similarly, $f_B(\vec{r}, \vec{v})$ is the thermal equilibrium average of the same function:

$$f_B(\vec{r}, \vec{v}) = \left\langle \sum_{j=1}^{N} \delta(\vec{R}_j - \vec{r})\delta(\vec{V}_j - \vec{V}) \right\rangle$$

$$= n(m/2\pi k_B T)^{3/2} \exp(-mv^2/2k_B T). \quad (6)$$

Here, the angular brackets (without a superscript zero) denote an equilibrium average; and n is the number density.

In analogy to Eq. (4), we define the instantaneous phase-space deviation function $\phi(\vec{r}, \vec{v}; t)$ by

$$\phi(\vec{r}, \vec{v}; t) = (f_B)^{-1}(f - f_B). \quad (7)$$

The average of this equation is identical with Eq. (4).

Now we want to write an equation of motion for $\phi(\vec{r}, \vec{v}; t)$. Since this equation is linear in ϕ, and its average over some initial ensemble is the linearized Boltzmann equation, this equation must be

$$\frac{\partial \phi}{\partial t} + (\vec{v} \cdot \vec{\nabla}_r - J)\phi = F(\vec{r}, \vec{v}; t), \quad (8)$$

where $F(\vec{r}, \vec{v}; t)$ is the random-force term. The ensemble average of F must vanish,

$$\langle F(\vec{r}, \vec{v}; t)\rangle^{(0)} = 0. \quad (9)$$

Equation (8) is our Boltzmann-Langevin equation.

It is more convenient to work in the Fourier representation of Eq. (8)

$$\frac{\partial}{\partial t}\phi(\vec{k},\vec{v};t)+(i\vec{k}\cdot\vec{v}-J)\phi(\vec{k},\vec{v};t)=F(\vec{k},\vec{v};t). \quad (10)$$

The Fourier transform of $\phi(\vec{r},\vec{v};t)$ is $\phi(\vec{k},\vec{v};t)$, explicitly,

$$\phi(\vec{k},\vec{v};t)=\frac{1}{f_B}\left(\sum_{j=1}^{N} e^{i\vec{k}\cdot\vec{R}_j(t)}\delta(\vec{v}_j(t)-\vec{v})-f_B\delta(\vec{k})\right); \quad (11)$$

and the Fourier transform of the random force is $F(\vec{k},\vec{v};t)$.

Also, it is convenient to define the linear operator $L(\vec{k},\vec{v})$ as follows:

$$L(\vec{k},\vec{v}) = i\vec{k}\circ\vec{v}-J, \quad (12)$$

so that Eq. (10) becomes

$$\frac{\partial\phi}{\partial t}+L\phi = F(t)\ . \quad (13)$$

Now we are ready to discuss the correlation function (or second moment) of the random force. The first step is to solve Eq. (13) formally, as a linear operator equation, to give

$$\phi(\vec{k},\vec{v};t) = e^{-tL}\phi(\vec{k},\vec{v};0)+\int_0^t ds\, e^{-(t-s)L}F(\vec{k},\vec{v};s). \quad (14)$$

The equilibrium correlation function (or second moment) of the solution is the following:

$$\langle\phi(\vec{k}_1,\vec{v}_1)\phi(\vec{k}_2,\vec{v}_2)\rangle=\lim_{t\to\infty}\int_0^t ds_1\int_0^t ds_2 \times\exp[-(t-s_1)L(\vec{k}_1,\vec{v}_1)]\exp[-(t-s_2)L(\vec{k}_2,\vec{v}_2)] \times\langle F(\vec{k}_1,\vec{v}_1;s_1)F(\vec{k}_2,\vec{v}_2;s_2)\rangle. \quad (15)$$

Next, we introduce the assumption that the random force is Markovian and write

$$\langle F(\vec{k}_1,\vec{v}_1;s_1)F(\vec{k}_2,\vec{v}_2;s_2)\rangle = 2B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2)\delta(s_1-s_2)\ , \quad (16)$$

where $B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2)$ is some unknown function. On substitution into Eq. (15), and after one integration, we get

$$\langle\phi(\vec{k}_1,\vec{v}_1)\phi(\vec{k}_2,\vec{v}_2)\rangle=\lim_{t\to\infty}\int_0^t ds\,\exp\{-(t-s)\times[L(\vec{k}_1,\vec{v}_1)+L(\vec{k}_2,\vec{v}_2)]\}\,2B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2). \quad (17)$$

In order to get B explicitly, one operates on both sides of Eq. (17) with $[L(\vec{k}_1,\vec{v}_1)+L(\vec{k}_2,\vec{v}_2)]$, and one uses the identity

$$(L_1+L_2)e^{-(t-s)(L_1+L_2)}=\frac{\partial}{\partial s}e^{-(t-s)(L_1+L_2)}, \quad (18)$$

where, for simplicity, we abbreviate $L_1=L(\vec{k}_1,\vec{v}_1)$, etc.

