Reciprocal Bootstrap Model

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Through the use of a nonlinear equation for the Bethe-Salpeter amplitude, the reciprocal bootstrap model of the $N(I,J^P=\frac{1}{2},\frac{1}{2}^+)$ and $\Delta(I,J^P=\frac{3}{2},\frac{3}{2}^+)$ baryons is studied. The results are compared with some of the consequences of the N/D method, the linear Bethe-Salpeter equation, and an approximation to the nonlinear equation.

I. INTRODUCTION

According to the reciprocal bootstrap model of the $N(I,J^P=\frac{1}{2},\frac{1}{2}^+)$ and $\Delta(I,J^P=\frac{3}{2},\frac{3}{2}^+)$ baryons, these particles are bound states of themselves and pions.[1,2] This is known to be a greatly oversimplified model; there are many other important components, and the customary static approximation is rather crude. Nevertheless, since this is the simplest bootstrap model that is at least qualitatively reasonable, it is often used for illustrating and testing new techniques and concepts. In this paper, we examine the N—Δ model through the use of a nonlinear equation[3] for the Bethe-Salpeter amplitude. The results are compared with some of the consequences of the N/D method,[4] the linear Bethe-Salpeter equation,[5] and an approximation to the nonlinear equation.[6]

II. EQUATIONS FOR THE STATIC MODEL

Let the index i denote a baryon multiplet of mass M_i. Each M_i is assumed to be real (below threshold). Denote the propagator by $S_i(E)$, where E is the energy; $S_i(E)$ has a pole of unit residue at $E=M_i$. The vertex functions Γ_{ij} are approximated by the following form:

$$\Gamma_{ij}(x,y)=2\phi_{ij}(x)\phi_{ji}(y)/g_{ij}-\phi_{ij}(x)-\phi_{ji}(y)+g_{ij}\,, \quad (1)$$

where

$$\phi_{ij}(x)=\Gamma_{ij}(x,0)=\Gamma_{ji}(0,x)\,,\quad g_{ij}=\Gamma_{ij}(0,0)=g_{ji}\,,$$

and the energies of the baryons are denoted by $E=M_i-x$, etc. The motivations for (1) and for possible generalizations are discussed in Ref. 3.

The coupled-channel Bethe-Salpeter equations are

$$\phi_{ij}(x)=\sum_{kl}\int v\,de\,I_{jk}{}^{il}(x,y,M_i)S_k(M_k-y)\phi_{ki}(y) \quad (y=M_k-M_i+e)\,, \quad (2)$$

where $v=v(e)$ is the phase-space factor for a meson of energy e, including the standard damping factor; the contribution of exchange of the lth baryon type to the interaction kernel is

$$I_{jk}{}^{il}(x,y,E)=\Gamma_{jl}(x,z)S_l(M-z)\Gamma_{lk}(z,y)C_{kl}{}^{ji} \quad (z=x+y+E+M_1-M_j-M_k)\,, \quad (3)$$

where the $C_{kl}{}^{ji}$ are Racah coefficients (crossing matrix elements).

It is convenient to write $\phi_{ij}(x)=g_{ij}\Phi_{ij}(x)$, where $\Phi_{ij}(0)=1$, and to consider separately the problems of determining the coupling constants g_{ij} and the "form factors" Φ_{ij}. On the mass shell, we can write (2) in the form of a nonlinear equation for the coupling constants

$$g=\Sigma_{kl}D_{kl}{}^{ji}C_{kl}{}^{ji}g_{jl}g_{lk}g_{ki}\,, \quad (4)$$

where the $D_{kl}{}^{ji}$ are certain complicated integrals over the form factors and the propagators satisfying the relation

$$D_{kl}{}^{ji}=D_{lk}{}^{ij}. \quad (5)$$

A similar equation can be obtained by considering the extremum principle that is equivalent to (2) when the interaction is considered as

$$0=\delta(\Sigma_j w_j{}^i g_{ji}{}^2\bar{K}_j{}^i-\Sigma_{jkl}w_j{}^i\bar{I}_{jk}{}^{il}g_{ji}g_{ki})\,, \quad (6)$$

