This falloff is quite rapid and no such effect has been seen as yet in the data. This indicates trouble with the model unless something drastic happens to the asymptotic behavior of $G(p_1^2,t)$ as t is varied from the physical mass of the exchanged particle to $t=-1/2\omega(2\omega-1)$. When $\omega\gtrsim 1$, this extrapolation is not large and we cannot theoretically justify such a change in behavior.

It may be that the data are not yet in the asymptotic region. In present experiments, $-p_1^2$ is not large compared to the above mentioned extrapolation of t. Assuming that this is the case, and that the model is still applicable, we can make a simple prediction on final-momentum distributions.

If we define the final-particle ordering for multiparticle-production events by decreasing lab momentum (decreasing α_i), we should find most events with $s_1=(q_1+q_2)^2\gtrsim -p_1^2$, whereas further $s_i=(q_i+q_{i+1})^2$ will tend to lower values. Furthermore, as we increase the lab energy ν with fixed $\omega\neq 0$, the average multiplicity should decrease to two. These are effects that may begin to show up at nonasymptotic energies and should be looked for.

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Overlapping Resonances in Three-Meson States*†

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We consider a system of three pseudoscalar mesons in which one of the mesons can form a two-particle resonance with either of the other two. The models we discuss include the Lee model, approximations to the Lee model, and a fully relativistic isobar model. As an example for the nonstatic model we discuss the 3π state containing two overlapping, identical ρ resonances. There have been claims that resonance projections can lead to enhancements in the three-particle mass when there are overlapping two-particle resonances. We show that these enhancements are caused by approximations which are not actually resonance projections, and we show for our models that properly made resonance projections do not lead to enhancements. We discuss briefly some alternatives to the isobar model for treating overlapping resonances.

I. INTRODUCTION

In this paper we consider an example of overlapping resonances, namely, a state containing three pseudoscalar mesons in which one of the mesons can form a two-particle resonance with either of the other two. Experimentally, such a state usually occurs as a subsystem for a final state in a meson-nucleon production reaction, and experimental data are becoming available with good enough statistics to allow a detailed study of such subsystems. Two cases of particular interest are the 3π system with two identical π's either of which can form a ρ with the third π, and the charged $K\pi\pi$ system which contains the appropriate quantum numbers for one ρ and (at least) one K^*. Much of the interest in these particular systems comes from the fact that there are experimentally observed enhancements in the three-particle mass spectrum for both these cases near the overlap threshold, called the $A_1(1080)$ and the $K^*(1300)$, respectively. One of the motivations of this work was to investigate the possibility that such enhancements could be caused merely by the resonance overlap.

In Sec. II A we discuss overlapping resonances in the Lee model and in other static models. The Lee model is of particular interest because it presents the overlapping resonance situation within the context of a completely soluble field theory, and the static kinematics is useful for gaining understanding of some of the mechanisms involved in overlapping resonances. In Sec. II B we develop a nonstatic isobar model using the helicity-state formalism introduced by Wick.[1] The model developed here, although equivalent to other approaches,[2–4] is particularly simple to work with when calculating total or differential cross sections. As an example we discuss the 3π case referred to above.

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[1] G. C. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962).

[2] J. Werle, *Relativistic Theory of Interactions* (John Wiley & Sons, Inc., New York, 1966).

[3] T. W. Ludlam, Ph.D. dissertation, Yale University, 1969 (unpublished).

[4] C. Y. Chien *et al.*, Phys. Letters **28B**, 143 (1968).

It has been shown[5,6] for a large class of models (including most of those developed in Sec. II) that overlapping resonances will not lead to an enhancement in the three-particle mass. However, there have been some claims[7–9] that enhancements occur in these models if one makes a projection which involves treating the resonances as though they were stable particles, or if one only includes the resonance bands of the Dalitz plot when calculating the three-particle mass spectrum. In Sec. III we define what is meant by a resonance projection, and we discuss the significance of resonance cuts when there are overlapping resonances. We show that the enhancements found in Refs. 7–9 result either from an incorrect application or from an incorrect interpretation of these projection techniques.

Our approximations to the Lee model, the isobar model, and the resonance approximations are all based on a form of the amplitude which is linear in the two resonances. In Sec. IV we discuss briefly some alternatives to this linear form.

II. MODELS

A. Static Models

1. Background

A static model is one in which some of the particles obey relativistic kinematics while others are treated in their static limit. In terms of a particle's three-momentum $\mathbf{p}$ and its mass m, the particle's energy p^0 is given by

$$p^0=(\mathbf{p}\cdot\mathbf{p}+m^2)^{1/2}=m+\mathbf{p}\cdot\mathbf{p}/2m+\cdots. \quad (1)$$

The static limit is obtained by keeping only the first term on the far-right-hand side of Eq. (1) so that the energy is independent of the three-momentum and is constant. The applicability of the static limit in physical processes in particle physics is limited to a few special cases such as low-energy pion-nucleon scattering where taking the nucleon as static is equivalent to ignoring the nucleon recoil.

We now indicate some simplifications which occur when one works with static kinematics. For a three-particle state with all particles treated relativistically we obtain the following well-known relation among the two-particle invariant masses $s_{ij}^{1/2}$, the individual particle masses m_j, and the three-particle mass M:

$$s_{12}+s_{32}+s_{31}=M^2+m_1^2+m_2^2+m_3^2. \quad (2)$$

It follows from Eq. (2) that two of the three invariant masses are independent. If the three-particle state contains the possibility of two-particle resonances either between No. 1 and No. 2 or between No. 3 and No. 2, the transition amplitude to that state will have a nontrivial dependence on both s_{12} and s_{32}, so that in calculating a cross section, the nontrivial part of the integration will be at least two-dimensional. Also, the range in M for which s_{12} and s_{32} can simultaneously be at their resonance values (the overlap region in M) may be quite large; for example, for a 3π state with two ρ resonances, the overlap region is $1.1\leq M\leq 3.9$ BeV. Thus if one is looking for structure due to the overlap, effects may be spread out over a 2-BeV range in M.

Now we take as static both No. 2 and the two-particle resonances. Equation (2) is replaced by

$$P^0=p_1^0+m_2+p_3^0, \quad (3)$$

from which it follows that if P^0, the total energy, is given, only one energy, p_1^0 or p_3^0, is independent. Resonances are now characterized by particular values of p_1^0 or p_3^0. Thus there will be only a single nontrivial integration in the sum over final states. Also, the overlap region in P^0 reduces to a single point (neglecting the resonance widths). It would therefore appear that any structure due to overlapping resonances would be more likely to occur in static models than in nonstatic models.

Another simplification which occurs is that the Dalitz plot which is an area in the nonstatic case reduces to a line in the static case.

2. Lee Model and Other Static Models

The basic Lee model contains three fields, V, N, and θ, where V and N represent static particles, and θ represents a particle with relativistic kinematics. By restricting the dynamics to $V\rightleftarrows N\theta$, the Lee model is an exactly soluble field theory. Chen-Cheung and Sommerfield[7] have extended the basic Lee model to include two V fields. For appropriate choices of the Lee model parameters, one can have both a stable V particle (called the V) and an unstable V particle (called the V^*). For such a choice, the renormalized momentum-space propagator $\hat{G}_{\bar{V}}(p)$ for a definite linear combination of the two V fields (called $\bar{V}$) has a pole in p^0 at $p^0=m$, with m below the $N\theta$ threshold μ,[10] and a pole on the second sheet in p^0 at $p^0=\beta\simeq m^*-\frac{1}{2}i\Gamma$, where $\Gamma>0$ and $m^*>\mu$. The parametrization for the Lee model used in Ref. 7 leads to $m\ll\mu$, $m^*\gg\mu$, and $\Gamma/m^*=0.2$. Thus, $N\theta$ elastic scattering will be dominated by the V^*.

In the $(V\theta)$ sector, there are the following two well-defined processes:

(b) $\theta+V\rightarrow\theta+V$,
(c) $\theta+V\rightarrow\theta+N+\theta$.

