Modified Tamm-Dancoff Approach to Charged-Scalar Theory*

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A scheme for truncating the τ-function equations appropriate to charged-scalar-meson theory is presented and solved in the one- and two-meson approximations. The basis of the truncation scheme is a restriction of the number of mesons occurring in bare intermediate states. The resulting equations are modified so that the same diagrams which account for the dressing of the nucleon also account for meson-nucleon scattering. A meson-nucleon bound state is present, for large enough coupling constant, in the two-meson approximation. It occurs in the channel where the interaction takes place entirely through exchange diagrams. It is found that the masses and cutoff function of the theory can be chosen in such a way that the probability of finding the physical particle as a bare particle decreases in going from the one-meson to the two-meson approximation.

INTRODUCTION

In order to understand the role of field theory in describing strong interactions, the situation with regard to dynamical bound states must be clarified. There is now a large body of literature on meson-nucleon scattering that is based on the classic work of Chew and Low.[1] In this work, it is assumed that there are no bound states lying between the single-nucleon state and the meson-nucleon continuum. The general analysis of solutions of the Low equation in the one-meson approximation by Castillejo, Dalitz, and Dyson[2] showed that meson-nucleon scattering cannot be determined from this equation in the presence of bound states. In this work, the authors conclude: "It appears that the Low equation expresses the information we can deduce from the fixed space dependence of the interaction, knowing nothing about the internal structure of the scatterer." Recently, a two-meson solution to the Low equation for charged-scalar-meson theory has been obtained.[3] In this work, it is also assumed that there are no bound states.

Two basic questions immediately arise. Are there necessarily bound states present in meson-nucleon theory when one takes into account multimeson intermediate states? If these bound states do occur, what is the proper equation determining meson-nucleon scattering? In order to investigate the possibility of bound states, including those with the quantum number of the nucleon, we treat the single-particle nucleon states on an equal footing with the meson-nucleon scattering states. For simplicity, we work with scalar rather than pseudoscalar mesons, and consider the τ-function equations that describe the single-nucleon propagators as well as the meson-nucleon scattering and production amplitudes. In order to truncate these equations, we use a Tamm-Dancoff treatment,[4] where the number of mesons occurring in bare intermediate states is restricted. The truncated equations are then modified so that, for a meson-nucleon state, the nucleon is dressed only by those diagrams that are allowed according to the order of truncation. In other words, if one meson is already present with a nucleon, that nucleon is dressed according to the $N-1$ truncation scheme, if we are working in the N truncation scheme. The advantage of this method is that in any order of truncation, the nucleons are dressed by the same set of diagrams that are responsible for meson-nucleon scattering. The disadvantage is the absence of crossing symmetry.[5] This is not considered to be a serious problem, since, at this stage, our emphasis is on the presence of dynamical bound states rather than scattering amplitudes.

In Sec. I, the Hamiltonian is written down, and the field equations and commutation relations are given. Then the τ functions appropriate to the lowest nontrivial sector with one baryon are defined, and the equations determining these functions are written down.

In Sec. II, the truncation method is introduced and the one-meson scheme is applied to the lowest sector. It is immediately seen that the truncated system of equations must be modified if the nucleon propagator is to have the proper analytic structure. The necessary modification is made and explained in the context of a solvable model for which the truncated system of equations is exact. Solutions to the one-meson equations are given.

In Sec. III, the two-meson scheme is applied to the two lowest nontrivial sectors with one baryon. A symbolic representation of the τ-function equations is given. Using this representation, the equations for the nucleon propagator and meson-nucleon four-point function are reduced to the expansion of these functions in terms of Feynman diagrams. The necessary modification of the two-meson equations is made and explained

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[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[2] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[3] J. B. Bronzan, J. Math. Phys. **7**, 1351 (1966).
[4] I. Tamm, J. Phys. (USSR) **9**, 449 (1945); S. M. Dancoff, Phys. Rev. **78**, 382 (1950); F. J. Dyson, *ibid.* **90**, 994 (1953).
[5] Crossing symmetry is meant in the usual sense as the relation or set of relations arising from the symmetry of sets of Feynman diagrams when incoming and outgoing meson lines are exchanged. See M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

using these Feynman diagrams. A general rule is given for modifying the N-meson equations. Then the solution to the modified two-meson equations is obtained. In solving for the renormalization constants, it is found that the parameters of the theory can be picked in such a manner that the wave-function renormalization constants in the two-meson scheme are less than in the one-meson scheme. This result suggests the possibility of an interesting theorem regarding elementary and composite particles in field theory and this point is discussed. From the solutions for the τ functions, it is seen that there can be no bound state in the channel[6] where the interaction proceeds through uncrossed diagrams. On the other hand, a bound state does occur (for large enough coupling constant) in the channel where the interaction proceeds entirely through crossed (exchange) diagrams. Finally, the reason for the absence of crossing symmetry in the modified truncation scheme is pointed out.

In the concluding section, we discuss the relation between the method that is presented in this paper for solving meson-nucleon theory, with emphasis on the spectrum, and the method based on the work of Chew and Low for calculating meson-nucleon scattering. Then we summarize our results and indicate the direction of future work.

I. HAMILTONIAN AND τ-FUNCTION EQUATIONS

We add a term $N\to V+\bar{\theta}$ to the conventional Lee-model interaction[7] $V\to N+\theta$ to obtain the renormalized Hamiltonian[8]

$$\begin{aligned} H=&m_V Z_V\psi_V^\dagger\psi_V+m_N Z_N\psi_N^\dagger\psi_N+\sum_k \omega(a_k^\dagger a_k+b_k^\dagger b_k)\\ &+\delta m_V Z_V\psi_V^\dagger\psi_V+\delta m_N Z_N\psi_N^\dagger\psi_N\\ &+g\sum_k [u(\omega)/(2\omega)^{1/2}][\psi_V^\dagger\psi_N a_k+\psi_N^\dagger a_k^\dagger\psi_V\\ &+\psi_N^\dagger\psi_V b_k+\psi_V^\dagger b_k^\dagger\psi_N], \end{aligned} \tag{1}$$

where $\omega=(k^2+\mu^2)^{1/2}$ and μ is the meson mass. In order to have stable V and N particles, we insist that

$$m_V<m_N+\mu \quad\text{and}\quad m_N<m_V+\mu. \tag{2}$$

The usual commutation relations hold:

$$\{Z_V^{1/2}\psi_V,Z_V^{1/2}\psi_V^\dagger\}=\{Z_N^{1/2}\psi_N,Z_N^{1/2}\psi_N^\dagger\}=1 \tag{3a}$$

and

$$[a_k,a_{k'}^\dagger]=[b_k,b_{k'}^\dagger]=\delta_{kk'}. \tag{3b}$$

[6] We use the expressions "sector" and "channel" interchangeably.

[7] M. S. Maxon and R. B. Curtis, Phys. Rev. **137**, B996 (1965), hereafter denoted as I.

[8] Since many of the results obtained in Refs. 7 and 11 will be used in this work, we use the notation developed in these articles. To obtain the more conventional notation such as that used in Ref. 3, make the replacement $V\to p$, $N\to n$, $\theta\to\pi^+$, $\bar{\theta}\to\pi^-$, $m_V=m_N=m$, and therefore $Z_V=Z_N=Z$.

We will call the particle that is associated with the b field the $\bar{\theta}$ particle.

There are two conserved quantities in this theory:

$$Q_1=Z_V\psi_V^\dagger\psi_V+Z_N\psi_N^\dagger\psi_N \tag{4a}$$

and

$$Q_2=Z_V\psi_V^\dagger\psi_V+\sum_k (a_k^\dagger a_k-b_k^\dagger b_k). \tag{4b}$$

Thus the model breaks up into sectors designated by the integer eigenvalues of Q_1 and Q_2. It is obvious from the form of Q_2 that any number of θ-$\bar{\theta}$ pairs may be added to a given state to obtain another state with the same quantum numbers. Therefore, there are an infinite number of states spanning each sector.