where

$$\bar{K}_j{}^i=\int\Phi_{ji}(x)^2S_j(M_j-x)v\,de\,,$$

$$I_{jk}{}^{il}=\int\int\Phi_{ji}(x)S_j(M_j-x)I_{jk}{}^{il}(x,y,M_i)S_k \times(M_k-y)\Phi_{ki}(y)v\,de\,v'de' \quad (7)$$

$$(x=M_j-M_i+e,\quad y=M_k-M_i+e)\,.$$

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[3] R. E. Cutkosky, Phys. Rev. **154**, 1375 (1967).
[4] R. F. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1331 (1965); Phys. Rev. Letters **13**, 497 (1964).
[5] R. F. Sawyer, Phys. Rev. **142**, 991 (1966).
[6] K. Y. Lin and R. E. Cutkosky, Phys. Rev. **130**, B205 (1965); F. S. Chen-Cheung, *ibid.* **158**, B1529 (1967).

The w_j^i are weights related to the multiplicities. If we vary with respect to the g's only, and then exhibit explicitly the quadratic dependence of I_{jk}^{il} upon the g's, we obtain

$$g_{ji}=\Sigma_{kl}\hat{D}_{kl}^{ji}g_{jl}g_{lk}g_{ki}C_{kl}^{ji}, \quad (8)$$

where

$$\hat{D}_{kl}^{ji}=\bar{I}_{jk}^{il}/\bar{K}_j^i C_{kl}^{ji}g_{jl}g_{lk}. \quad (9)$$

Equations (4) and (8) necessarily give the same solution. In some earlier works[7] it was assumed that $\hat{D}=D$, which is a simple way to guarantee identity of the solutions. This hypothesis is a useful one, as it leads to additional restrictions on the D's and K's. We have found, however, in the present model, that it is only good as a rough approximation.

With the notation $\gamma_j^i=g_{ji}(w_j^i\bar{K}_j^i)^{1/2}$, we can write (6) or (8) in the standard form for an eigenvalue equation

$$\gamma_j^i=\Sigma_k\lambda^i U_{jk}^i\gamma_j^i, \quad (\lambda^i=1), \quad (10)$$

where U_{jk}^i can be interpreted as an "effective potential" for states with the quantum numbers of (i), as averaged with the appropriate form factors. It is constructed as follows:

$$U_{jk}^i=\Sigma_l w_j^i C_{kl}^{ji}V_{jk}^{il}g_{jl}g_{lk}, \quad (11)$$

where the factor $V_{jk}^{il}=(\bar{K}_j^i/\bar{K}_k^i)^{1/2}\hat{D}_{kl}^{ji}$ contains the dynamical information. Note that the Racah coefficients obey the symmetry relation $w_j^iC_{kl}^{ji}=w_k^iC_{jl}^{ki}$.

Equation (10) provides a simple way to estimate the strength of the forces associated with quantum numbers for which there are supposed to be no bound states. In addition, we can estimate the dependence of output masses on the masses of constituents and exchanged particles from the values of the V's.

The normalization conditions on the Bethe-Salpeter amplitudes are

$$N_i=1=\int v\,de\,\Sigma_j w_j^i\phi_{ji}(x)^2S_j'(M_j-x) + \int\int v\,de\,v'de'\Sigma_{jkl}w_j^i\phi_{ji}(x)S_j(M_j-x) \times I_{jk}^{il}(x,y,M_i)'S_k(M_k-y)\phi_{ki}(y), \quad (12)$$

where x and y have the same meaning as in (7), and where the primes on S and I denote differentiation with respect to $-E$.

III. SINGLE-MULTIPLET MODEL

We reexamine here the degenerate $[SU(4)]$ version of the N—Δ model (cf. Refs. 3 and 6) in order to see how sensitively it depends on the spectral function of the baryon propagator

$$S(E)=(M-E)^{-1}+\int_1^\infty \rho(e)\,de\,(M+e-E)^{-1}. \quad (13)$$

The static model thus involves two arbitrary functions v and ρ, although in most treatments the second has been omitted. It is well known that the static model really depends on only one characteristic parameter of the function v—a cutoff energy. In our analysis we consider