The production process (c) has a three-particle final state with two overlapping two-particle resonances [see

[5] R. D. Amado, Phys. Rev. **158**, 1414 (1967).
[6] C. Schmid, Phys. Rev. **154**, 1363 (1967).
[7] F. S. Chen-Cheung and C. M. Sommerfield, Phys. Rev. **152**, 1401 (1966). See Ref. 33 for some comments and corrections.
[8] A. M. Gleeson and W. J. Meggs, Nuovo Cimento **55**, 584 (1968).
[9] A. M. Gleeson and W. J. Meggs, Nuovo Cimento **62A**, 181 (1969).

[10] The mass of the θ is μ. Because of the static kinematics and the selection rule $V\rightleftarrows N\theta$, the mass of the N can be taken to be zero with no loss of generality. The residue of $-i\hat{G}_{\bar{V}}(p)$ at the pole at $p^0=m$ is g^2.

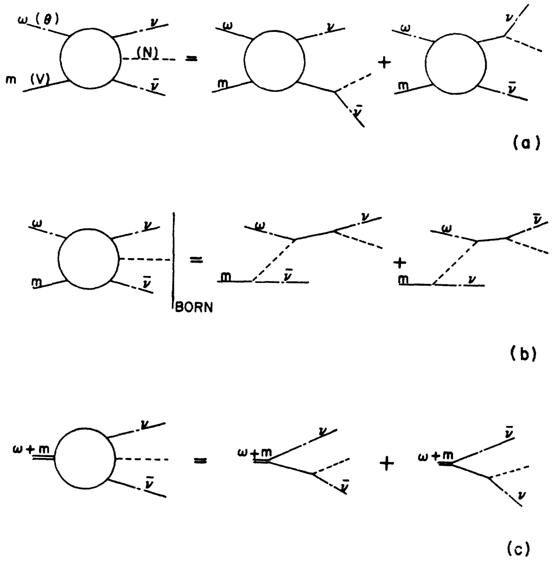

FIG. 1. Static production models. (a) The amplitude for the exact Lee model; (b) the amplitude for the Born approximation to the Lee model; (c) the amplitude for the simplified static model. ω, ν, $\bar{\nu}$, and m are the energies of the particles.

Fig. 1(a)]. The cross section for (c) can be written in the form[7]

$$\sigma_{(c)}(\omega)=\frac{1}{\pi}\int_{\mu}^{\bar{\mu}} d\nu \frac{|L(\omega,\nu,\bar{\nu})|^2 k(\nu)k(\bar{\nu})}{|\Delta(\nu)|^2|\Delta(\bar{\nu})|^2}, \qquad (4)$$

where $\Delta(\omega)=g^2[-i\hat{G}_{V}(p)]^{-1}$, $k(\omega)=[\omega^2-\mu^2]^{1/2}$, $\bar{x}=\omega+m-x$; ω, ν, and $\bar{\nu}$ are the energies of the initial and final θ's; and $L(\omega,\nu,\bar{\nu})=L(\omega,\bar{\nu},\nu)$ is a slowly varying function of ν. $\sigma_{(c)}(\omega)$ is plotted versus $(\omega-\omega_0)/\Gamma$ in Fig. 2, where $\omega_0=2m^*-m$ is the overlap energy.

There is a nearby double pole in $[\Delta(\nu)\Delta(\bar{\nu})]^{-1}$ when $\nu=m^*$ and $\bar{m}^*=m^*$. If L were slowly varying in ω as well as in ν, this double pole would lead to an enhancement in $\sigma_{(c)}(\omega)$ at the overlap energy. As we see in Fig. 2, no such enhancement occurs. Amado[5] explains this lack of an enhancement using an argument which is based on unitarity. We give a slight variation of Amado's argument in Appendix A.

There is no evidence of any structure in the exact expression $\sigma_{(c)}(\omega)$ which is not already included in the one-N exchange or Born approximation [see Fig. 1(b)]. We also make the approximation that $\Delta(\omega)$ is given for all ω by its approximate form for ω near m^*:

$$\Delta(\omega)|_{\omega \text{ near } m^*}\simeq Z^{*-1}(\omega-m^*+\tfrac{1}{2}i\Gamma). \qquad (5)$$

The resulting cross section for process (c) is included in Fig. 2, and we see that little has been lost by these approximations.

The part of the amplitude associated with the production process is slowly varying for the final-state θ energies near the resonance value, so as a further simplification we approximate the production by a point vertex, and we remove all information concerning the initial state aside from its total energy $P^0=\omega+m$ [see Fig. 1(c)]. The cross section $\sigma_{(c)}(\omega)$ is replaced by an effective cross section $\tau(\omega)$ which is essentially the total transition probability per unit time

$$\tau(\omega)=\tfrac{1}{2}\int_{\mu}^{\bar{\mu}} d\nu\, k(\nu)k(\bar{\nu})\left|\frac{1}{\Delta(\nu)}+\frac{1}{\Delta(\bar{\nu})}\right|^2, \qquad (6)$$

where $\Delta(\omega)$ is given by Eq. (5). $\tau(\omega)$ is also included in Fig. 2. Again, there is no structure in the overlap region, and with the simplified dynamics we can see this explicitly. We write

$$\frac{1}{\Delta(\nu)}+\frac{1}{\Delta(\bar{\nu})}=\frac{\bar{m}^*-m^*+i\Gamma}{Z^*\Delta(\nu)\Delta(\bar{\nu})}. \qquad (7)$$

The residue of the pole at $\nu=\beta\simeq m^*-\frac{1}{2}i\Gamma$ in Eq. (7) is given by

$$\frac{\bar{m}^*-m^*+i\Gamma}{\Delta(\bar{\beta})}=Z^*\frac{\bar{m}^*-m^*+i\Gamma}{\bar{m}^*-m^*+i\Gamma}=Z^*.$$

Thus there is no double pole in the integrand and hence nothing to single out the overlap energy.

The other curves in Fig. 2 are cross sections for the quasi-process

(d) $\theta+V\to\theta+V^*$.

These are discussed in Sec. III.

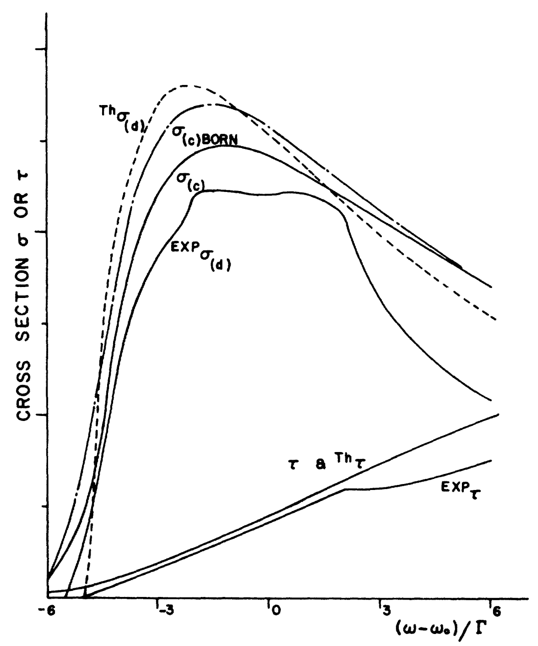

FIG. 2. Cross sections for the static model production processes along with some resonance projections. The Lee model curves are labeled σ and the simplified static model curves are labeled τ.

B. Nonstatic Model

1. Development of the Model

The model we now discuss is a fully relativistic version of the simplified static model of the previous section. We limit the model to a single value of the total angular momentum and parity, J^η, because we are primarily interested in coherent effects and experimental situations in which a single eigenstate is believed to be dominant. Also, we introduce no structure in M, the three-particle mass, that does not follow directly from the final-state interactions and kinematics.