The result of this is

$$(L_1+L_2)\langle\phi_1\phi_2\rangle=\lim_{t\to\infty}\int_0^t ds\,\frac{\partial}{\partial s}\exp[-(t-s)(L_1+L_2)] \times 2B(1,2)=2B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2). \quad (19)$$

(In order to get the second equality in the last equation, we use a property of the Boltzmann collision operator, that the real part of L is positive.)

Equation (19) contains the equilibrium correlation function of $\phi(\vec{k},\vec{v})$. This can be evaluated easily in the limit of low density,

$$\langle\phi(\vec{k}_1,\vec{v}_1)\phi(\vec{k}_2,\vec{v}_2)\rangle=[f_B(v_2)]^{-1}\delta(\vec{v}_1-\vec{v}_2)\delta(\vec{k}_1+\vec{k}_2). \quad (20)$$

Therefore, we can write

$$2B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2) =(L_1+L_2)[f_B(v_2)]^{-1}\delta(\vec{v}_1-\vec{v}_2)\delta(\vec{k}_1+\vec{k}_2). \quad (21)$$

By using the identity

$$(L_1+L_2)\delta(\vec{v}_1-\vec{v}_2)\delta(\vec{k}_1+\vec{k}_2)=2\delta(\vec{k}_1+\vec{k}_2)J(\vec{v}_1)\delta(\vec{v}_1-\vec{v}_2), \quad (22)$$

we obtain the following simple expression for B:

$$B(\vec{k}_1,\vec{v}_1,\vec{k}_2,\vec{v}_2)=\delta(\vec{k}_1+\vec{k}_2)[f_B(v_2)]^{-1}J(\vec{v}_1)\delta(\vec{v}_1-\vec{v}_2). \quad (23)$$

Thus, the correlation function of the random force is

$$\langle F(\vec{k}_1,\vec{v}_1;t_1)F(\vec{k}_2,\vec{v}_2;t_2)\rangle=2\delta(\vec{k}_1+\vec{k}_2) \times\delta(t_1-t_2)[f_B(v_2)]^{-1}J(\vec{v}_1)\delta(\vec{v}_1-\vec{v}_2)\ . \quad (24)$$

In later calculations, the Laplace transform of this expression with respect to time will be needed:

$$\langle F(\vec{k}_1,\vec{v}_1;\epsilon_1)F(\vec{k}_2,\vec{v}_2;\epsilon_2)\rangle=2\delta(\vec{k}_1+\vec{k}_2) \times\delta(\epsilon_1+\epsilon_2)[f_B(v_2)]^{-1}J(\vec{v}_1)\delta(\vec{v}_1-\vec{v}_2), \quad (25)$$

where, e.g.,

$$F(\vec{k}_1,\vec{v}_1;\epsilon_1)=\int_0^\infty dt_1\, e^{-\epsilon_1 t_1}F(\vec{k}_1,\vec{v}_1;t_1). \quad (26)$$

This concludes our discussion of the properties of the random-force term in the Boltzmann-Langevin equation.

III. SOLUTION OF BOLTZMANN-LANGEVIN EQUATION

The formal solution of the Boltzmann-Langevin equation was already written in Eq. (14). It consists of two parts; the first is

$$\phi_1(\vec{k}, \vec{v}; t) = \exp[-tL(\vec{k}, \vec{v})]\phi(\vec{k}, \vec{v}; 0), \tag{27}$$

where $\phi(\vec{k}, \vec{v}; 0)$ is the initial value of $\phi(\vec{k}, \vec{v})$. This is, in fact, the solution of the averaged Boltzmann equation, and it can be evaluated in any of the known approximations, such as the Chapman-Enskog approximation.

The second part of the formal solution is the fluctuating part:

$$\phi^1(\vec{k}, \vec{v}; t) = \int_0^t ds\, e^{-(t-s)L(\vec{k}, \vec{v})} F(\vec{k}, \vec{v}; s). \tag{28}$$

Its average is, of course, zero, but its second moment does not vanish.