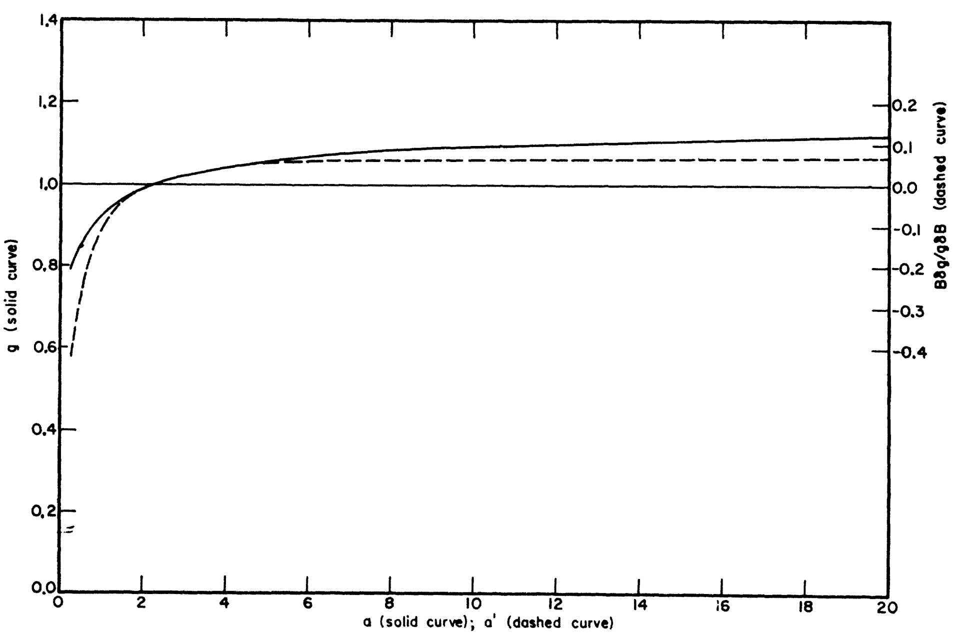

FIG. 1. Dependence of the coupling constant on the form of the continuum contribution to the propagator. Here the unit for a is the mass of the meson.

[7] R. E. Cutkosky and M. Leon, Phys. Rev. 138, B667 (1965).

the form

$$\rho(e)=B\delta(e-a)+B'\delta(e-a'). \qquad (14)$$

We first took $B'=0$; then a represents an average energy of the continuum contribution to $S(E)$, and B represents the net strength of this continuum. For a given value of a, the parameter B is fixed by imposing the normalization condition (10) on the bound-state solution.

We took for $v(x)$ the expression

$$v(x)=(x^2-1)^{3/2}(1+x^2/\Lambda^2)^{-2}. \qquad (15)$$

Just as in other formulations of the static-bootstrap model, the solution depends very strongly on Λ. In Fig. 1 we show some results for $\Lambda=5$ [which is approximately right, in units of the $SU(4)$-average meson mass]. For $SU(4)$, the Racah coefficient is $C=47/63$.

For given B and a, we solved Eq. (2) by the iterative procedure described in Ref. 3; then B was found from (12). In Fig. 1, g is plotted against a. Observe that g is insensitive to a, provided a is big enough. (Because only the relative values of g are interesting, we normalized to the value at $a=2.2$, which is near the peak of v.) Next, for $a=2.2$, we reduced B by 10%, and, for various a', we adjusted B' in accordance with (12). The results, graphed in Fig. 1, indicate again that the form of the solution is very insensitive to the form of $\rho(e)$ for large e. The results also show that the shape of $\rho(e)$ near threshold is quite significant; this part of ρ would, however, be determinable, in a more extensive study, by consideration of the low-energy scattering given by the potential (3). The parameter B (and in the second case, B') varied almost linearly with a (a'), and in such a way that the low-energy form of S remained nearly invariant.

We conclude that the static-bootstrap model really involves only one important adjustable parameter, namely, the damping parameter Λ. In order to have a solution, it is necessary to consider a more general form for the propagator than is customary, but when the high-energy form of the propagator is adjusted in accordance with the supposition that a bootstrap solution exists, it does not matter how this adjustment is made.