An effective transition amplitude for describing interactions among three pseudoscalar particles can be defined as follows:

$$T=\int \prod_{n=1}^{3}\left(\frac{d^3p_n'}{2p_n'^0}\right)T'(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')\langle \mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'\rangle\,, \quad (8)$$

where the function T' includes any correlations among the particles. With a normalization convention for the states given by Eq. (B1) (see Appendix B), no correlations among the particles leads to

$$T'\propto T\propto 1 \quad \text{(no correlations)}, \qquad (9)$$

where the proportionality constants are unity if there are no identical particles. For our purposes, two other sets of variables introduced by Wick[1] are more useful than the three three-momenta for describing the three-particle system: $\{P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}\}$ and $\{P,J,\Lambda,s_{12},j,\lambda\}$, where the variables are defined below in Appendix B. We differ from Wick in our normalization convention for the states defined in terms of these sets of variables because we want Eq. (9) to continue to hold when the right-hand side of Eq. (8) is written in terms of either of these sets. Gleeson and Meggs[8,9] have used a model which is similar to ours except that they used Wick's normalization conventions so that their function corresponding to T has some extraneous momentum dependence.

The parity operator $\tilde{\eta}$ has the following effect on the angular momentum state[2]:

$$\tilde{\eta}|P,J,\Lambda,s_{12},j,\lambda\rangle=(-1)^{J+1}|P,J,\Lambda,s_{12},j,-\lambda\rangle\,,$$

where we have evaluated P in the (12)3 center-of-momentum frame so that it is invariant under parity, and we have used the fact that the three particles are pseudoscalars. Thus the angular momentum state is an eigenstate of parity only if $\lambda=0$. In this case, the eigenvalue η is $(-1)^{J+1}$, which is usually called the natural parity. One can, of course, form eigenstates of parity by taking linear combinations of the states of the form $|\lambda\rangle\pm|-\lambda\rangle$ where the $+$ leads to a natural-parity eigenstate and the $-$ to the opposite parity. Unless $J=0$ or $j=0$, the choice of a particular linear combination will depend upon one's choice of a dynamic model. Note that we had no such ambiguity in the static case.

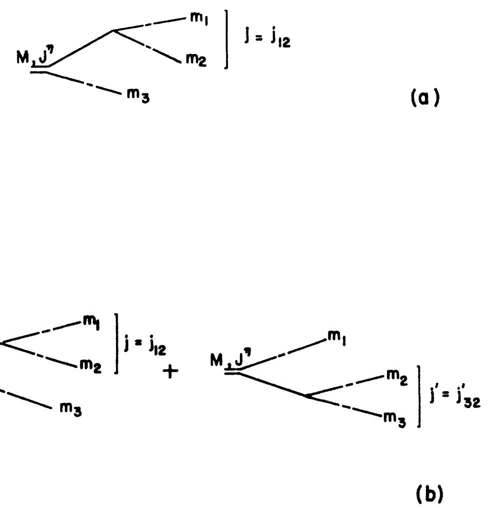

FIG. 3. Amplitudes for the nonstatic model.

We first consider the case in which there are no identical particles and there is a two-particle resonance with spin j in the (12) subsystem [see Fig. 3(a)]. Using some of the definitions given in Appendix B, we can write the effective transition amplitude for this final state as

$$\begin{aligned}T&=T_{J^\eta j}(M,s_{12},\theta_{12},R_3)\\&=\sum_{J'j'\Lambda'\lambda'}\int d^4P'\int ds_{12}'\frac{p_{12}'q_3'}{4m_{12}'M'}A_{J'\Lambda'j'\lambda'}{}^{\eta}(M',s_{12}')\\&\quad\times\delta_{JJ'}\delta_{jj'}\langle P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}|P',J',\Lambda',s_{12}',j',\lambda'\rangle\\&=N_JN_j\sum_{\Lambda\lambda}A_{J^\eta\Lambda j\lambda}(M,s_{12})\bar{D}_{\Lambda}{}^{J}{}_{\lambda}(R_3)d_{\lambda}{}^{j}{}_{0}(\theta_{12})\,,\end{aligned} \qquad (10)$$

where A is a weight function which includes the two-particle resonance propagator, the parity projection, and any additional dependence on M and s_{12} which follows from the specific dynamics. We assume that the weight function can be written in the form

$$A_{J^\eta\Lambda j\lambda}(M,s_{12})=f_j(s_{12})B_{J^\eta j\lambda}(q_3)\,.$$

Using either two-particle unitarity or a comparison of two-particle and one-particle phase space,[11] we find that f is a well-defined function given by

$$f_j(s_{12})=\frac{[(m_{12}/p_{12})\,\mathrm{Im}\Delta_j(s_{12})]^{1/2}}{\Delta_j(s_{12})}\,,$$

where Δ_j is the inverse propagator for the resonance. [For small p_{12}, $f_j(s_{12})\propto(p_{12})^j$.] The simplest dynamical model for the momentum dependence of B gives[12]

[11] J. D. Jackson, Nuovo Cimento **34**, 1644 (1964).

[12] Reference 13 gives a nonrelativistic derivation. Relativistically, one can get this by writing down the covariant interaction Hamiltonian with the lowest-order derivative coupling [see, for example, R. Blankenbecler, R. L. Sugar, and J. D. Sullivan, Phys. Rev. **172**, 1451 (1968)].

$$B_{J^\eta j\lambda}(q_3)=\sum_{L=|J-j|}^{J+j} b_{J^\eta j\lambda L}\left(\frac{q_3}{\hat{q}_3}\right)^L, \qquad (11)$$

where $\hat{q}_3$ is a constant with the dimensions of momentum so that the dimensionality of B does not depend upon L. Nonrelativistically, L is just the relative orbital angular momentum between No. 3 and the (12) system. It has been shown by Fabri[13] that for low energies, the lowest possible values of L compatible with a given J, j, and η would be favored. Dalitz[14] obtained essentially the same results without introducing L by arguing that only the lowest powers of the $|\mathbf{p}_i|$ should occur which were compatible with a given J. Zemach[15] has rephrased Dalitz's model in relativistic terms, and Werle[2] has done the same for Fabri's model using the results of MacFarlane[16] and McKerrell,[17] who showed that L could be defined in a covariant fashion. L states (or canonical states) can be written as the following linear combination of helicity states[2,17]:

$$|P,J,\Lambda,s_{12},j,L\rangle = (N_L/N_J)\sum_\lambda \langle LOj\lambda|J\lambda\rangle|P,J,\Lambda,s_{12},j,\lambda\rangle, \qquad (12)$$

where $\langle LOj\lambda|J\lambda\rangle$ is a Clebsch-Gordan coefficient. A useful property of L states is that they are eigenstates of parity with eigenvalue $(-1)^{j+L+1}$. If only a single value of L contributes to T, Eq. (11) reads

$$B_{J^\eta j\lambda}(q_3)|_{\text{single }L}=(N_L/N_J)\langle LOj\lambda|J\rangle(q_3/\hat{q}_3)^L. \qquad (13)$$

We now consider the case in which there are resonances in both the (12)3 and the (32)1 channels [see Fig. 3(b)]. If Nos. 1 and 3 are not identical, there is a second term in Eq. (10) corresponding to the second diagram in Fig. 3(b). The two terms add coherently, and there is an arbitrary relative magnitude and phase

$$T=T_{J^\eta j_{12}}(M,s_{12},\theta_{12},R_3)+Xe^{iY}T_{J^\eta j_{32}'}(M,s_{32},\theta_{32},R_1). \qquad (14)$$

We have included the subscripts on j and j' to emphasize that they refer to different systems, so that even if $j=j'$, the system will not be in an eigenstate of j. If Nos. 1 and 3 are identical, the symmetrization in the state normalization in 1 and 3 leads to a second term in T which is the same as Eq. (10) except that 1 and 3 are interchanged. The resulting expression for the total amplitude is a specific case of Eq. (14) in which $j=j'$, $X=1$, and $Y=0$.

By analogy with Eq. (6) for the simplified static model, we define an effective cross section

$$\tau_{J^\eta}(M)=\tfrac{1}{4}\int\frac{p_{12}q_3}{m_{12}M}ds_{12}\int d(\cos\theta_{12})\int dU_{R_3}|T|^2. \qquad (15)$$

[13] E. Fabri, Nuovo Cimento **11**, 479 (1954).
[14] R. H. Dalitz, Phil. Mag. **44**, 1068 (1953).
[15] C. Zemach, Phys. Rev. **133**, B1201 (1964).
[16] A. J. MacFarlane, J. Math. Phys. **4**, 490 (1963).
[17] A. McKerrell, Nuovo Cimento **34**, 1289 (1964).