We list the first few sectors in order to familiarize the reader with the notation:

(q_1,q_2)	States
(0,0)	$\lvert 0\rangle, \lvert\theta\bar{\theta}\rangle,\cdots,\lvert n(\theta\bar{\theta})\rangle,\cdots$
(0,1)	$\lvert\theta\rangle, \lvert\theta,(\theta\bar{\theta})\rangle,\cdots,\lvert\theta,n(\theta\bar{\theta})\rangle,\cdots$
(1,1)	$\lvert V\rangle, \lvert V,(\theta\bar{\theta})\rangle,\cdots,\lvert V,n(\theta\bar{\theta})\rangle,\cdots$ $\lvert N\theta\rangle, \lvert N\theta,(\theta\bar{\theta})\rangle,\cdots,\lvert N\theta,n(\theta\bar{\theta})\rangle,\cdots$
(1,2)	$\lvert V\theta\rangle, \lvert V\theta,(\theta\bar{\theta})\rangle,\cdots,\lvert V\theta,n(\theta\bar{\theta})\rangle,\cdots$ $\lvert N2\theta\rangle, \lvert N2\theta,(\theta\bar{\theta})\rangle,\cdots,\lvert N2\theta,n(\theta\bar{\theta})\rangle,\cdots$

(5)

where $n(\theta\bar{\theta})$ denotes n θ-$\bar{\theta}$ pairs. From Eq. (1), we see that the Hamiltonian has the symmetry

$$H\begin{pmatrix}V\leftrightarrow N\\ \theta\leftrightarrow\bar{\theta}\end{pmatrix}=H. \tag{6}$$

Under this transformation, we have

(q_1,q_2)	States
$(0,1)\to(0,-1)$	$\lvert\bar{\theta}\rangle, \lvert\bar{\theta},(\theta\bar{\theta})\rangle,\cdots,\lvert\bar{\theta},n(\theta\bar{\theta})\rangle,\cdots$
$(1,1)\to(1,0)$	$\lvert N\rangle, (N,(\theta\bar{\theta})\rangle,\cdots,\lvert N,n(\theta\bar{\theta})\rangle,\cdots$ $\lvert V\bar{\theta}\rangle, \lvert V\bar{\theta},(\theta\bar{\theta})\rangle,\cdots,\lvert V\bar{\theta},n(\theta\bar{\theta})\rangle,\cdots$
$(1,2)\to(1,-1)$	$\lvert N\bar{\theta}\rangle, \lvert N\bar{\theta},(\theta\bar{\theta})\rangle,\cdots,\lvert N\bar{\theta},n(\theta\bar{\theta})\rangle,\cdots$ $\lvert V2\bar{\theta}\rangle, \lvert V2\bar{\theta},(\theta\bar{\theta})\rangle,\cdots,\lvert V2\bar{\theta},n(\theta\bar{\theta})\rangle,\cdots$

(7)

Since Z_V and Z_N will be expressed in terms of m_V and m_N, we only need to make the replacement $m_V\leftrightarrow m_N$ in the solutions for the sectors listed in (5) to obtain the solutions to the corresponding sectors[9] indicated in (7).

Since there is no interaction unless a nucleon is present, the physical states spanning the sectors (0,0), (0,1), and (0,−1), can be taken to be the bare states. Therefore, the lowest nontrivial sector is the (1,1) sector. We shall denote this as the V-$N\theta$ sector to be consistent with the notation of I.

[9] If we restrict ourselves to sectors where $q_1=1$ (one nucleon), and use the notation of Ref. 8, then the (1,1) and (1,2) sectors correspond to the scalar-meson–nucleon systems with total electric charge +1 and +2, respectively. Making the replacement $m_V\leftrightarrow m_N$ in the solutions for these sectors yields the solution to the (1,0) and (1,−1) sectors. These sectors correspond to scalar-meson–nucleon systems with total electric charge 0 and −1, respectively.

From (1) and (3), we obtain the field equations

$$Z_V\left(i\frac{d}{dt}-m_{0,V}\right)\psi_V(t)=g\sum_k\frac{u(\omega)}{(2\omega)^{1/2}}\times\psi_N(t)[a_k(t)+b_k^\dagger(t)], \quad (8a)$$

$$Z_N\left(i\frac{d}{dt}-m_{0,N}\right)\psi_N(t)=g\sum_k\frac{u(\omega)}{(2\omega)^{1/2}}\times\psi_V(t)[a_k^\dagger(t)+b_k(t)], \quad (8b)$$

$$\left(i\frac{d}{dt}-\omega\right)a_k(t)=g\frac{u(\omega)}{(2\omega)^{1/2}}\psi_N^\dagger(t)\psi_V(t), \quad (8c)$$

and

$$\left(i\frac{d}{dt}-\omega\right)b_k(t)=g\frac{u(\omega)}{(2\omega)^{1/2}}\psi_V^\dagger(t)\psi_N(t), \quad (8d)$$

where

$$m_{0,V}=m_V+\delta m_V. \quad (8e)$$

The τ functions appropriate to the V-$N\theta$ sector are

$$\tau_V^1(s)\equiv\langle 0|T[\psi_V(s)\psi_V^\dagger(0)]|0\rangle, \quad (9a)$$

$$\tau_V^2(s;\omega)\equiv\frac{(2\omega)^{1/2}}{u(\omega)}\times\langle 0|T[\psi_N(s)a_k(s)\psi_V^\dagger]|0\rangle, \quad (9b)$$

$$\tau_V^3(s;\omega)\equiv\frac{(2\omega)^{1/2}}{u(\omega)}\langle 0|T[\psi_V(s)\psi_N^\dagger a_k^\dagger]|0\rangle, \quad (9c)$$

$$\tau_V^4(s;\omega,\omega')\equiv\frac{(4\omega\omega')^{1/2}}{u(\omega)u(\omega')}\langle 0|T[\psi_V(s)a_k(s)\times b_{k'}(s)\psi_V^\dagger]|0\rangle, \quad (9d)$$

$$\tau_V^6(s;\omega,\omega',\omega'')\equiv\frac{(8\omega\omega'\omega'')^{1/2}}{u(\omega)u(\omega')u(\omega'')}\times\langle 0|T[\psi_N(s)a_k(s)a_{k'}(s)\times b_{k''}(s)\psi_V^\dagger]|0\rangle, \quad (9e)$$

$$\tau_V^8(s;\omega,\omega',\omega'',\omega''')\equiv\frac{(16\omega\omega'\omega''\omega''')^{1/2}}{u(\omega)u(\omega')u(\omega'')u(\omega''')}\times\langle 0|T[\psi_V(s)a_k(s)a_{k'}(s)\times b_{k''}(s)b_{k'''}(s)\psi_V^\dagger]|0\rangle, \quad (9f)$$

and so on.

From Eqs. (8) and (3), the τ-function equations are

$$\left(i\frac{d}{ds}-m_{0,V}\right)\tau_V^1(s)=\frac{i}{Z_V}\delta(s)+\frac{g}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\tau_V^2(s;\omega), \quad (10a)$$

$$\left(i\frac{d}{ds}-m_{0,N}-\omega\right)\tau_V^2(s;\omega)=\frac{g}{Z_N}\tau_V^1(s)+\frac{g}{Z_N}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\tau_V^4(s;\omega,\omega'), \quad (10b)$$

$$\left(i\frac{d}{ds}-m_{0,V}-\omega-\omega'\right)\tau_V^4(s;\omega,\omega')=\frac{g}{Z_V}\tau_V^2(s;\omega)+\frac{g}{Z_V}\sum_{k''}\frac{u^2(\omega'')}{2\omega''}\tau_V^6(s;\omega,\omega'',\omega'), \quad (10c)$$

$$\left(i\frac{d}{ds}-m_{0,N}-\omega-\omega'-\omega''\right)\tau_V^6(s;\omega,\omega',\omega'')=\frac{g}{Z_N}[\tau_V^4(s;\omega,\omega'')+\tau_V^4(s;\omega',\omega'')]+\frac{g}{Z_N}\sum_{k'''}\frac{u^2(\omega''')}{2\omega'''}\tau_V^8(s;\omega,\omega',\omega'',\omega'''), \quad (10d)$$

and so on.

Let us define

$$\hat{\tau}^\alpha(W;\omega,\omega',\cdots)\equiv\frac{1}{i}\int_{-\infty}^{\infty}ds\,e^{iWs}\hat{\tau}^\alpha(s;\omega,\omega',\cdots). \quad (11)$$

The τ-function equations in momentum space become

$$(W-m_{0,V})\hat{\tau}_V^1(W)=\frac{1}{Z_V}+\frac{g}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\hat{\tau}_V^2(W;\omega), \quad (12a)$$

$$(W-m_{0,N}-\omega)\hat{\tau}_V^2(W;\omega)=\frac{g}{Z_N}\hat{\tau}_V^1(W)+\frac{g}{Z_N}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\hat{\tau}_V^4(W;\omega,\omega'), \quad (12b)$$

$$(W-m_{0,V}-\omega-\omega')\hat{\tau}_V^4(W;\omega,\omega')=\frac{g}{Z_V}\hat{\tau}_V^2(W;\omega)+\frac{g}{Z_V}\sum_{k''}\frac{u^2(\omega'')}{2\omega''}\hat{\tau}_V^6(W;\omega,\omega'',\omega'), \quad (12c)$$

$$(W-m_{0,N}-\omega-\omega'-\omega'')\hat{\tau}_V^6(W;\omega,\omega',\omega'')=\frac{g}{Z_N}[\hat{\tau}_V^4(W;\omega,\omega'')+\hat{\tau}_V^4(W;\omega',\omega'')]+\frac{g}{Z_N}\sum_{k'''}\frac{u^2(\omega''')}{2\omega'''}\hat{\tau}_V^8(W;\omega,\omega',\omega'',\omega'''), \quad (12d)$$

and so on. To solve this sector exactly, we must solve this infinite set of equations. In the next section, we present a truncation scheme that restricts the number of θ's and $\bar{\theta}$'s that can occur in bare intermediate states, and apply the lowest-order scheme to the V-$N\theta$ sector.