For further applications it is easier to work with the Laplace transform

$$\phi^1(\vec{k}, \vec{v}; \epsilon) = [\epsilon + L(\vec{k}, \vec{v})]^{-1} F(\vec{k}, \vec{v}; \epsilon) . \tag{29}$$

The correlation function of $\phi^1(\vec{k}, \vec{v}; \epsilon)$ is the following:

$$\begin{aligned}\langle\phi^1(\vec{k}_1, \vec{v}_1; \epsilon_1)\phi^1(\vec{k}_2, \vec{v}_2; \epsilon_2)\rangle &= \{[\epsilon_1 + L(\vec{k}_1, \vec{v}_1)] \\ &\times [\epsilon_2 + L(\vec{k}_2, \vec{v}_2)]\}^{-1}\langle F(\vec{k}_1, \vec{v}_1, \epsilon_1) F(\vec{k}_2, \vec{v}_2; \epsilon_2)\rangle \\ &= 2\delta(\vec{k}_1 + \vec{k}_2)\delta(\epsilon_1 + \epsilon_2)\{[\epsilon_1 + L(\vec{k}_1, \vec{v}_1)] \\ &\times [\epsilon_2 + L(\vec{k}_2, \vec{v}_2)]\}^{-1} f_B{}^{-1}(v_2) J(v_1)\delta(\vec{v}_1 - \vec{v}_2).\end{aligned} \tag{30}$$

For hydrodynamic calculations it is convenient to use the dimensionless velocity, defined as

$$\vec{c} = (m/2k_BT)^{1/2}\vec{v} . \tag{31}$$

The correlation function in terms of this variable is

$$\begin{aligned}\langle\phi^1(\vec{k}_1, \vec{c}_1; \epsilon_1)\phi^1(\vec{k}_2, \vec{c}_2; \epsilon_2)\rangle &= 2\delta(\vec{k}_1 + \vec{k}_2)\delta(\epsilon_1 + \epsilon_2)(2k_BT/m)^{1/2}(\pi^{3/2}/n) \\ &\times \{[\epsilon_1 + (2k_BT/m)^{1/2} L(\vec{k}_1, \vec{c}_1)][\epsilon_2 + (2k_BT/m)^{1/2} L(\vec{k}_2, \vec{c}_2)]\}^{-1} e^{c_2^2} J(c_1)\delta(c_1 - c_2) .\end{aligned} \tag{32}$$

IV. FLUCTUATIONS OF HYDRODYNAMIC VARIABLES

Expressions for the average pressure tensor and the average heat current can be evaluated from approximate solutions of the Boltzmann equation. The coefficients of viscosity and heat conductivity are extracted from these expressions. By using the Boltzmann-Langevin equation, one can find expressions also for the fluctuating parts of the pressure tensor and the heat current, and evaluate their correlation functions.

The pressure tensor is defined as[3-5]

$$\overleftrightarrow{P} = m\int d^3U\, \vec{U}\vec{U} f_B \phi, \tag{33}$$

where $\vec{U} = \vec{v} - \langle\vec{v}\rangle$. In order to get the fluctuating part of the pressure tensor one has to put ϕ^1 into Eq. (33) instead of ϕ. On doing this, using the dimensionless velocity $\vec{c}$ [Eq. (31)], one gets the following expression for the Fourier Laplace transform of the fluctuating pressure tensor:

$$\overleftrightarrow{P}^1(\vec{k}, \epsilon) = 2k_B Tn\pi^{-3/2} \int d^3c\, \vec{c}\vec{c}\, e^{-C^2} \phi^1(\vec{k}, \vec{c}; \epsilon). \tag{34}$$

The correlation function of this quantity is

$$\langle \overleftrightarrow{P}^1(\vec{k}_1, \epsilon_1) \overleftrightarrow{P}^1(\vec{k}_2, \epsilon_2)\rangle = (2k_BTn)^2\pi^{-3} \int d^3c_1 \int d^3c_2 \vec{c}_1\vec{c}_1\vec{c}_2\vec{c}_2 e^{-C_1^2} e^{-C_2^2} \langle \phi^1(\vec{k}_1, \vec{c}_1; \epsilon_1)\phi^1(\vec{k}_2, \vec{c}_2; \epsilon_2)\rangle. \tag{35}$$