As written in Eq. (14), T is a function of both the (12)3 set and the (32)1 set. When calculating τ, this causes no problem for the incoherent part of $|T|^2$ because we need only choose the variables of integration appropriately. However, both sets appear in a nonseparable way in the interference term, so that we must express the (32)1 set in terms of the (12)3 set. In Appendix B we write expressions for $s_{32}=s_{32}(M,s_{12},\cos\theta_{12})$ and $\cos\theta_{32}=\cos\theta_{32}(M,s_{12},s_{32})$ so that the only function left to be rewritten in terms of the (12)3 basis is $\bar{D}_{\Lambda}{}^{J}{}_{\lambda}(R_1)$. This can be done by proceeding as Wick did in calculating what he calls the "recoupling coefficient."[1] Our situation is slightly different from the one he considered in that he calculated $\langle(12)3|(23)1\rangle$, which involves a cyclic permutation of the particle indices, while we want an interchange of Nos. 1 and 3.[18] Some of the algebra for this calculation is given in Appendix B, and the final result for T is

$$T_{J^\eta}(M,(12)3\text{ set}) =N_J\sum_{\Lambda\lambda\lambda'}\bar{D}_{\Lambda}{}^{J}{}_{\lambda'}(R_3)\{N_jf_j(s_{12})\delta_{\lambda\lambda'}B_{J^\eta j\lambda}(q_3)d_{\lambda}{}^{j}{}_{0}(\theta_{12}) +Xe^{iY}N_{j'}f'_{j'}(s_{32})B'_{J^\eta j'\lambda}(q_1)d_{\lambda}{}^{J}{}_{\lambda'}(\chi)d_{0}{}^{j'}{}_{\lambda}(\theta_{32})\}. \qquad (16)$$

The primes on B and f in the second term are there because these functions might differ from the B and f in the first term.

Using Eq. (14), we see that the integration over U_{R_3} in Eq. (15) is trivial. Because of the orthogonality of the D's, the double sums on Λ and λ' reduce to single sums, with the resulting sum on Λ giving a factor of $2J+1$. Equation (15) then becomes

$$\tau_{J^\eta}(M)=\frac{2J+1}{8M}\int\frac{p_{12}q_3}{m_{12}}ds_{12}\int d(\cos\theta_{12}) \times\sum_{\lambda'}|(2j+1)^{1/2}f_j(s_{12})\xi_J{}^{\lambda'}B_{J^\eta j\lambda'}(q_3)d_{\lambda'}{}^{j}{}_{0}(\theta_{12}) +Xe^{iY}\sum_\lambda[2j'+1]^{1/2}f'_{j'}(s_{32}) \times B'_{J^\eta j'\lambda}(q_1)d_{\lambda}{}^{J}{}_{\lambda'}(x)d_{0}{}^{j'}{}_{\lambda}(\theta_{32})|^2, \qquad (15')$$

where

$$\xi_J{}^\lambda=1,\quad J\geq|\lambda|$$
$$=0,\quad \text{otherwise.}$$

The two-dimensional integral in Eq. (15′) can be transformed quite easily into an integral over the usual Dalitz plot variables s_{12} and s_{32} using

$$\frac{1}{M}\int\frac{p_{12}q_3}{m_{12}}ds_{12}\int d(\cos\theta_{12})=\frac{1}{2M^2}\int ds_{12}\int ds_{32}. \qquad (17)$$

Thus the quantity $\sum_{\lambda'}|\;\;|^2$ in Eq. (15′) is just the Dalitz plot density.

[18] This was not taken into account in Refs. 8 and 9.

The above results are easily extended to the case of three overlapping two-particle resonances in a three-particle system. Following the procedure used above one can reexpress the third resonance term in T in the same basis as the other two. Some of the angles and relations are given in Appendix B for the case in which the third resonance is defined in the (31)2 basis. Note that because of the linear form of the amplitude, the resonances can only interfere two at a time even though there are three of them. Thus, we would not expect τ for the three-resonance case to have any appreciably different structure from τ for the two-resonance case.[6]

2. Another Form of the Model

Two groups of experimentalists[3,4] have used a model which is equivalent to ours for studying overlapping two-particle resonances in the charged $K\pi\pi$ system near 1.3 BeV. For the two-resonance case, their total transition amplitude takes the same form as Eq. (14) above but, instead of Eq. (10) for the individual channel amplitudes, they have terms of the form

$$T'_{J^\eta jL}(M,s_{12},\theta_{12}^c,\phi_{12}^c,\theta_3^c,\phi_3^c) =\frac{N_L}{N_J}f_j(s_{12})\left(\frac{q_3}{\hat{q}_3}\right)^L \sum_{\Lambda n} \langle L, \Lambda-n, j, n|J\Lambda\rangle \times Y_L^{\Lambda-n}(\theta_3^c,\phi_3^c)Y_j^n(\theta_{12}^c,\phi_{12}^c). \quad (18)$$

All quantities are defined as before except that the coordinate system for measuring the angles appearing in Eq. (18) is fixed in space. This is not the case for the helicity-inspired system defined in Appendix B and used in the previous subsection. In the helicity frame, only θ_{ik} is an observable, while in the "canonical" frame used in Eq. (18) all the angles can be determined experimentally.

A nonrelativistic form of Eq. (18) was derived by Fabri[13] and the above form was derived by Werle[2] starting from L states. We have derived Eq. (18) starting from helicity states, and we have shown explicitly the equivalence between Eq. (18) and Eq. (10) for B given by Eq. (13).[19] The canonical framework is much more cumbersome to work with than the helicity framework when calculating cross sections because there is no easy way to reexpress the (32)1 set of variables in terms of the (12)3 set using angles defined in the canonical framework.

3. Example with Two Identical Particles: 3π System

As an example we consider the 3π system in which the first and third pions are identical and the overlapping resonances are ρ mesons.[20]

If we set $J=j=0$, our model reduces to that of Chang,[21] who reported an enhancement in the three-pion mass near the overlap threshold. Repeating Chang's calculations, we find no enhancement, and we conclude, as did Sweig,[22] that Chang's calculation is erroneous.[23]

We now give the ρ its correct spin, and we set $J^\eta=1^+$, which is the popular favorite for the $A_1(1080)$.[24] The choice for B in Eq. (11) is now model-dependent[24]: Hard-pion techniques lead to predictions for $A_1\to\rho\pi$ to be dominated by s waves[25] so that only $L=0$ contributes to B; quark model calculations of pion emission by quarks, neglecting recoil corrections, predict[26] that $A_1\to\rho\pi$ is purely transverse—only $\lambda=\pm1$; and chiral $SU(2)\times SU(2)$ predicts[27] that the decay should be longitudinal, which means only $\lambda=0$ survives for each $\rho\pi$ part of the amplitude. A difficulty in reaching a decision concerning the correctness of any of these alternatives is that there is enough additional structure in the 3π spectrum in the A_1 region so that it is not certain that there *is* an A_1.[28]

Most experimental analyses approach the A_1 from the point of view that if there is an A_1, and if it has $J^\eta=1^+$, then it is characterized by an s wave for each $\rho\pi$ system even though there could be some d wave. The justification for this assumption is essentially a carryover of the nonrelativistic angular momentum barrier that inhibits higher orbital angular momenta relative to lower ones. There is, however, a group at SLAC[29] which favors the longitudinal mode of decay on the basis of their data and method of analysis. The formalism we have developed, particularly with respect to our treatment of the two-particle resonance and the form of B in Eq. (11), is based on the simplest possible momentum dependence consistent with two-particle unitarity and the threshold behavior, so that the formalism is probably on its firmest footing when one uses only the lowest possible value of L. Keeping more than one L means that results will necessarily be dependent on the choice of the normalizing factor $\hat{q}$ of Eq. (11). This will have an effect of setting the scale between contributions for different values of L. We therefore concentrate on s-wave dominance.