II. ONE-MESON MODIFIED TAMM-DANCOFF APPROXIMATION

Let us expand $\tau_V{}^4(s;\omega,\omega')$, defined in Eq. (9d), in a complete set of intermediate states. We obtain

$$\tau_V{}^4(s;\omega,\omega')=\frac{(4\omega\omega')^{1/2}}{u(\omega)u(\omega')}\theta(s) \times\sum_n \langle 0|\psi_V a_k b_{k'}|n\rangle\langle n|\psi_V{}^\dagger(-s)|0\rangle, \quad (13)$$

where we choose the set $\{|n\rangle\}$ to consist of bare states. The bare V and N-θ states are annihilated by the operator $b_{k'}$. Therefore, states which contribute must contain at least one θ-$\bar{\theta}$ pair. In this manner it is simple to see that restricting the sum over intermediate states to those which contain N mesons (m θ's and n $\bar{\theta}$'s, where $m+n=N$) implies that $\tau_V{}^{2N+2+p}=0$, where $p=0,1,2,\cdots$. This is just the Tamm-Dancoff approximation.[4] Let us now apply the one-meson scheme to the V-$N\theta$ sector.

For $N=1$, $\tau_V{}^4(s;\omega,\omega')$ and all higher-order τ's will vanish. In this approximation, the system of equations (12) becomes

$$(W-m_{0,V})\hat{\tau}_V{}^1(W)=\frac{1}{Z_V}+\frac{g}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\hat{\tau}_V{}^2(W;\omega) \quad (14a)$$

and

$$(W-m_{0,N}-\omega)\hat{\tau}_V{}^2(W;\omega)=\frac{g}{Z_N}\hat{\tau}_V{}^1(W). \quad (14b)$$

If we solve Eqs. (14a) and (14b) as they stand, there will be a branch cut in $\hat{\tau}_V{}^1(W)$ for

$$m_{0,N}+\mu\leq W\leq\infty. \quad (15)$$

Expanding $\hat{\tau}_V{}^1(W)$ in a complete set of physical states, and using the asymptotic condition on the V field,[7] we obtain

$$\hat{\tau}_V{}^1(W)=\frac{1}{(W-m_V+i\epsilon)}+\sum_k\frac{|\langle 0|\psi_V|N\theta_\omega(\pm)\rangle|^2}{(W-m_N-\omega+i\epsilon)} +\sum_k\sum_{k'}\frac{|\langle 0|\psi_V|V\theta_\omega\bar{\theta}_{\omega'}(\pm)\rangle|^2}{(W-m_V-\omega-\omega'+i\epsilon)}+\cdots, \quad (16)$$

where $|N\theta_\omega(\pm)\rangle$ denotes the "in" or "out" N-θ scattering states defined in I. From Eq. (16), we see that the analytic structure of the V propagator should consist of a pole at $W=m_V$ with residue $+1$, and cuts for $m_N+\mu\leq W\leq\infty$, $m_V+2\mu\leq W\leq\infty$, etc. If we solved the infinite set of equations (12), the solution for $\hat{\tau}_V{}^1(W)$ would certainly exhibit this analytic structure.

When we truncate the set (12) according to the one-meson approximation, we not only lose the cuts corresponding to multimeson production, but also handle the N particle as if it were bare. The only states present in the V-$N\theta$ sector in the one-meson approximation are the $|V\rangle$ and $|N\theta_\omega\rangle$ states, and the N must be handled as a bare particle, since it is already accompanied by one meson. Therefore, with respect to the V-$N\theta$ system, the interaction $N\to V+\bar{\theta}$ is absent in the one-meson approximation. In fact, by comparing with Eqs. (41a) and (41b) of I, it can be seen that Eqs. (14a) and (14b) are just the equations of the conventional Lee model. Therefore, we set

$$m_{0,N}=m_N \quad (17a)$$

and

$$Z_N=1 \quad (17b)$$

in Eqs. (14a) and (14b) to obtain the modified one-meson Tamm-Dancoff equations

$$(W-m_{0,V})\hat{\tau}_V{}^1(W)=\frac{1}{Z_V}+\frac{g}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\hat{\tau}_V{}^2(W;\omega), \quad (18a)$$

$$(W-m_N-\omega)\hat{\tau}_V{}^2(W;\omega)=g\hat{\tau}_V{}^1(W). \quad (18b)$$

From I, the solution to these equations is

$$\hat{\tau}_V{}^1(W)=(W-m_V+i\epsilon)^{-1}[1-\beta(W)]^{-1}, \quad (19)$$

where

$$\beta(W)=g^2(W-m_V)\times\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)^2(W-m_N-\omega+i\epsilon)} =\frac{g^2}{4\pi^2}(W-m_V)\times\int_\mu^\infty\frac{d\omega\, u^2(\omega)(\omega^2-\mu^2)^{1/2}}{(m_V-m_N-\omega)^2(W-m_N-\omega+i\epsilon)}. \quad (20)$$

The renormalization constants are

$$\delta m_V=-\frac{g^2}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)} \quad (21a)$$

and

$$Z_V=1-g^2\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m\ -m_N-\omega)^2}. \quad (21b)$$

The τ function appropriate to N-θ scattering is

$$\tau_{N\theta}{}^1(s;\omega,\omega')\equiv\frac{(4\omega\omega')^{1/2}}{u(\omega)u(\omega')}\langle 0|T[\psi_N(s)a_k(s)\psi_N{}^\dagger a_{k'}{}^\dagger]|0\rangle. \quad (22)$$

The modified one-meson Tamm-Dancoff equations determining this τ function are

$$(W-m_{0,V})\hat{\tau}_V{}^3(W;\omega)=\frac{g}{Z_V}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\times\hat{\tau}_{N\theta}{}^1(W;\omega',\omega)\,, \quad (23a)$$

$$(W-m_N-\omega)\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')=\frac{2\omega\delta_{kk'}}{u^2(\omega)}+g\hat{\tau}_V{}^3(W;\omega')\,. \quad (23b)$$

The solution to these equations, using (21a) and (21b), is

$$\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{(W-m_N-\omega)}+\frac{g^2}{(W-m_N-\omega)(W-m_N-\omega')(W-m_V)[1-\beta(W)]}\,, \quad (24)$$

which, of course, agrees with the expression found in I. Using the reduction formula of I, we find the S-matrix element for N-θ scattering, using (24), to be

$$S_{N\theta}{}^{kk'}=\delta_{kk'}+2\pi i\delta(\omega-\omega')\frac{u^2(\omega)}{2\omega}\times\frac{g^2}{(m_V-m_N-\omega)[1-\beta(\omega+m_N)]}\,, \quad (25)$$

where

$$\beta(\omega+m_N)=\frac{g^2}{4\pi^2}(m_V-m_N-\omega)\times\int_\mu^\infty\frac{d\omega' u^2(\omega')(\omega'^2-\mu^2)^{1/2}}{(m_V-m_N-\omega')^2(\omega'-\omega-i\epsilon)}\,. \quad (26)$$

Collecting results, the solutions to the V-$N\theta$ sector in the one-meson approximation are

$$\hat{\tau}_V{}^1(W)=(W-m_V+i\epsilon)^{-1}[1-\beta(W)]^{-1} \quad (27)$$

and

$$\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{(W-m_N-\omega)}+g^2(W-m_N-\omega)^{-1}(W-m_N-\omega')^{-1}(W-m_V)^{-1}\times[1-\beta(W)]^{-1}. \quad (28)$$

The results for the N-$V\bar{\theta}$ sector in the one-meson approximation are obtained from the above results by making the replacement $m_V\leftrightarrow m_N$, as discussed in Sec. I. Therefore,

$$\hat{\tau}_N{}^1(W)=(W-m_N+i\epsilon)^{-1}[1-\beta_N(W)]^{-1} \quad (29)$$

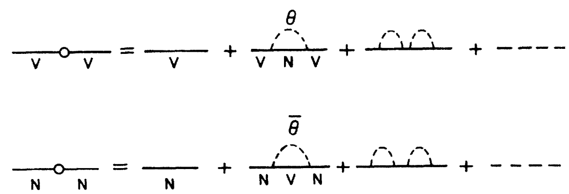

FIG. 1. V and N propagators in the one-meson approximation.

and

$$\hat{\tau}_{V\bar{\theta}}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{(W-m_V-\omega)}+g^2(W-m_V-\omega)^{-1}(W-m_V-\omega')^{-1}(W-m_N)^{-1}\times[1-\beta_N(W)]^{-1}, \quad (30)$$

where

$$\beta_N(W)=g^2(W-m_N)\times\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m_N-m_V-\omega)^2(W-m_V-\omega+i\epsilon)}=\frac{g^2}{4\pi^2}(W-m_N)\times\int_\mu^\infty\frac{d\omega u^2(\omega)(\omega^2-\mu^2)^{1/2}}{(m_N-m_V-\omega)^2(W-m_V-\omega+i\epsilon)}\,. \quad (31)$$

The perturbation expansion of $\hat{\tau}_V{}^1(W)$ and $\hat{\tau}_N{}^1(W)$ is shown in Fig. 1. By inspection, it is obvious that the N particle occurring in the N-θ intermediate state is treated as a bare particle in the one-meson approximation.