IV. RESULTS

In this section, we shall present the results of the reciprocal bootstrap model for the N and Δ. We denote N by the subscript 0, and Δ by 1. The mass difference $\Delta-N=\delta a$ and the three coupling constants g_{i+j} are to be determined from the model. Both propagators are given by Eqs. (13) and (14), with $B'=0$ and with the same value for a that is taken to be 1. An iteration procedure was started by assuming some initial values for B, δ, and also for the form factors Φ_{ij} when one of the baryons is on the mass shell. The three coupling constants g_0, g_1, and g_2 are then determined from Eq. (4). Equations (2), (4), and (12) were then solved (Fig. 2), by a complicated iterative method, for the g_{i+j}, Φ_{ij}, B, and δ. The mass difference is $\delta=0.385$. The coupling constants are $g_0=0.87$, $g_1=0.6$, and $g_2=0.79$; the relative strength of the continuum contribution of the propagator B is 2.5. In Tables I and II, we compare the results with that from the simple static-bootstrap model in Ref. 4 and with that from Ref. 6.

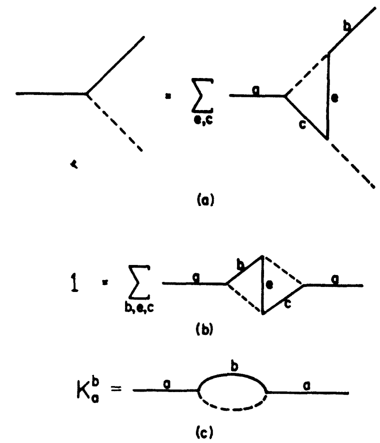

FIG. 2. (a) Vertex equations for Eqs. (2) and (4). (b) Normalization equations for Eq. (12). The notations are $D_{ab}{}^{ec}=D_{ki}{}^{ji}$ and $V_{bc}{}^{ae}=V_{jk}{}^{il}$. (c) The graph representation for $K_a{}^b$.

TABLE I. Comparison of the coupling-constant ratios and the mass differences in this work, simple Bethe-Salpeter bootstrap model (B.S.) ,and strong-coupling models (S.C.). $g_{NN\pi}$ (model of row 1)/$g_{NN\pi}$ (model of row 2) = 1, and $g_{NN\pi}$ (model of row 2)/$g_{NN\pi}$ (model of row 3) = 0.9. The experimental value of the first column is $(g_{\Delta N\pi}/g_{NN\pi})^2=0.5$.

	$(g_{\Delta N\pi}/g_{NN\pi})^2$	$g_{NN\pi}/g_{\Delta\Delta\pi}$	$g_{\Delta N\pi}/g_{\Delta\Delta\pi}$	$\delta=\Delta-N$
This work	0.475	1.1	0.75	0.385
B.S. model,[a] $N-\Delta$	0.36	1.5	0.9	0.311
B.S. model,[b]				
$I_{max}=J_{max}=\infty$	0.5	1	0.7	0
S.C. model[c]	0.5	1	0.7	0
$SU(4)$,[d] **20** multiplet	0.31	1	0.75	0

[a] Reference 6, Lin *et al.*
[b] Reference 6, Chen-Cheung.
[c] V. Singh, Phys. Rev. **144**, 1275 (1966); see also Ref. b above.
[d] Reference 6, Lin *et al.*

TABLE II. $\partial V/\partial m$ of this work estimated by taking the average of the V's in Table IV. $\partial V/\partial m$ for B.S. models are calculated from the changes in the propagators only, neglecting the induced changes in the vertex function and using $B=0$. D.F. is the result from Ref. 4. B.S. stands for linear Bethe-Salpeter model. The subscripts e, c, and a to mass m indicate the corresponding propagator; see Fig. 2(b) for illustration.

	$\partial V/\partial m_e$	$\partial V/\partial m_c$	$\partial V/\partial m_a$	$\partial m_a/\partial m_e$	$\partial m_a/\partial m_c$
This work	0.43	0.51	0.94	0.46	0.54
L-C model[a]	1	0	1	1	0
B.S. $(B=0)$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
B.S. $(B=\infty)$	$\frac{1}{3}B^{-1}$	$\frac{1}{2}B^{-1}$	$\frac{5}{6}B^{-1}$	$\frac{2}{5}$	$\frac{3}{5}$
B.S. $(B=2.5)$	0.4	0.75	1.14	0.36	0.65
D.F.[b]				−1	2

[a] Reference. 6
[b] Reference 4.