For $L=0$, $J=1$, and $j=1$, Eq. (13) gives

$$B_{1^+1\lambda}(q)|_{L=0}=(1/\sqrt{3})\langle 001\lambda|1\lambda\rangle=1/\sqrt{3},$$

[19] J. A. Snoke, Ph.D. dissertation, Yale University, 1969 (unpublished).

[20] We have chosen $m^*=770$ MeV and $\Gamma=128$ MeV for the ρ.

[21] N. P. Chang, Phys. Rev. Letters **14**, 806 (1965).

[22] M. Sweig, University of Chicago Report No. EFINS 65-70 (unpublished).

[23] See Ref. 19 for more details.

[24] H. Harari, Rapporteur Talk, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968), p. 195.

[25] H. Schnitzer and S. Weinberg, Phys. Rev. **170**, 1638 (1968).

[26] H. J. Lipkin, Phys. Rev. **159**, 1303 (1967).

[27] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

[28] N. Barash-Schmidt *et al.*, Rev. Mod. Phys. **41**, 109 (1969).

[29] J. Ballam *et al.*, Phys. Rev. Letters **21**, 934 (1968).

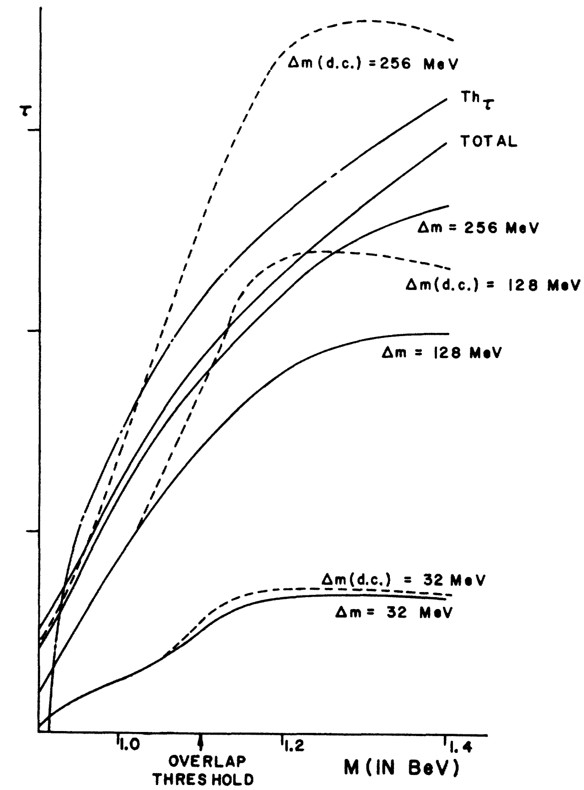

FIG. 4. Effective cross sections for the 3π model with $J^\eta = 1^+$ and each $\rho\pi$ system in a relative s wave.

and

$$\tau_1^+(M)|_{s\text{ wave }\rho\pi} = \frac{3}{8M}\int ds_{12}\frac{p_{12}q_3}{m_{12}}\int d(\cos\theta_{12}) \times\sum_\lambda |f_1(s_{12})d_{\lambda}{}^1{}_0(\theta_{12})+f_1(s_{32})d_0{}^1{}_\lambda(\theta_{32}+\chi)|^2$$

$$= \frac{3}{8M}\int ds_{12}\frac{p_{12}q_3}{m_{12}}\int d(\cos\theta_{12}) \times\{|f_1(s_{12})|^2+|f_1(s_{32})|^2 +2\,\mathrm{Re}[f_1(s_{12})\bar{f}_1(s_{32})]\cos(\theta_{12}+\theta_{32}+\chi)\}. \quad (19a)$$

If we use the simplest form of $\Delta_1(s)$ consistent with two-particle unitarity,[11]

$$\Delta_1(s) = s - m^* + im^*\Gamma m^* p^3/(sp^{*2})^{3/2},$$

where $p^* = p(s = m^{*2})$, Eq. (19a) becomes

$$\tau_1^+(M)|_{s\text{ wave }\rho\pi} = \frac{3m^{*2}\Gamma}{8Mp^{*3}}\int ds_{12}\frac{p_{12}q_3}{m_{12}}\int d(\cos\theta_{12}) \times\left\{\frac{p_{12}^2}{|\Delta_1(s_{12})|^2}+\frac{p_{32}^2}{|\Delta_1(s_{32})|^2} +2p_{12}p_{32}\,\mathrm{Re}\left[\frac{1}{\Delta_1(s_{12})\bar{\Delta}_1(s_{32})}\right]\cos(\theta_{12}+\theta_{32}+\chi)\right\}. \quad (19b)$$

$\tau_1^+(M)$ is plotted versus M in Fig. 4 (labeled "total"). Note that the curve is smooth throughout with no peaking at the overlap threshold (which is also the A_1 region).

It is interesting to calculate some of the two-particle projections for this model and to compare them with the projections for the hypothetical case in which there is a 3π system with only a single ρ. Curves for $d\tau/ds_{12}$ versus s_{12} and for $d\tau/d\Omega_{12}$ versus $\cos\theta_{12}$ are given in Fig. 5 for three different values of the three-particle mass; $M=1.1$, 1.555, and 1.875 BeV using Eq. (19b) for τ. These differential cross sections can be obtained from experimental data by including both of the two-particle masses or angles which occur for each event. The two-particle mass spectra are given in Fig. 5(a), (b), and (c) for s in the interval $[m^{*2}-2m^*\Gamma,\ m^{*2}+2m^*\Gamma]$. The dashed curves are the distributions for the single-ρ case. We see that while there are some differences in the shape of the curves for the two models, the peak position and the width are not appreciably changed. The angular distribution for the single-ρ case would be isotropic, or a straight horizontal line in Fig. 5(d), (e), and (f), and we see that $d\tau/d\Omega_{12}$ (labeled "total") differs considerably from this for all three values of M. Also included in these subfigures are some angular distributions which do not treat the identical particles symmetrically; we include them because such curves were given by Bouchiat and Flamand.[30] In the curves labeled "resonance projection," m_{12} is set equal to m^*, and in the curves labeled "one resonance band," we have only integrated s_{12} over the region $m^{*2}-2m^*\Gamma \le s_{12} \le m^{*2}+2m^*\Gamma$. We see that these projections have somewhat less structure than the "total" curves, but they are still not isotropic. Thus the two-particle mass distributions in the resonance region are only slightly affected by approximating our coherent model by an incoherent model, whereas the angular distributions are affected to a fairly significant extent. The analysis of Ref. 29 has not used a coherent model, calling into question the interpretation of the $\rho\pi$ angular distribution therein.

The longitudinal or zero helicity model would contain contributions from both $L=0$ and $L=2$ ($L=1$ has the wrong parity). To compute B_λ for this case, one first decomposes a $\lambda=0$ helicity state into L states, multiplies each L state by $(q/\hat{q})^L$, and then rewrites the L states in terms of helicity states. The result is

$$B_{1+1\lambda}(q)|_{\text{longit}} = \tfrac{1}{3}\sum_L [2L+1]\langle L010|10\rangle\langle L01\lambda|1\lambda\rangle[q/\hat{q}]^L. \quad (20)$$

It is only for $q=\hat{q}$ that the right-hand side of (20) reduces to $\delta_{\lambda 0}$ so that our results look like a zero-helicity model. If there is an A_1 with central mass $M=M^*$, and if $\hat{q}$ is defined to be $\hat{q}=\gamma(M^{*2},m^{*2},m^2)/2M^*$, then the

[30] C. Bouchiat and G. Flamand, Nuovo Cimento 23, 13 (1962).

transition amplitude for each $\rho\pi$ system becomes

$$T_{1+1}(M^*,s_{12}^*,\theta_{12},R_3)\,|_{\rm longit} = (3/4\pi)\sum_{\Lambda} f_1(s_{12}^*)\bar{D}_{\Lambda}{}^1{}_0(R_3)\cos\theta_{12}.$$

Thus it is only when the A_1 and ρ are on shell that $\lambda=0$ and our dynamical approximation for B lead to a $\cos^2\theta$ prediction for each $\rho\pi$ system.