Finally, we remark that the V-θ system has no interaction in the one-meson approximation. Let us now proceed to the two-meson scheme.

III. TWO-MESON MODIFIED TAMM-DANCOFF APPROXIMATION

V-$N\theta$ Sector

For $N=2$, $\tau_V{}^6(s;\omega,\omega')$ and all higher τ functions will vanish. In this approximation, the system of equations (12) becomes

$$(W-m_{0,V})\hat{\tau}_V{}^1(W)=\frac{1}{Z_V}+\frac{g}{Z_V}\sum_k\frac{u^2(\omega)}{2\omega}\hat{\tau}_V{}^2(W;\omega)\,, \quad (32a)$$

$$(W-m_{0,N}-\omega)\hat{\tau}_V{}^2(W;\omega)=\frac{g}{Z_N}\hat{\tau}_V{}^1(W)+\frac{g}{Z_N}\sum_k\frac{u^2(\omega')}{2\omega'}\hat{\tau}_V{}^4(W;\omega,\omega')\,, \quad (32b)$$

$$(W-m_{0,V}-\omega-\omega')\hat{\tau}_V{}^4(W;\omega,\omega')=\frac{g}{Z_V}\hat{\tau}_V{}^2(W;\omega)\,. \quad (32c)$$

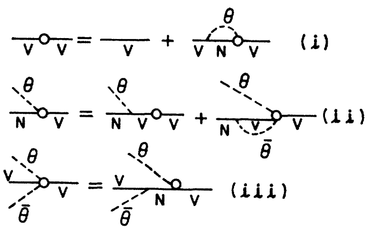

FIG. 2. Two-meson equations for the V propagator.

We represent Eqs. (32) symbolically in Fig. 2. Using these diagrams, it is simple to obtain the perturbation expansion for $\hat{\tau}_V{}^1(W)$. This is shown in Fig. 3. We see that the N particle that occurs in the intermediate state is dressed according to the one-meson approximation.

In order to write the appropriate two-meson equations for N-θ scattering, we must define

$$\tau_{N\theta}{}^2(s;\omega,\omega',\omega'')\equiv\frac{(8\omega\omega'\omega'')^{1/2}}{u(\omega)u(\omega')u(\omega'')}\times\langle 0|T[\psi_V(s)a_k(s)b_{k'}(s)\psi_N{}^\dagger a_{k''}{}^\dagger]|0\rangle. \quad (33)$$

Then the truncated equations are

$$(W-m_{0,V})\hat{\tau}_V{}^3(W;\omega)=\frac{g}{Z_V}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\hat{\tau}_N{}^1(W;\omega',\omega), \quad (34a)$$

$$(W-m_{0,N}-\omega)\hat{\tau}_N{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{Z_N}+\frac{g}{Z_N}\hat{\tau}_V{}^3(W;\omega')+\frac{g}{Z_N}\sum_{k''}\frac{u^2(\omega'')}{2\omega''}\hat{\tau}_{N\theta}{}^2(W;\omega,\omega'',\omega'), \quad (34b)$$

and

$$(W-m_{0,V}-\omega-\omega')\hat{\tau}_{N\theta}{}^2(W;\omega,\omega',\omega'')=\frac{g}{Z_V}\hat{\tau}_{N\theta}{}^1(W;\omega,\omega''), \quad (34c)$$

where $\hat{\tau}_V{}^3(W;\omega)$ and $\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')$ are defined by Eqs. (9c) and (22). The symbolic representation of Eqs. (34) is shown in Fig. 4.

If we carry out the reduction of the equations shown in Fig. 4 to the perturbation expansion of $\hat{\tau}_{N\theta}{}^2(W;\omega,\omega')$ in the same manner that we did for $\hat{\tau}_V{}^1(W)$, the results to sixth order are shown in Fig. 5. By comparing Figs. 1, 3, and 5, we see that the external N particle in Fig. 5 is dressed according to the one-meson approximation, while the intermediate V is dressed according to the two-meson approximation. We also note that, if we close off the meson lines in the diagrams of Fig. 5, we get all the diagrams of Fig. 3 except for the bare-V term. This is what we mean when we say that the same set of diagrams that dress the V particle are responsible for N-θ scattering. Therefore, we are naturally led to the following rule for modifying the truncated equations from an examination of the sets of Feynman diagrams which contribute in the N-meson scheme.

We modify the τ-function equations of the N-meson truncation scheme by dressing a nucleon according to the $N-M$ modified truncation scheme when it appears in a state with M mesons.

To demonstrate this rule, consider the two-meson equations (32) for the V propagator. The function $\hat{\tau}_V{}^4(W;\omega,\omega')$ represents the V-$V\theta\bar{\theta}$ vertex. By comparing Eq. (32c) and its symbolic representation in Fig. 2, Eq. (iii), we see that the V particle associated with the parameters $m_{0,V}$, Z_V in Eq. (32c) is accompanied by a θ-$\bar{\theta}$ pair. Therefore, in the two-meson approximation, it must be handled as a bare particle. This is analogous to the situation in the one-meson scheme of Sec. II, where the N particle was handled as a bare particle in the calculation of the V-$N\theta$ vertex, because it was always accompanied by a θ particle. Therefore, in Eq. (32c), we set

$$m_{0,V}=m_V \quad (35a)$$

and

$$Z_V=1. \quad (35b)$$

Making this replacement, we obtain

$$\hat{\tau}_V{}^4(W;\omega,\omega')=\frac{g\hat{\tau}_V{}^2(W;\omega)}{(W-m_V-\omega-\omega'+i\epsilon)}. \quad (36)$$

Substituting this result into (32b), we obtain

$$\left[W-m_{0,N}-\omega-\frac{g^2}{Z_N}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{1}{(W-m_V-\omega-\omega'+i\epsilon)}\right]\times\hat{\tau}_V{}^2(W;\omega)=\frac{g}{Z_N}\hat{\tau}_V{}^1(W). \quad (37)$$

$\hat{\tau}_V{}^2(W;\omega)$ represents the V-$N\theta$ vertex, so that the N associated with the parameters $m_{0,N}$, Z_N in Eq. (37) is accompanied by one meson. Therefore, we dress the N particle according to the one-meson approximation.

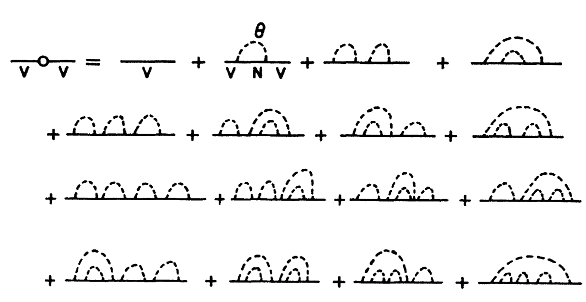

FIG. 3. V propagator to eighth order in the two-meson approximation.