An expression for the correlation function of the fluctuating distribution function was already derived [Eq. (32)]; by substituting it in Eq. (35) we get, after one integration,

$$\langle\overleftrightarrow{P}^1(\vec{k}_1, \epsilon_1)\overleftrightarrow{P}^1(\vec{k}_2, \epsilon_2)\rangle = (2m^2n/\pi^{3/2})(2k_BT/m)^{5/2}\delta(\vec{k}_1 + \vec{k}_2)\delta(\epsilon_1 + \epsilon_2)$$

$$\times \int d^3c\, \vec{c}\,\vec{c}\, e^{-c^2}\{[\epsilon_1 + (2k_BT/m)^{1/2}L(\vec{k}_1,\vec{c})][\epsilon_2 + (2k_BT/m)^{1/2}L(\vec{k}_2\vec{c})]\}^{-1} J(\vec{c})\vec{c}\,\vec{c}. \tag{36}$$

From the properties of the linearized collision operator it is known that $J(c)[c^2]=0$; therefore, we may write

$$J[2\vec{c}\,\vec{c}] = J[2(\vec{c}\,\vec{c} - \tfrac{1}{3}c^2\overleftrightarrow{I})]. \tag{37}$$

In order to remain in the framework of the first Chapman-Enskog approximation we have to evaluate Eq. (36) in the limits of $k, \epsilon \to 0$. Doing this, and using Eq. (37), we get from Eq. (36) the following:

$$\langle \overleftrightarrow{P}^1(\vec{k}_1,\epsilon_1)\,\overleftrightarrow{P}^1(\vec{k}_2,\epsilon_2)\rangle = (m^2n/\pi^{3/2})(2k_BT/m)^{3/2}\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\int d^3c\,\vec{c}\,\vec{c}\,e^{-C^2}[J(c)]^{-1}[2(\vec{c}\,\vec{c}-\tfrac{1}{3}c^2\overleftrightarrow{I})]. \tag{38}$$

The viscosity coefficient can be expressed as follows[3-5]:

$$\eta = mn(2k_BT/m)^{1/2}(4/15\pi^{1/2})\int dc\, c^6 e^{-C^2} S(c), \tag{39}$$

where $S(c)$ is the solution of the following integral equation:

$$J[(\vec{c}\,\vec{c} - \tfrac{1}{3}c^2\overleftrightarrow{I})S(c)] = 2(\vec{c}\,\vec{c} - \tfrac{1}{3}c^2\overleftrightarrow{I}). \tag{40}$$

The correlation function, expressed in terms of the function $S(c)$, has the following form:

$$\langle \overleftrightarrow{P}^1(\vec{k}_1,\epsilon_1)\overleftrightarrow{P}^1(\vec{k}_2,\epsilon_2)\rangle = (m^2n/\pi^{3/2})(2k_BT/m)^{3/2}\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\int d^3c\,\vec{c}\,\vec{c}\,e^{-C^2}(\vec{c}\,\vec{c}-\tfrac{1}{3}c^2\overleftrightarrow{I})S(c). \tag{41}$$

Performing the angular integrations and using Eq. (39) for the viscosity we get, after some rearrangements, the final result:

$$\langle P_{ij}{}^1(\vec{k}_1,\epsilon_1)P_{lm}{}^1(\vec{k}_2,\epsilon_2)\rangle = 2k_BT\eta\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}-\tfrac{2}{3}\delta_{ij}\delta_{lm}). \tag{42}$$

This becomes, after inversion of the Fourier and Laplace transforms,

$$\langle P_{ij}{}^1(\vec{R}_1,t_1)P_{lm}{}^1(\vec{R}_2,t_2)\rangle = 2k_BT\eta\delta(\vec{R}_1-\vec{R}_2)\delta(t_1-t_2)(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}-\tfrac{2}{3}\delta_{ij}\delta_{lm}). \tag{43}$$

These expressions are equivalent to the results derived by Landau and Lifshitz.[8]

The correlation function of the fluctuating heat current can be derived in a similar way. One begins with the following expression for the heat current[3-5]:

$$\vec{q} = \tfrac{1}{2}m\int d^3U\,\vec{U}\,U^2 f_B\phi. \tag{44}$$

Expressed in the dimensionless velocity c, the Laplace transform of the fluctuating heat current has the following form:

$$\vec{q}^{\,1}(\vec{k},\epsilon) = (mn/2\pi^{3/2})(2k_BT/m)^{3/2}\int d^3c\,\vec{c}\,c^2 e^{-C^2}\phi^1(\vec{k},\vec{c};\epsilon). \tag{45}$$