A similar analysis for the transverse case leads to a $\sin^2\theta$ prediction for each $\rho\pi$ system only when both the A_1 and the ρ are on shell.

4. *Example with No Identical Particles*: $K\pi\pi$ *System*

Another case currently of interest is the charged $K\pi\pi$ system for M near 1.3 BeV. When this system contains a charged K, there are two overlapping resonances, a ρ and a K^*, and when there is a neutral K, there can be three overlapping resonances, a ρ and two K^*'s.[3,4]

When there is a charged K and the particles are ordered $\{K^\pm, \pi^\pm, \pi^\mp\}$, the above definitions of f and B (at least for $L=0$) are such that if the $K\pi\pi$ system transforms like a K under $SU(3)$, pure f coupling (d coupling) [corresponding to an octet with $C=+1$ (-1)] leads to $X=1$ and $Y=0$ (π) in Eq. (14). We have found[31] that for X near unity, the 2π-mass distribution is quite sensitive to Y. This fact could prove quite useful in testing the hypothesis[32] that the $K^*(1300)$ enhancement contains two 1^+ resonances belonging to octets with opposite C.

III. RESONANCE APPROXIMATIONS

A. Theoretical Resonance Approximations

A resonance projection entails treating the resonance as though it were a stable particle. This can be accomplished in the models discussed in the previous section by keeping only the lowest-order term in Γ/m^* when calculating the cross section. Two attempts[7,8] at making resonance projections in situations in which there were overlapping, identical resonances have led to anomalous peaks in the three-particle mass spectrum—anomalous in that these peaks do not occur in the total cross section. Elsewhere,[33] we have shown that the anomalous peak in the Lee model[7] resulted from an invalid approximation in the calculation of the cross section for the quasiprocess (d). We now show that the anomalous peaks in both Refs. 7 and 8 can be attributed to higher order terms in Γ/m^*.

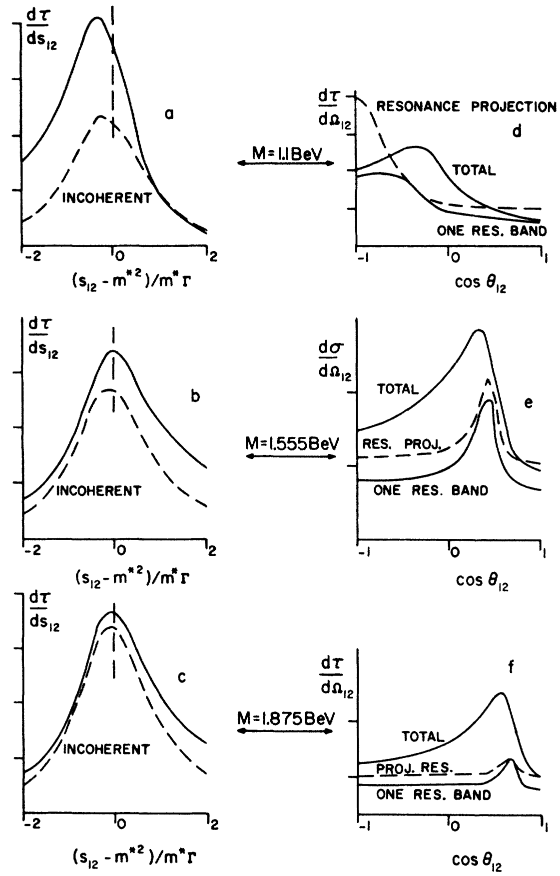

FIG. 5. Differential cross sections for the model used in Fig. 4.

If there is only a *single* two-particle resonance in a three-particle state, the resonance projection takes the form

$$\mathrm{Im}\Delta_j(s)/|\Delta_j(s)|^2 \to \pi\delta(s-m^{*2}) \qquad (21a)$$

for the nonstatic model of Sec. II B, and

$$\mathrm{Im}\Delta(\omega)/|\Delta(\omega)|^2 \to \pi Z^*\delta(\omega-m^*) \qquad (21b)$$

for the static models of Sec. II A. The anomalous peaks of Refs. 7 and 8 follow if one applies these projections *directly* to Eq. (4) or to Eq. (15′). However, if one writes $|T|^2$ for an overlapping resonance model in the form

$$|T|^2 = \underbrace{|T((12)3)|^2+|XT((32)1)|^2}_{\text{(incoherent)}} + \underbrace{2\,\mathrm{Re}\{T((12)3)Xe^{-iY}\bar{T}((32)1)\}}_{\text{(interference)}}, \qquad (22)$$

a little algebra shows that the interference term of Eq. (22) is of a higher order in Γ/m^* than the incoherent part. Thus the interference term must be dropped before applying the projection Eq. (21a) or Eq. (21b).[34] A resonance projection for the case of two overlapping,

[31] T. W. Ludlam and J. A. Snoke (to be published).

[32] *Meson Spectroscopy*, edited by C. Baltay and A. Rosenfeld (W. A. Benjamin, Inc., New York, 1968), p. 209.

[33] J. A. Snoke, Phys. Rev. 179, 1620 (1969).

[34] Another way of saying this is that in the Dalitz plot for a three-particle final state containing two two-particle resonances, the density in the overlap region is approximately a factor of 2 times the density in the nonoverlap part of the resonance bands.

identical resonances is therefore the same as the resonance projection for the case of no identical particles and a single resonance.[35] These arguments are easily generalized to the case of overlapping, nonidentical resonances if one uses the fact that the identical resonance amplitude can be viewed as a specific choice of X and Y in Eq. (14). The resonance cross sections resulting from these projections for the Born approximation to the Lee model,[36] the simplified static model, and our nonstatic model are, respectively, given by

$$^{\mathrm{Th}}\sigma_{(d)}(\omega)_{\mathrm{Born}}=\frac{g^2\Gamma k(\bar{m}^*)\theta(\bar{m}^*-\mu)}{2k(m^*)k(\omega)(\omega-m^*)},$$

$$^{\mathrm{Th}}\tau(\omega)=(2\pi Z^*/\Gamma)k(m^*)k(\bar{m}^*)\theta(\bar{m}^*-\mu),$$

$$^{\mathrm{Th}}\tau_J{}^{\eta}(M)=\frac{\pi(2J+1)}{4M}\Big[\theta(M-m_3-m_{12}{}^*)q_3{}^*\times\sum_{\lambda=-j}^{j}|\xi_J{}^{\lambda}B_{J^{\eta}j\lambda}(q_3{}^*)|^2+\theta(M-m_1-m_{32}{}^*)X^2q_1{}^*\times\sum_{\lambda=-j'}^{j'}|\xi_J{}^{\lambda}B'_{J^{\eta}j'\lambda}(q_1{}^*)|^2\Big].$$

These cross sections are plotted along with the corresponding total cross sections. We see that the resonance cross sections have no anomalous structure, and that they are quite reasonable approximations to the total cross sections.

B. Resonance Cuts as Experimental Resonance Projections

Working from data, a two-particle-resonance cut is made by accepting (rejecting) an event if the appropriate two-particle mass is inside (outside) an interval $[m^*-\kappa\Gamma,\ m^*+\kappa'\Gamma]$, where κ and κ' are usually chosen so that the selection region has a width Δm of about 2Γ. If the final state contains two identical, overlapping resonances, the cut is most naturally defined as selecting an event if either of the appropriate two-particle masses falls in some designated region.

The reason for such a cut is that it provides a simple, model-independent way of suppressing events in the final state which do not result from processes which produce the resonance. There is the implicit assumption that the background can be taken to be incoherent to the resonance production; when the coherence cannot be ignored, the interpretation of these cuts becomes model dependent. Such is the case with the differential cross sections shown in Fig. 5 where the coherence of the second resonance has a large effect particularly on the angular distributions.

Resonance cuts can be simulated in our models by restricting the region of integration to the selection region. As the width of the selection region decreases, the effect of a resonance cut approaches a multiple of the result of applying the resonance projection Eq. (21a) or Eq. (21b) *directly* to the expressions for the cross section. The resonance-cut procedure is therefore often interpreted as a simple form of a resonance projection. It should be clear from the discussion in the preceding subsection that this interpretation is not valid when there are overlapping resonances; the $\Delta m\to 0$ limit of the resonance cut in this case is *not* the theoretical resonance projection, but instead a projection which leads to anomalous peaks in the three-particle mass spectrum.