From Eqs. (18a), (18b), and (29) of Sec. II, we have

$$[\hat{\tau}_N{}^1(W)]^{\text{1-meson}} = Z_N{}^{-1}\left[W - m_{0,N} - \frac{g^2}{Z_N}\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(W-m_V-\omega+i\epsilon)}\right]^{-1} = (W-m_N+i\epsilon)^{-1}[1-\beta_N(W)]^{-1}. \quad (38)$$

Substituting the values for $m_{0,N}$, Z_N from the one-meson approximation [Eqs. (21a) and (21b), with $m_V \leftrightarrow m_N$, $Z_V \leftrightarrow Z_N$] into Eq. (37) yields

$$\hat{\tau}_V{}^2(W;\omega) = \frac{g\hat{\tau}_V{}^1(W)}{(W-\omega-m_N+i\epsilon)[1-\beta_N(W-\omega)]}. \quad (39)$$

Substituting the result (39) into (32a), we obtain

$$\hat{\tau}_V{}^1(W) = Z_V{}^{-1}\left[W - m_{0,V} - \frac{g^2}{Z_V}\sum_k \frac{u^2(\omega)}{2\omega} \times \frac{1}{(W-m_N-\omega+i\epsilon)[1-\beta_N(W-\omega)]}\right]^{-1} \quad (40)$$

as the solution for the V propagator in the two-meson approximation. Comparing this result with the one-meson result from Eq. (18)

$$[\hat{\tau}_V{}^1(W)]^{\text{1-meson}} = Z_V{}^{-1}\left[W - m_{0,V} - \frac{g^2}{Z_V}\sum_k \frac{u^2(\omega)}{2\omega} \times \frac{1}{(W-m_N-\omega+i\epsilon)}\right]^{-1}, \quad (41)$$

we see that the only difference is that the N particle in the intermediate N-θ state is dressed according to the one-meson approximation in Eq. (40), whereas it is handled as a bare particle in Eq. (41). This can also be seen by comparison of Figs. 1 and 3.

From inspection of Eq. (40), we see that $\hat{\tau}_V{}^1(W)$ will have branch cuts for

$$m_N+\mu \leq W \leq \infty \quad (42a)$$

and

$$m_V+2\mu \leq W \leq \infty\,. \quad (42b)$$

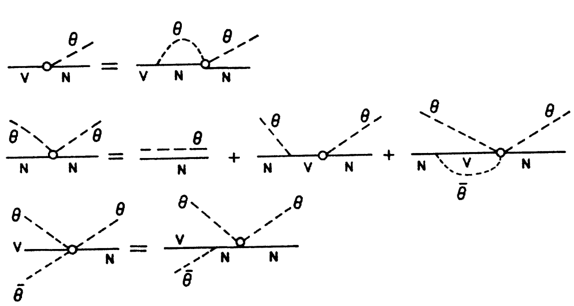

FIG. 4. Two-meson equations for the N-θ four-point function.

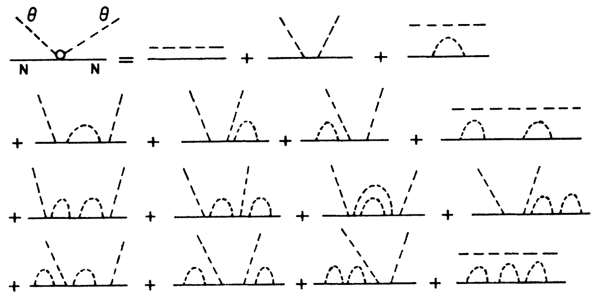

FIG. 5. N-θ four-point function to sixth order in the two-meson approximation.

These two cuts arise from the contribution to the propagator from the N-θ and $V\theta\bar{\theta}$ intermediate states, respectively.

We now determine the renormalization constants by insisting that $\hat{\tau}_V{}^1(W)$ have a simple pole at $W=m_V$ with residue $+1$. After a bit of algebra, we obtain

$$(\delta m_V)^{\text{2-meson}} = -\frac{g^2}{Z_V}\sum_k \frac{u^2(\omega)}{2\omega} \times \frac{1}{(m_V-m_N-\omega)[1-\beta_N(m_V-\omega)]} \quad (43a)$$

and

$$(Z_V)^{\text{2-meson}} = 1 - g^2\sum_k \frac{u^2(\omega)}{2\omega} \times \left[\frac{1-\beta_N(m_V-\omega)}{(m_V-m_N-\omega)} + \frac{d\beta_N(m_V-\omega)}{d\omega}\right] \Big/ (m_V-m_N-\omega)[1-\beta_N(m_V-\omega)]^2, \quad (43b)$$

where, according to Eq. (31),

$$\beta_N(m_V-\omega) = g^2(\omega+m_N-m_V) \times \sum_{k'} \frac{u^2(\omega')}{2\omega'}\frac{1}{(m_N-m_V-\omega')^2(\omega+\omega')}. \quad (44)$$

We denote the renormalization constants in Eqs. (43a) and (43b) as $(\delta m_V)^{\text{2-meson}}$ and $(Z_V)^{\text{2-meson}}$ in order to distinguish them from the same quantities determined in the one-meson approximation [Eqs. (21a) and (21b)]. To ensure that

$$0 \leq (Z_V)^{\text{2-meson}} \leq 1\,, \quad (45a)$$

we define $(g_c{}^2)^{\text{2-meson}}$ from the equation

$$[Z_V(g_c{}^2)^{\text{2-meson}}] = 0\,, \quad (45b)$$

and restrict the coupling constant to lie in the range

$$0 \leq g^2 \leq (g_c{}^2)^{\text{2-meson}} \quad (46)$$

in the two-meson scheme. We obtain

$$(g_c^{-2})^{2\text{-meson}}=\sum_k \frac{u^2(\omega)}{2\omega} \times\left[\frac{1-\beta_N^{\max}(m_V-\omega)}{(m_V-m_N-\omega)}+\frac{d\beta_N^{\max}(m_V-\omega)}{d\omega}\right]\Big/ (m_V-m_N-\omega)[1-\beta_N^{\max}(m_V-\omega)]^2. \quad (47)$$

The function $\beta_N^{\max}$ in Eq. (47) is defined as in Eq. (44), with $g^2=(g_c^2)^{2\text{-meson}}$. It is impossible to solve Eq. (47) analytically for $(g_c^2)^{2\text{-meson}}$ in order to compare directly with $(g_c^2)^{1\text{-meson}}$. Instead, we will now show that it is possible to choose the parameters $u(\omega)$, m_V, and m_N in such a manner that

$$(g_c^2)^{2\text{-meson}}<(g_c^2)^{1\text{-meson}}, \quad (48a)$$

or, from (45b),

$$(Z_V)^{2\text{-meson}}<(Z_V)^{1\text{-meson}}. \quad (48b)$$

From Eqs. (21a) and (21b), we choose $(g_c^2)^{1\text{-meson}}$ to be the smaller of

$$(g_c^{-2})_V{}^{1\text{-meson}}=\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)^2} \quad (49a)$$

and

$$(g_c^{-2})_N{}^{1\text{-meson}}=\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(m_N-m_V-\omega)^2}. \quad (49b)$$

If we choose the parameters m_V and m_N so that

$$m_V>m_N, \quad (50a)$$

then

$$(g_c^2)_V{}^{1\text{-meson}}<(g_c^2)_N{}^{1\text{-meson}}. \quad (50b)$$

With the choice (50a), we must take

$$(g_c^{-2})^{1\text{-meson}}=(g_c^{-2})_V{}^{1\text{-meson}} =\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)^2}. \quad (51)$$

From Eq. (50b), and the relation

$$(Z_{V,N})^{1\text{-meson}}=1-g^2/[(g_c^2)_{V,N}]^{1\text{-meson}}, \quad (52a)$$

we have

$$(Z_V)^{1\text{-meson}}<(Z_N)^{1\text{-meson}} \quad (52b)$$

for a given value of g lying in the range

$$0\leq g^2\leq (g_c^2)^{1\text{-meson}}. \quad (52c)$$

If Eq. (48) is satisfied, we will have

$$[Z_N((g_c^2)^{2\text{-meson}})]^{1\text{-meson}} >[Z_V((g_c^2)^{2\text{-meson}})]^{1\text{-meson}} >[Z_V((g_c^2)^{2\text{-meson}})]^{2\text{-meson}}=0. \quad (53)$$

Using Eqs. (47) and (51), we obtain

$$(g_c^{-2})^{2\text{-meson}}-(g_c^{-2})^{1\text{-meson}}=-\sum_k \frac{u^2(\omega)}{2\omega}\frac{d}{d\omega}f(\omega), \quad (54a)$$

where

$$f(\omega)=\frac{\beta_N^{\max}(m_V-\omega)}{[1-\beta_N^{\max}(m_V-\omega)](\omega+m_N-m_V)}. \quad (54b)$$

Using (44), with $g^2=(g_c^2)^{2\text{-meson}}$, and (54b), we obtain

$$[1-\beta_N^{\max}(m_V-\omega)]^2\frac{d}{d\omega}f(\omega) =\left[(g_c^2)^{2\text{-meson}}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{1}{(\omega'+m_V-m_N)^2(\omega+\omega')}\right]^2 -(g_c^2)^{2\text{-meson}}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{1}{(\omega'+m_V-m_N)^2(\omega+\omega')^2}. \quad (55)$$

The function $[1-\beta_N^{\max}(m_V-\omega)]^{-2}$ rises from the value 1, at $\omega=m_V-m_N<\mu$, to the value

$$[Z_N^{-2}((g_c^2)^{2\text{-meson}})]^{1\text{-meson}}$$

as $\omega\rightarrow\infty$. Now,

$$[Z_N((g_c^2)^{2\text{-meson}})]^{1\text{-meson}}$$

is certainly small, so that

$$[Z_N^{-2}((g_c^2)^{2\text{-meson}})]^{1\text{-meson}}$$

will be quite large. We choose the cutoff function $u(\omega)$ so that the major contribution to the integrals in Eqs. (54a) and (55) comes from the region where $\omega\gg\omega'$. This can be done since the integrands of (55) fall off more rapidly in ω' than the integrand of (54) falls off in ω. Thus

$$\omega^2[1-\beta_N^{\max}(m_V-\omega)]^2\frac{d}{d\omega}f(\omega) \simeq[1-Z_N((g_c^2)^{2\text{-meson}})]^2-[1-Z_N((g_c^2)^{2\text{-meson}})]. \quad (56)$$

From (53),

$$1-Z_N((g_c^2)^{2\text{-meson}})<1, \quad (57)$$

so that

$$\omega^2[1-\beta_N^{\max}(m_V-\omega)]^2\frac{d}{d\omega}f(\omega)<0. \quad (58)$$

Using (58) and (54a), we obtain (48a).