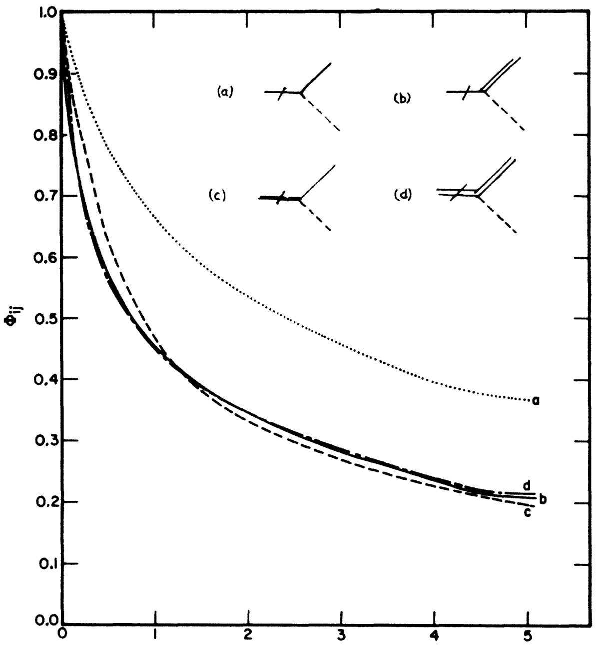

FIG. 3. Insert: Various form factors $\Phi_{ij}(x)$. x is the amount by which the second baryon is away from the mass shell. The first baryon is on the mass shell. x is in units of a. Form factors (a), (b), (c), and (d) plotted in one graph for better comparison.

The form factors Φ_{ij} are plotted against x in Fig. 3 (one of the baryons is on the mass shell).

From the result of this work, the "effective potential" as defined in Eq. (10) can be obtained. First we list the values of $\hat{D}_{kl}{}^{ji}$ and $D_{kl}{}^{ji}$ in Table III. The values of $V_{jk}{}^{il}$ are then presented in Table IV. Variations of $V_{jk}{}^{il}$ with respect to baryon mass in various propagators are listed in Table II and are compared with those from several other models.

Some general conclusions from this work are the following. Although some of the input (the form of the baryon propagators and the structure of the off-mass-shell vertex functions, for example) is quite different from what is usually assumed in the static model; the general nature of the results is rather similar. However, results pertaining to perturbation of the model, especially by the method of Ref. 4 and, to a lesser extent, by the method of Refs. 6, may not always be justified, as is indicated by Table II. Compared with the method of Ref. 6, it is mainly the normalization equations, not the mass-shell vertex equations, which are affected by the more complicated structure of the vertices and propagators.

A pessimistic note is that the technical problem of finding the solution in the two-multiplet case was much harder than it was in the single-multiplet case. Further progress will have to be made on this technical problem before a more realistic model calculation along these lines can be attempted. (For interesting related articles, see Ref. 8.)

TABLE III. Values of $D_{kl}{}^{ji}$ as defined in Eq. (4) and values of $\hat{D}_{kl}{}^{ji}$ as defined in Eq. (9). D_0 is the value calculated according to the L-C model (Ref. 6) normalized to unity for 0000. For illustration of $D_{ab}{}^{ec}$, see Fig. 2(a), right-hand side of the equality.

aceb	$\hat{D}_{ab}{}^{ec}$	$D_{ab}{}^{ec}$	$(D_0)_{ab}{}^{ec}$
0000	0.79	1.26	1.0
0010	0.39	0.54	0.62
0100	0.61	0.54	0.62
0110	0.33	0.26	0.23
1000	1.36	1.64	1.38
1100	0.79	0.81	1.0
1010	0.67	0.67	1.0
1110	0.58	0.39	0.62
0001	1.46	1.64	1.38
0011	1.16	0.81	1.0
0101	0.95	0.67	1.0
0111	0.58	0.39	0.62
1001	1.70	2.18	1.77
1011	0.92	0.99	1.38
1101	1.25	0.99	1.38
1111	0.77	0.57	1.0

TABLE IV. Effective potentials as defined in Eq. (11). For illustration of $V_{bc}{}^{ae}$, see Fig. 2(b), right-hand side of equality.

abeca	$V_{bc}{}^{ae}$
00000	1
01000	0.9
00100	0.87
01010	0.86
01100	0.81
01110	0.74
10001	1.82
11001	1.47
10101	1.22
11101	1.1
11111	1

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[8] K. V. Vasavada and Y. S. Kim, Phys. Rev. **152**, 1259 (1966).