In Fig. 2 we have included curves for resonance-cut cross sections (labeled Exp) for the Lee model and for the simplified static model with $\Delta m=2\Gamma$. There is some structure in the overlap region, but no peaks. In Fig. 4 we have included curves for different choices of Δm for the 3π model with $J^{\eta}=1^+$ and each $\rho\pi$ system in a relative s wave. As Δm decreases, there is an enhancement near the overlap threshold which becomes more pronounced, which is what we would expect from our discussion above.

A mistake which has been made in the calculations of resonance-cut cross sections when there are overlapping resonances[7,9] is to integrate over one resonance band and then to double the result. This has the effect of counting the overlap region twice. For static models, this leads to an anomalous peak.[7,19,33] For the nonstatic model, this procedure introduces structure, but the effect is not as pronounced as in the static case [curves labeled Δm(d.c.) in Fig. 4].

IV. OTHER MODELS

We do not claim that the nonstatic model developed in Sec. II B is more than a reference or a first approximation to any physical situation; its main attractions are that it is easy to work with and it treats the resonances coherently. It has been suggested[5,37] that the linear or isobar model is not the appropriate starting point for dealing with overlapping resonances, so we now consider the proposed alternatives.

The scattering amplitude for process (c) of the Lee model discussed above can either be written as the sum of two terms each containing a single resonance propagator $[\Delta(\nu)]^{-1}$, or it can be written as a single term containing a product of the two resonance propagators. Amado[5] has proposed that the dynamics of a state which contains overlapping resonances are more clearly presented in the product form of the amplitude, which implies that this is the form upon which any approximations should be made.[38] Our models (except for the

[35] The fact that there are two terms in the incoherent part of $|T|^2$ in Eq. (22) is cancelled by the factor of $\frac{1}{2}$ which we put in the sum over final states when calculating the cross section because of the identical particles.

[36] This is also the theoretical resonance approximation for the exact Lee model. See Refs. 33 and 19.

[37] C. Lovelace, Phys. Letters **28B**, 264 (1968).

[38] I. J. R. Aitchison and C. Kacser [Phys. Rev. **173**, 1700 (1968)] show that Amado's hypothesis is model-dependent.

exact Lee model) and the resonance projections can be viewed as approximations based on the linear form of an exact scattering amplitude. We find for the Lee model (which is one of Amado's examples) that when the coupling gets strong enough that the Born approximation is no longer valid, then Γ/m^* is much too large for the two-particle interaction to be called a resonance. We hesitate to make any general pronouncements based on this result because of the limited applicability of the Lee model.

For the nonstatic case, the Veneziano model[39] may provide an alternative approach to that of our isobar model and Born approximations. It is not clear at present how fundamental the differences actually are between isobar models and the corresponding Veneziano models since Boguta,[40] using an isobar model, claims to have reproduced the results of Lovelace[37] and Altarelli and Rubinstein[41] which were obtained with a version of the Veneziano model. Also, the amplitude used by Lovelace can be written as a slowly varying form factor times an isobar model near the resonance overlap.

ACKNOWLEDGMENT

I wish to thank Professor C. M. Sommerfield for suggesting the topic and for many valuable discussions during my investigation.

APPENDIX A: PROOF OF NO ENHANCEMENT FOR PRODUCTION PROCESS (c) IN LEE MODEL

Unitarity in the $(V\theta)$ sector in the Lee model leads to[7]

$$\frac{|u(\omega)|^2}{k(\omega)}\,\mathrm{Im}F(\omega,\omega)=\sigma_{(b)}(\omega)+\sigma_{(c)}(\omega)\,, \tag{A1}$$

where $F(\omega,\omega)$ is the scattering amplitude for process (b), and $u(\omega)$ is a cutoff function. Equation (A1) can be rewritten in a form analogous to Eq. (19) of Ref. 5:

$$\mathrm{Im}F(\omega,\omega)=\frac{|u(\omega)|^2k(\omega)}{4\pi}|F(\omega,\omega)|^2 +\frac{1}{|C(\omega)|^2}\int_\mu^{\bar{\mu}} d\nu\frac{N(\omega,\nu,\bar{\nu})}{|\Delta(\nu)|^2|\Delta(\bar{\nu})|^2}\,, \tag{A2}$$

where $C(\omega)=2-K(\omega)Q(\omega)$, with $K(\omega)$ and $Q(\omega)$ defined in Ref. 7. [The function $C(\omega)$ corresponds to $\Delta(E)$ in Eq. (19) of Ref. 5.] If the integral in Eq. (A2) resonates at $\omega=\omega_0$, then so will $F(\omega,\omega)$ and hence so will $\sigma_{(b)}(\omega)$. However, as Amado[5] points out, if there were a resonance in $\sigma_{(b)}(\omega)$, one would hope that it would show up in $1/C(\omega)$ since this is the three-body part of the amplitude for (c). If both the integral and $1/C(\omega)$ resonate, the contribution to $F(\omega,\omega)$ will be doubly resonant contrary to assumption. Amado then argues that the fact that there is no enhancement in $\sigma_{(b)}(\omega)$ is caused by a corresponding de-enhancement factor in our $|C(\omega)|^{-2}$, and he relates this de-enhancement to the mechanism by which the Peierls enhancement[42] is cancelled in the Lee model. It is at this point that we differ with Amado: We believe the cancellation of the possible resonance from the integral in Eq. (A2) should be disassociated from the cancellation of the Peierls singularity. The function $1/C(\omega)$ was shown by Pagnamenta[43] to contain both the Peierls singularity and a nearby zero which cancelled that singularity, and using the computer we found $|C(\omega)|^{-2}$ to be essentially constant for the parametrization of the Lee model used in Ref. 7. Thus, the function $N(\omega,\nu,\bar{\nu})$, while slowly varying in ν, has the proper ω behavior so that the integrand in Eq. (A2) has no double pole.

To help clarify this final point, we refer to the calculation at the end of Sec. II A 2 in which we showed explicitly how the de-enhancement occurred for the simplified static model. Note that the Peierls singularity does not occur in this model and that $N(\omega,\nu,\bar{\nu})$ is independent of ν.

APPENDIX B: APPENDIX TO SEC. II B

For spinless particles with nonzero masses, a three-particle system can be fully described in terms of the three three-momenta $\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}$ or any equivalent set of nine independent variables. For the above set we define the state $|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle$ with a normalization given by[44]

$$\langle\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'\rangle=\prod_{n=1}^{3}[2p_n{}^0\delta^{(3)}(\mathbf{p}_n-\mathbf{p}_n')]. \tag{B1}$$

An equivalent set of variables to the above which we will find useful is the set proposed by Wick,[1] $\{P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}\}$. P is the total four-momentum which in the over-all center-of-momentum frame [the (12)3 c.m.f.] reduces to $\mathbf{P}=0$ and $P^0=(-P^\mu P_\mu)^{1/2}=M$. In this frame the "standard orientation" has the three momenta lying in the xz plane with $\hat{z}=-\hat{p}_3$ and $\hat{p}_1\cdot\hat{x}\geq 0$. The angles (θ_3,ϕ_3) give the actual orientation of $\mathbf{P}_{12}\equiv\mathbf{p}_1+\mathbf{p}_2=-\mathbf{p}_3$ with respect to this coordinate system. The invariant two-particle mass $(s_{12})^{1/2}$ is defined by $s_{12}=(-P_{12}{}^\mu P_{12\mu})$. In the (12) c.m.f., the angles (θ_{12},ϕ_{12}) give the orientation of $\hat{p}_1$ relative to the standard orientation defined as above except that here

[39] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[40] J. Boguta, Physikalisches Institut Universitat Bonn Report, 1969 (unpublished).

[41] G. Altarelli and H. R. Rubinstein, Phys. Rev. **183**, 1469 (1969).