We have shown that it is possible to choose the parameters occurring in charged-scalar theory in such a manner that $(g_c^2)^{2\text{-meson}}<(g_c^2)^{1\text{-meson}}$ or, for a given value of g, it is less probable to find the physical V particle as a bare V, when $V\theta\bar{\theta}$ states as well as $N\theta$ are included in the calculation. This is certainly a reasonable situation and suggests an interesting possibility:

As we include more and more mesons in the calculation, it is less and less probable to find the physical particle as a bare particle, for a given value of the coupling constant. This would certainly be in the spirit of constructing a theory in which none of the strongly interacting particles would be more elementary than another.[10]

Substituting (43a) and (43b) into (40), we obtain the V propagator in the two-meson approximation

$$\hat{\tau}_V{}^1(W)=(W-m_V+i\epsilon)^{-1}[1-\gamma(W)]^{-1}, \quad (59)$$

where

$$\gamma(W)\equiv\gamma_1(W)+\gamma_2(W)-\gamma_3(W) \quad (60a)$$

and

$$\gamma_1(W)\equiv(W-m_V)g^2\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)^2(W-m_N-\omega+i\epsilon)[1-\beta_N(m_V-\omega)]}, \quad (60b)$$

$$\gamma_2(W)\equiv(W-m_V)g^2\sum_k\frac{u^2(\omega)}{2\omega}\left[g^2\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{(W-m_N-2\omega-\omega')}{(m_N-m_V-\omega')(\omega+\omega')^2(W-m_V-\omega-\omega')}\right]\Big/ (m_V-m_N-\omega)(W-m_N-\omega)[1-\beta_N(m_V-\omega)], \quad (60c)$$

$$\gamma_3(W)\equiv(W-m_V)g^2\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(W-m_N-\omega)[1-\beta_N(m_V-\omega)][1-\beta_N(W-\omega)]}\times\left\{g^2\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{g^2\sum_{k''}[u^2(\omega'')/2\omega''][(m_N-m_V-\omega'')(\omega+\omega'')(W-m_V-\omega-\omega'')]^{-1}}{(m_N-m_V-\omega')(\omega+\omega')(W-m_V-\omega-\omega')}\right\}. \quad (60d)$$

Note that

$$\gamma_1(W)\xrightarrow[W\to\infty]{}g^2\sum_k\frac{u^2(\omega)}{2\omega}\times\frac{1}{(m_V-m_N-\omega)^2[1-\beta_N(m_V-\omega)]}, \quad (61a)$$

$$\gamma_2(W)\xrightarrow[W\to\infty]{}g^2\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)[1-\beta_N(m_V-\omega)]}\times\left[g^2\sum_{k'}\frac{u^2(\omega')}{2\omega'}\frac{1}{(m_N-m_V-\omega')(\omega+\omega')^2}\right], \quad (61b)$$

and

$$\gamma_3(W)\xrightarrow[W\to\infty]{}0. \quad (61c)$$

Comparing with (43b), we see that

$$1-\gamma(W)\xrightarrow[W\to\infty]{}(Z_V)^{2\text{-meson}}. \quad (62)$$

Also, it can be seen from (60) that $\gamma_1(W)$ has a cut for $m_N+\mu\leq W\leq\infty$, so that it contains part of the contribution due to N-θ states. $\gamma_2(W)$ and $\gamma_3(W)$ have cuts for $m_N+\mu\leq W\leq\infty$ and $m_V+2\mu\leq W\leq\infty$, thereby containing contributions due to both N-θ and $V\theta\bar{\theta}$ states, Furthermore, the function $\gamma(W)$ behaves in the same manner as the function $\beta(W)$, for $W<m_N+\mu$, where it is real. Therefore, there can be no bound state in the V-$N\theta$ sector in the two-meson approximation. In fact, by comparing Eqs. (59) and (27), it is seen that the V propagators in the one- and two-meson schemes are very much alike. This also follows by comparing Figs. 1 and 3, where it is seen that the Feynman diagrams in both schemes have no crossed lines. It turns out that crossed diagrams appear in this channel for the first time in the three-meson scheme.

Let us now turn to the scattering and production processes $N+\theta\leftrightarrow N+\theta$ and $N+\theta\to V+\theta+\bar{\theta}$. In addition to the V-$N\theta$ vertex $\tau_V{}^3(s;\omega)$ and the N-θ four-point function $\tau_{N\theta}{}^1(s;\omega,\omega')$ defined in (9c) and (22), we have

$$\tau_{N\theta}{}^2(s;\omega,\omega',\omega'')\equiv\frac{(8\omega\omega'\omega'')^{1/2}}{u(\omega)u(\omega')u(\omega'')}\times\langle 0|T[\psi_V(s)a_k(s)b_{k'}(s)\psi_N{}^\dagger a_{k''}{}^\dagger]|0\rangle. \quad (63)$$

The two-meson modified equations for the Fourier transforms of these functions are

$$(W-m_{0,V})\hat{\tau}_V{}^3(W;\omega)=\frac{g}{Z_V}\sum_{k'}\frac{u^2(\omega')}{2\omega'}\hat{\tau}_{N\theta}{}^1(W;\omega',\omega), \quad (64a)$$

$$(W-m_{0,N}-\omega)\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{Z_N}+\frac{g}{Z_N}\hat{\tau}_V{}^3(W;\omega')+\frac{g}{Z_N}\sum_{k''}\frac{u^2(\omega'')}{2\omega''}\hat{\tau}_{N\theta}{}^2(W;\omega,\omega'',\omega'), \quad (64b)$$

and

$$(W-m_V-\omega-\omega')\hat{\tau}_{N\theta}{}^2(W;\omega,\omega',\omega'')=g\hat{\tau}_{N\theta}{}^1(W;\omega,\omega''), \quad (64c)$$

[10] W. Heisenberg, in *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958); G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), Sec. 1.

where $m_{0,N}$ and Z_N in Eq. (64b) are calculated in the one-meson approximation. We solve these equations by substituting (64c) into (64b) and then using (38) to eliminate $m_{0,N}$ and Z_N. We then substitute the result into (64a) and use (40) and (59) to eliminate $m_{0,V}$ and Z_V. In this way we obtain

$$\hat{\tau}_V{}^3(W;\omega)=g(W-m_V)^{-1}(W-m_N-\omega)^{-1}[1-\beta_N(W-\omega)]^{-1}[1-\gamma(W)]^{-1}, \tag{65a}$$

$$\hat{\tau}_{N\theta}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{(W-m_N-\omega)[1-\beta_N(W-\omega)]}+\frac{g^2}{(W-m_N-\omega)[1-\beta_N(W-\omega)](W-m_N-\omega')[1-\beta_N(W-\omega')](W-m_V)[1-\gamma(W)]}, \tag{65b}$$

and

$$\hat{\tau}_{N\theta}{}^2(W;\omega,\omega',\omega'')=g\hat{\tau}_{N\theta}{}^1(W;\omega,\omega'')/(W-m_V-\omega-\omega'+i\epsilon) \tag{65c}$$

as the solutions to the system (64). Note the similarity between the expression for the N-θ four-point function in the two-meson approximation (65b) and in the one-meson approximation (24). The reason for this similarity is seen by inspecting Fig. 5. The only difference between the Feynman expansion for $\tau_{N\theta}{}^1(s;\omega,\omega')$ in the one- and two-meson approximations is that the N particle is dressed by the iterated bubble diagram whenever it appears in the two-meson case, while it is bare in the one-meson case. There are no crossed diagrams, so that the same kind of dynamics operates in the V-$N\theta$ sector in the one- and two-meson schemes. This situation will change in the three-meson approximation where crossed diagrams appear for the first time in this sector.