Its correlation function is

$$\langle\vec{q}^{\,1}(\vec{k}_1,\epsilon_1)\vec{q}^{\,1}(\vec{k}_2,\epsilon_2)\rangle = (m^2n^2/4\pi^3)(2k_BT/m)^3$$

$$\times\int d^3c_1\int d^3c_2\,\vec{c}_1c_1{}^2\,\vec{c}_2c_2{}^2 e^{-c_1{}^2}e^{-c_2{}^2}\langle\phi^1(\vec{k}_1,\vec{c}_1;\epsilon_1)\phi^1(\vec{k}_2,\vec{c};\epsilon_2)\rangle. \tag{46}$$

Substituting the expression for the distribution correlation function and performing one integration, we get

$$\langle \vec{q}^{\,1}(\vec{k}_1{}^1, \epsilon_1)\vec{q}^{\,1}(\vec{k}_2, \epsilon_2)\rangle\rangle = (m^2 n/4\pi^{3/2})(2k_B T/m)^{7/2}\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\int d^3c\,\vec{c}c^2 e^{-c^2}$$
$$\times\{[\epsilon_1+(2k_BT/m)^{1/2}L(\vec{k}_1,\vec{c})][\epsilon_2+(2k_BT/m)^{1/2}L(\vec{k}_2,\vec{c})]\}^{-1}2J(c)\vec{c}c^2. \quad (47)$$

The vector $\vec{c}$ is an eigenfunction of $J(c)$ with a zero eigenvalue; therefore, one is permitted to write

$$J(\vec{c}c^2) = J(\vec{c}(c^2-\tfrac{5}{2})) . \quad (48)$$

Using Eq. (48) and taking the limits $k, \epsilon \to 0$, we get, from Eq. (47), the following:

$$\langle \vec{q}^{\,1}(\vec{k}_1, \epsilon_1)\vec{q}^{\,1}(\vec{k}_2, \epsilon_2)\rangle\rangle = (m^2 n/2\pi^{3/2})(2k_BT/m)^{5/2}\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\int d^3c\,\vec{c}\vec{c}c^2 e^{-c^2}[J(c)]^{-1}[\vec{c}(c^2-\tfrac{5}{2})] \quad (49)$$

The coefficient of heat conductivity can be expressed in the following manner[3-5]:

$$\lambda = \tfrac{4}{3}nk_B/\pi^{1/2}T(2k_BT/m)^{1/2}\int dc\,c^6 e^{-c^2}R(c), \quad (50)$$

where $R(c)$ is the solution of the following integral equation:

$$J[\vec{c}R(c)] = \vec{c}(c^2-\tfrac{5}{2}). \quad (51)$$

When the correlation function [Eq. (49)] is expressed in terms of $R(c)$, we get

$$\langle \vec{q}(\vec{k}_1, \epsilon_1)\vec{q}^{\,1}(\vec{k}_2, \epsilon_2)\rangle\rangle = (m^2 n/2\pi^{3/2})(2k_BT/m)^{5/2}\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\int d^3c\,\vec{c}\vec{c}c^2 e^{-c^2}\vec{c}R(c). \quad (52)$$

Performing the angular integrations and using the expression for the coefficient of heat conductivity, we get

$$\langle q_i{}^1(\vec{k}_1, \epsilon_1)q_j{}^1(\vec{k}_2, \epsilon_2)\rangle\rangle = 2k_BT^2\lambda\delta(\vec{k}_1+\vec{k}_2)\delta(\epsilon_1+\epsilon_2)\delta_{ij} . \quad (53)$$

On inverting the Fourier and Laplace transforms, this becomes

$$\langle q_i{}^1(\vec{R}_1, t_1)q_j{}^1(\vec{R}_2, t_2)\rangle = 2k_BT^2\lambda\delta(\vec{R}_1-\vec{R}_2)\delta(t_1-t_2)\delta_{ij} . \quad (54)$$

The last two expressions are equivalent to results of Landau and Lifshitz.[8]

In conclusion, we find that the fluctuating part of the solution of the Boltzmann-Langevin equation gives rise to fluctuations in the pressure tensor and the heat current, and we find that correlation functions of these fluctuations are related, in a known way, to the coefficients of viscosity and thermal conductivity.

Note added in proof. Many of the results reported in this article are contained in a report by Fox [Ronald F. Fox, Ph.D. dissertation, Rockefeller University, 1969 (unpublished)], although the methods used are somewhat different. We are grateful to Professor G.E. Uhlenbeck for calling our attention to this work.

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