[42] R. L. Peierls, Phys. Rev. Letters **6**, 641 (1961). Also see Ref. 6.

[43] A. Pagnamenta, Nucl. Phys. **87**, 801 (1967). His main example actually has the wrong sign for the imaginary part of the inverse propagator to be interpreted as a resonance. Some minus-sign errors hide this fact.

[44] This is the normalization when there are no identical particles. When the No. 1 and No. 3 are identical, there will be a second term which is the same as the first except that 1(3) is replaced by 3(1) in all the unprimed variables.

it is defined with reference to the (12) c.m.f. Note that the two standard orientations have the same $\hat{z}$ and are related by a Lorentz transformation along $\hat{z}$. This choice makes it possible to treat the $\hat{z}$ projection of the two-particle angular momentum $j=j_{12}$ [measured in the (12) c.m.f.] as a helicity for the (12) system. The following identity is easily proved:

$$\int \prod_{i=1}^{3} \frac{d^3p_i}{2p_i^0} = \frac{1}{4}\int d^4P \int_{(m_1+m_2)^2}^{(M-m_3)^2} ds_{12} \frac{p_{12}q_3}{m_{12}M} \times \int d(\cos\theta_{12}) \int d\phi_{12} \int d(\cos\theta_3) \int d\phi_3 , \quad \text{(B2)}$$

where[45]

$$m_{12}^2 = s_{12}, \quad p_{12} = \gamma(s_{12}, m_1^2, m_2^2)/2m_{12},$$

and

$$q_3 = \gamma(M^2, s_{12}, m_3^2)/2M,$$

with

$$\gamma(a,b,c) = [a^2+b^2+c^2-2(ab+ac+bc)]^{1/2}.$$

We introduce the following notation:

$$\int_{-1}^{1} d(\cos\theta_{12}) \int_0^{2\pi} d\phi_{12} \equiv \int d\Omega_{12}$$

and

$$\int_0^{2\pi} d\phi_3 \int_{-1}^{1} d(\cos\theta_3) \int_0^{2\pi} d\phi_{12} = \int d\Omega_3 \int_0^{2\pi} d\phi_{12} \equiv \int dU_{R_3}.$$

The last grouping is useful because the rotation (operator)

$$\tilde{R}_3 \equiv \tilde{R}(\phi_3, \theta_3, \phi_{12}) = e^{-i\tilde{J}_z\phi_3} e^{-i\tilde{J}_y\theta_3} e^{-i\tilde{J}_z\phi_{12}}$$

maps a state in the standard orientation in the (12)3 c.m.f. given by $|P,0,0,s_{12},\theta_{12},0\rangle$ into the state $|P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}\rangle$. Equation (B2) and the requirement that Eq. (9) holds fix the normalization of these states to be[44]

$$\langle P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12} | P',\theta_3',\phi_3',s_{12}',\theta_{12}',\phi_{12}'\rangle = (4M'm_{12}'/p_{12}'q_3')\delta^{(4)}(P-P')\delta(s_{12}-s_{12}') \times \delta(\Omega_3-\Omega_3')\delta(\Omega_{12}-\Omega_{12}'),$$

where

$$\delta(\Omega-\Omega') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi').$$

A third set of variables is the angular momentum set $\{P,J,\Lambda,s_{12},j,\lambda\}$, where J is the three-particle angular momentum measured in the over-all c.m.f., $j=j_{12}$, while Λ and λ are, respectively, the $\hat{z}$ projections of J and j measured with respect to the standard orientation. Because of the definition of the coordinate systems, Λ and λ are helicities. We define the angular momentum state as a projection on the angle state

$$|P,J,\Lambda,s_{12},j,\lambda\rangle = N_J N_j \int dU_{R_3} \int d(\cos\theta_{12}) \times \bar{D}_{\Lambda}{}^{J}{}_{\lambda}(R_3) d_{\lambda}{}^{j}{}_{0}(\theta_{12}) | P,\theta_3,\phi_3,s_{12},\theta_{10},\phi_{10}\rangle,$$

where $N_L = [(2L+1)/4\pi]^{1/2}$ and $\bar{D}^J(R_3) = \bar{D}^J(\phi_3,\theta_3,\phi_{12})$ is the complex conjugate of the rotation matrix.[46] Equation (B3) also preserves Eq. (9), and using the orthogonality relation for the D's and d's, we find[44]

$$\langle P,J,\Lambda,s_{12},j_{12'},\lambda_{12} | P',J',\Lambda',s_{12}',j_{12}',\lambda_{12}'\rangle = \frac{4M'm_{12}'}{p_{12}'q_3'} \delta^{(4)}(P-P')\delta(s_{12}-s_{12}')\delta_{JJ'}\delta_{\Lambda\Lambda'}\delta_{j_{12}j_{12}'}\delta_{\lambda_{12}\lambda_{12}'}.$$

A couple of steps in the algebra for deriving the recoupling coefficient $\langle(32)1|(12)3\rangle$ are[19]

$$\bar{D}_{\Lambda}{}^{J}{}_{\lambda}(R_1) = \sum_{\mu} \bar{D}_{\Lambda}{}^{J}{}_{\mu}(R_3) D_{\lambda}{}^{J}{}_{\mu}(\pi,\chi,0) = (-1)^{\lambda} \sum_{\mu} \bar{D}_{\Lambda}{}^{J}{}_{\mu}(R_3) d_{\lambda}{}^{J}{}_{\mu}(\chi),$$

where χ is defined below. To get the final form of Eq. (16), we use the following symmetry relation for the d matrices:

$$(-1)^{\lambda} d_{\lambda}{}^{j'}{}_{0}(\theta_{32}) = d_0{}^{j'}{}_{\lambda}(\theta_{32}).$$

In deriving the recoupling coefficient for $\langle(32)2|(12)3\rangle$, we use

$$\bar{D}_{\Lambda}{}^{J}{}_{\lambda}(R_2) = \sum_{\mu} \bar{D}_{\Lambda}{}^{J}{}_{\mu}(R_3) d_{\mu}{}^{J}{}_{\lambda}(\chi'),$$

where χ' is defined below.

A little algebra leads to the following results for writing variables in the (32)1 basis or the (31)2 basis in terms of the (12)3 basis.

$$s_{32} = (2s_{12})^{-1}[(m_2^2-m_1^2)(M^2-m_3^2) - 4Mm_{12}p_{12}q_3\cos\theta_{12} + s_{12}(M^2+m_1^2+m_2^2+m_3^2-s_{12})],$$

$$s_{31} = (2s_{12})^{-1}[(m_1^2-m_2^2)(M^2-m_3^2) + 4Mm_{12}p_{12}q_3\cos\theta_{12} + s_{12}(M^2+m_1^2+m_2^2+m_3^2-s_{12})],$$

$$\cos\theta_{32} = -\hat{p}_3\cdot\hat{p}_1|_{(32)\text{ c.m.f.}} = (4Mm_{32}p_{32}q_1)^{-1}[(M^2-m_1^2)(m_2^2-m_3^2) + s_{32}(M^2+m_1^2+m_2^2+m_3^2-2s_{12}-s_{32})],$$

$$\cos\chi = \hat{p}_3\cdot\hat{p}_1|_{(12)3\text{ c.m.f.}} = (4M^2q_1q_3)^{-1}[s_{12}s_{32} + (M^2-m_1^2)s_{12} + (M^2-m_3^2)s_{32} - (M^2+2m_2^2-m_1^2-m_3^2)M^2 + m_1^2m_3^2],$$

$$\cos\theta_{31} = -\hat{p}_3\cdot\hat{p}_2|_{(31)\text{ c.m.f.}} = (\text{interchange 1 and 2 in } \cos\theta_{32}),$$

$$\cos\chi' = \hat{p}_3\cdot\hat{p}_2|_{(12)3\text{ c.m.f.}} = (\text{interchange 1 and 2 in } \cos\chi).$$

[45] $p_{12}=\frac{1}{2}|\mathbf{p}_1-\mathbf{p}_2|$ evaluated in the (12) c.m.f., and $q_3=|\mathbf{p}_3|$ evaluated in the (12)3 c.m.f.

[46] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).