Using the reduction formula from I, and the expressions (65b) and (65c), we can write down the S-matrix elements for N-θ scattering and $N\theta\to V\theta\bar{\theta}$ production. They are

$$S_{N\theta}{}^{kk'}=\delta_{kk'}+2\pi i\delta(\omega-\omega')\frac{u^2(\omega)}{2\omega}\frac{g^2}{(m_V-m_N-\omega)[1-\gamma(m_N+\omega)]} \tag{66}$$

and

$$S_{N\theta\to V\theta\bar{\theta}}{}^{k'';k,k'}=2\pi i\delta(m_N+\omega''-m_V-\omega-\omega')\frac{u(\omega)u(\omega')u(\omega'')}{(8\omega\omega'\omega'')^{1/2}}\times\frac{g^3}{(\omega'+m_V-m_N)[1-\beta_N(m_V+\omega')](m_V-m_N-\omega'')[1-\gamma(m_N+\omega'')]}. \tag{67}$$

It can be seen that the production amplitude (67) is essentially the product of N-$\theta_{\omega''}$ elastic scattering in the two-meson approximation (66) and $V\bar{\theta}_{\omega'}$ scattering in the one-meson approximation [Eq. (25) with $m_V\leftrightarrow m_N$, $\beta\leftrightarrow\beta_N$].

Now let us turn to the V-θ sector in the two-meson approximation.

V-θ Sector

Let us define the τ function appropriate to V-θ elastic scattering and $V\theta\to N2\theta$ production as

$$\tau_{V\theta}{}^1(s;\omega,\omega')\equiv\frac{(4\omega\omega')^{1/2}}{u(\omega)u(\omega')}\langle 0|T[\psi_V(s)a_k(s)\psi_V{}^\dagger a_{k'}{}^\dagger]|0\rangle \tag{68a}$$

and

$$\tau_{V\theta}{}^2(s;\omega,\omega',\omega'')\equiv\frac{(8\omega\omega'\omega'')^{1/2}}{u(\omega)u(\omega')u(\omega'')}\langle 0|T[\psi_N(s)a_k(s)a_{k'}(s)\psi_V{}^\dagger a_{k''}{}^\dagger]|0\rangle. \tag{68b}$$

The two-meson modified equations for the Fourier-transformed τ functions are

$$(W-m_{0,V}-\omega)\hat{\tau}_{V\theta}{}^1(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{Z_V}+\frac{g}{Z_V}\sum_{k''}\frac{u^2(\omega'')}{2\omega''}\hat{\tau}_{V\theta}{}^2(W;\omega'',\omega,\omega') \tag{69a}$$

and

$$(W-m_N-\omega-\omega')\hat{\tau}_{V\theta}{}^2(W;\omega,\omega',\omega'')=g[\hat{\tau}_{V\theta}{}^1(W;\omega',\omega'')+\hat{\tau}_{V\theta}{}^1(W;\omega,\omega'')], \tag{69b}$$

where $m_{0,V}$ and Z_V are given by the one-meson expressions (21a) and (21b). But these are precisely the equations for the V-θ sector of the conventional Lee model.[11] From II, the solution to these equations is

$$\begin{aligned}\hat{\tau}_{V\theta}{}^1(W+m_N;\omega,\omega') &= \frac{2\omega}{u^2(\omega)}\frac{\delta_{kk'}}{H^-(\omega)} + \frac{g^2}{(W-\omega-m_V+m_N)(W-\omega'-m_V+m_N)}\\ &\times\Bigg\{\frac{1}{(W-\omega-\omega'+i\epsilon)}\left[(m_V-m_N-\omega')I_W{}^+(\omega') - \frac{1}{[1-\beta(\omega'+m_N)][1-\beta(W-\omega'+m_N)]}\right]\\ &+\frac{(m_V-m_N-W+\omega')^2 I_W{}^+(W-\omega')}{(m_V-m_N-\omega')(\omega-\omega'-i\epsilon)} - \frac{(W-\omega-m_V+m_N)^2(W-2\omega')I_W{}^+(W-\omega)}{(m_V-m_N-\omega')(W-\omega-\omega'+i\epsilon)(\omega-\omega'-i\epsilon)}\\ &-\frac{G^+(W-m_V+m_N)[(W-\omega-m_V+m_N)I_W{}^+(W-\omega)+(2m_V-2m_N-W)I_W{}^+(W-m_V+m_N)]}{(m_V-m_N-\omega')(m_V-m_N-\omega)[1+G^+(W-m_V+m_N)I_W{}^+(W-m_V+m_N)]}\\ &\times\left[(m_V-m_N-\omega')I_W{}^+(\omega') - (m_V-m_N-W+\omega')I_W{}^+(W-\omega') + \frac{1}{[1-\beta(\omega'+m_N)][1-\beta(W-\omega'+m_N)]}\right]\Bigg\}, \qquad (70)\end{aligned}$$

where

$$G(z)\equiv(z-m_V+m_N)[1-\beta(z+m_N)], \qquad (71a)$$

$$H(z)\equiv G(W-z), \qquad (71b)$$

$$G^{\pm}(\omega)\equiv \lim_{\epsilon\to0^+} G(\omega\pm i\epsilon), \qquad (71c)$$

$$H^{\pm}(\omega)\equiv \lim_{\epsilon\to0^+} H(\omega\pm i\epsilon) = \lim_{\epsilon\to0^+} G(W-\omega\mp i\epsilon), \qquad (71d)$$

$$I_W(z)\equiv\frac{1}{\pi}\int_\mu^\infty d\omega'\,\mathrm{Im}\left[\frac{1}{G^+(\omega')}\right] \times\frac{1}{[1-\beta(W-\omega'+m_N)](\omega'-z)}. \qquad (72)$$

Using the diagrammatic representation of the Eqs. (69) and "calculating" the Feynman expansion for $\hat{\tau}_{V\theta}{}^1(W;\omega,\omega')$ up to sixth order, we obtain the perturbation expansion shown in Fig. 6. It can be seen from Fig. 6 that the interaction takes place entirely through crossed diagrams or exchange diagrams. Since exchanges of particles give rise to forces, we may expect that a V-θ bound state may exist, for large enough coupling constant. Indeed, this is precisely the case, and the V-θ bound state of the conventional Lee model is a well-known phenomenon.[12] It arises from the vanishing of the denominator $[1+G^+(W-m_V+m_N)I_W{}^+(W-m_V+m_N)]$ in Eq. (70) for the V-θ four-point function.

From II, we can write the S-matrix elements for $V\theta$ elastic scattering and $V\theta\to N2\theta$ production in the two-meson approximation. They are

$$S_{V\theta}{}^{kk'}=\delta_{kk'} +2\pi i\delta(\omega-\omega')\frac{u^2(\omega)}{2\omega}\frac{g^2}{G^+(\omega)}\left[\frac{1+G^+(\omega)A(\omega)}{1-G^+(\omega)A(\omega)}\right] \qquad (73)$$

and

$$S_{V\theta\to N2\theta}{}^{k'';k,k'} =2\pi i\delta(m_V+\omega''-m_N-\omega-\omega')\frac{u(\omega)u(\omega')u(\omega'')}{(8\omega\omega'\omega'')^{1/2}} \times\frac{\sqrt{2}g^3}{G^+(\omega)G^+(\omega'')[1-G^+(\omega'')A(\omega'')]}, \qquad (74)$$

where

$$A(\omega)\equiv -I_\omega{}^+(\omega)=\frac{1}{\pi}\int_\mu^\infty d\omega' \times\mathrm{Im}\left[\frac{1}{G^+(\omega')}\right]\frac{1}{G^+(m_V-m_N+\omega-\omega')}. \qquad (75)$$

The S-matrix element for N-2θ elastic scattering is given in II and will not be rewritten here.

An analysis of the denominator $1-G^+(\omega)A(\omega)$, carried out in II, shows that $G^+(\omega)A(\omega)$ passes through the value 1 at some $\omega=\omega_B$,

$$m_V-m_N<\omega_B<\mu, \qquad (76)$$

for

$$g_B{}^2\le g^2<g_c{}^2, \qquad (77)$$

where inequalities for $g_B{}^2$ are given in II. Therefore,

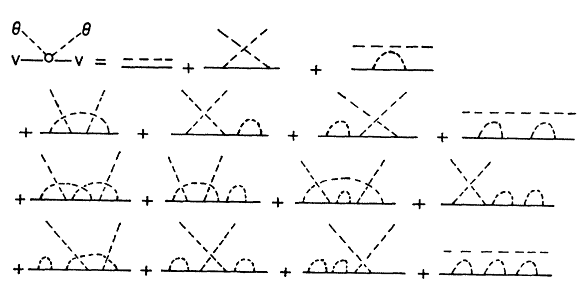

FIG. 6. V-θ four-point function to sixth order in the two-meson approximation.

[11] M. S. Maxon, Phys. Rev. **149**, 1273 (1966), hereafter denoted as II.

[12] See II and the references contained therein for discussions of the V-θ bound state.

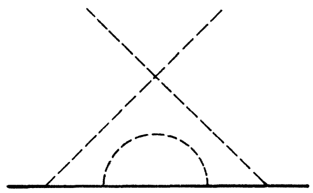

FIG. 7. Crossed diagram obtained from fourth term of Fig. 5.

if (77) is satisfied, a bound state of the V-θ system will occur with total energy $m_V+\omega_B$.

Now that we have obtained solutions in the V-$N\theta$ and V-θ sectors in the two-meson approximation, let us turn to the question of crossing symmetry.[5] The reader may have noticed that, although we have the proper poles and right-hand cuts in our τ functions and S-matrix elements, the left-hand cuts are entirely absent. This is due to the complete absence of crossing symmetry in the modified Tamm-Dancoff scheme. This can be seen most easily by comparing Figs. 5 and 6. Although there is an obvious correspondence between the perturbation-theory diagrams for N-θ (V-$\bar{\theta}$) scattering and V-θ (N-$\bar{\theta}$) scattering, it is not the simple exchange of incoming and outgoing mesons. If we perform this exchange on the second term in the expansion of Fig. 5, we obtain the second term in the expansion of Fig. 6. However, if we perform the exchange on the fourth term in Fig. 5, we obtain the diagram shown in Fig. 7, which has a three-meson intermediate state and is therefore absent in the two-meson-approximation diagrams of Fig. 6. Likewise, the fourth term of Fig. 6 goes into the diagram shown in Fig. 8, which is not included in the two-meson diagrams of Fig. 5 because it has a three-meson intermediate state.

Therefore, the absence of crossing symmetry in our scheme seems to be an inherent feature of truncating the τ-function equations by restricting the number of mesons occurring in intermediate states. We discuss the relationship of our method with meson-nucleon calculations which include crossing symmetry in the concluding section.

CONCLUSIONS

In this paper, we have presented a scheme for solving the infinite set of equations that arise for the τ functions of charged-scalar-meson theory. The basic idea of the scheme is to treat the single-nucleon states on an equal footing with meson-nucleon states. The advantage of the method is that we can investigate the spectrum of charged-scalar-meson theory, allowing for the possibility of dynamical bound states with the same quantum numbers as the nucleon. The disadvantage is the absence of crossing symmetry inherent in the finite-order truncation schemes. However, we do not feel that this poses a problem since we are primarily interested in the spectrum of the exact solution rather than an accurate description of scalar-meson–nucleon scattering. Let us explain this statement.

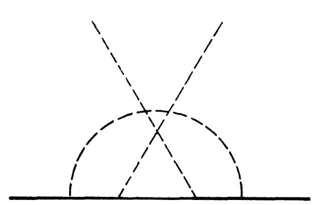

FIG. 8. Crossed diagram obtained from fourth term of Fig. 6.

Current methods for calculating meson-nucleon scattering, which take into account crossing symmetry, are based on the classic work of Chew and Low.[1] This work assumes that the spectrum of H (pseudoscalar mesons) includes a single-nucleon state followed by the meson-nucleon continuum. The resulting equation, describing P-wave pion-nucleon scattering, is solved in the one-meson approximation, and reproduces the essential feature of low-energy scattering, the (3,3) resonance, as well as determining the renormalized coupling constant from the experimental data ($f^2=0.08$). Since only one nucleon is observed experimentally, and a meson-nucleon scattering experiment is carried out with real nucleons (completely dressed), th e Chew-Low approach is certainly a reasonable one.

We have taken the attitude, in this paper, that the field theory describing the strong interaction between mesons and nucleons may have a spectrum that differs from the one usually assumed, namely, a single-nucleon state appearing as a simple pole, followed by the meson-nucleon continuum. In order to understand the role of field theory in describing the strong interactions, it is essential to establish the nature of the spectrum that arises from the exact solution to the theory (if one exists), and whether this spectrum has anything to do with the experimentally observed one. Therefore, we give up crossing symmetry in order to be able to handle the single-particle states on an equal footing with the single-particle–multimeson states.

We have obtained results in the one- and two-meson truncation schemes. It is found that the one-meson solution to the V-$N\theta$ sector and the two-meson solution to the V-θ sector are just the results obtained from the conventional Lee model.[7,11] These results are interpreted in the present context as demonstrating that a bound state ($V\theta$) occurs, for large enough coupling constant, in the channel in which the meson-nucleon interaction proceeds entirely through exchange diagrams.

The two-meson solution in the V-$N\theta$ sector is the simplest problem in our scheme in which the conventional Lee-model dynamics are inappropriate. The solution does not give rise to the possibility of a bound state ($N\theta$) as in the one-meson case. The reason for this is the complete absence of crossed diagrams as in the one-meson case. However, we find an interesting result in the course of solving for the renormalization constants. Namely, we can choose the parameters of the theory so that $[Z]^{2\text{-meson}} < [Z]^{1\text{-meson}}$ for a given value of the coupling constant. If this situation persists when we go to the higher-order truncation schemes, an interesting, and indeed reasonable, result arises. As we include more and more multimeson states, the connection between the physical particles and their fields becomes steadily weaker. This situation is certainly compatible with the point of view that, among the strongly interacting particles, no one particle is more elementary than another.[10]

Work on the three-meson solution is now being carried out. Results obtained so far explicitly show that the V-θ bound state, found in the two-meson approximation, makes its presence felt in the V-$N\theta$ sector in the three-meson scheme. Since there are crossed diagrams in this sector for the first time, the possibility of a bound state ($N\theta$) arises. Whether or not "enough" of the interaction proceeds through exchange diagrams to produce this bound state will be settled by the complete solution to the three-meson equations.

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Neutron Beta Decay in a Strong Magnetic Field

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The β decay of a neutron in a strong quantizing magnetic field is studied by making use of the exact Dirac wave function for the electron in a magnetic field of arbitrary strength. The general conclusion is that the lifetime is unaffected by the magnetic field for $H<10^{12}$ G. For $H>10^{12}$ G, the lifetime in a vacuum is appreciably decreased. This conclusion is still valid when the decay takes place in a degenerate electron gas at densities lower than 10^6 g/cm^3. At higher densities, the lifetime in a magnetic field tends to reach the value in the absence of the field, for any field strength.

1. INTRODUCTION

The properties of an electron gas in an external magnetic field, constant in time and homogeneous in space, have been extensively studied in a number of recent papers,[1] and results indicate that intense magnetic fields will be produced in the gravitational collapse of a star and in collapsed objects. Briefly, in a magnetic field the energy states of the electron are quantized and the properties of an electron gas are modified accordingly.

In this paper, we are interested in the modification of a β-decay process in an intense magnetic field. First, we compute the decay probability of a free neutron in a magnetic field; our results can be readily generalized for all elements whose lifetime and the energy of the electron from decay are known. This problem is of great interest, since it is now believed that in a gravitational collapse, fields as high as 10^{13} G can be produced. The nucleosynthesis process, which involves neutron captures, can be strongly affected by the presence of an intense magnetic field. In addition, strong magnetic fields could also have existed during the early phase of cosmological evolution. The modification of the neutron decay lifetime could also have caused a change in the helium production rate during the early evolution phase of our universe.

The most important effect of a magnetic field on the decay of a free electron is in the modification of the final state, the phase space. Strictly speaking, all states of the neutron and proton are affected by a magnetic field. However, the effect is smaller by a factor $(m/M_p)^2$, and for relatively small fields (compared to 10^{19} G), the modification of the proton and neutron states by magnetic fields can be entirely neglected.

The neutron mean life τ is calculated in two cases—namely, in a vacuum and in a highly degenerate magnetized electron gas—and the result is compared with the corresponding values of τ_0 in the absence of the magnetic field. The general result is that the neutron mean life in a magnetic field is decreased with respect to the "free" case, if the neutron is in a degenerate electron gas at a density lower than 10^6 g/cm^3.

Figure 1 shows the modified β spectrum for $\Theta=1$, and 0.1, where Θ is a measure of the magnetic field strength, $\Theta=H/H_q$, $H_q=m^2c^3/e\hbar=4.414\times10^{13}$ G.

Figures 2–4 show the neutron lifetime in a vacuum as a function of Θ, and in a degenerate gas as a function of the matter density. In both cases, the general behavior shows that the lifetime is decreased with respect to the free cases.

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[1] V. Canuto and H. Y. Chiu, Phys. Rev. **173**, 1210 (1968); **173**, 1220 (1968); **173**, 1229 (1968).