forward direction with the ρ and A_2 alone will still hold. Also we would like to mention that Sertorio and Toller[10] have also considered a model consisting of the ρ and a conspiring ρ' to fit the π-p charge-exchange data. The present model differs from theirs mainly in the universal slope of the trajectories, the $t=0$ ρ' intercept, and the behavior of the residues. Also recently Delbourgo and Salam[11] have considered a Regge-pole supermultiplet theory whereby due to the higher assumed symmetry the helicity-flip and -nonflip residues are related. They consider the ρ contribution alone, however, and therefore their model gives zero polarization and predicts zero differential cross section (instead of dips) where α_ρ is a negative integer or zero.

In conclusion we point out that the present model, in the same spirit as in Ref. 3, is an attempt to tie up several ideas (conspiracy, exchange degeneracy, and now Veneziano-type residues, etc.) in a consistent way. The widely considered Regge-pole treatment of photoproduction[12] and np charge exchange[13] is in terms of the pion conspiracy idea. On the other hand, exchange degeneracy implies that the B trajectory should also conspire (if the pion trajectory does). The co-conspirator of the B trajectory is the ρ' which we have utilized here. Finally, we point out that the present ideas imply that a conspiring A_2' (which is the co-conspirator of the pion) of intercept near zero, together with the A_2 trajectory, should be capable of explaining the $\pi^- p \to \eta n$ data. This reaction is the subject of a future investigation.

[10] L. Sertorio and M. Toller, Phys. Rev. Letters **19**, 1146 (1967).
[11] R. Delbourgo and A. Salam, Phys. Letters **28B**, 497 (1969).
[12] See, for example, A. Ahmadzadeh, R. J. Jacob, and B. P. Nigam, Phys. Rev. **178**, 2284 (1969).
[13] See, for example, F. Arbab and J. Dash, Phys. Rev. **163**, 1603 (1967).

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Two-Variable Expansion of the Scattering Amplitude for any Mass and Crossing Symmetry for Partial Waves

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A two-variable expansion of the scattering amplitude for the process $a+b \to c+d$ is proposed, where a, b, c, and d are spinless particles of arbitrary mass. It is diagonal in angular momentum, displays the threshold and pseudothreshold behavior of partial waves, and leads to sum rules which contain a *finite* number of partial waves due to the crossing symmetry of the collision amplitude. The results of our previous work are recovered when the masses are equal. The reaction $\pi+N \to \pi+N$ is treated with the inclusion of nucleon spin. The expansion is valid over the Dalitz plot for a decay amplitude. A simple method to derive sum rules which relate a finite number of partial waves without the use of the two-variable expansion is also outlined.

I. INTRODUCTION

THIS paper formulates a generalization of some previous work on two-variable expansions of scattering amplitudes[1] to processes which involve spinless particles of arbitrary mass. The original investigation dealt with a system where the masses of the four particles were equal. The amplitude was expressed as a sum of polynomials of the variables s, t, and u which were orthogonal and complete for a suitable scalar product over the Mandelstam triangle. The basis was diagonal in angular momentum, revealed the existence of an infinite sequence of finite dimensional "crossing matrices" for partial waves, and displayed their threshold behavior.

The investigation in Ref. 1 relies on the observation that there is a partial differential operator $\mathcal{O}$ in the variables s, t, and u which commutes with the angular mo-

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[1] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, and P. Ramond, *ibid.* **175**, 1974 (1968); (c) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, *ibid.* **176**, 1700 (1968); (d) A. P. Balachandran and J. Nuyts, Nucl. Phys. **B9**, 81 (1969). See also (e) W. Montgomery, L. O'Raifeartaigh, and P. Winternitz, Rutherford Laboratory report, 1969 (unpublished); (f) A. R. White, University of Cambridge report, 1969 (unpublished); (g) R. Z. Roskies, Yale report, 1969 (unpublished), and J. Math. Phys. (to be published). A closely related work is that of Ref. 2. For an alternative approach, see N. J. Vilenkin and J. A. Smorodinsky, Zh. Eksperim. i Teor. Fiz. **46**, 1793 (1964) [English transl.: Soviet Phys.—JETP **19**, 1209 (1964)]; P. Winternitz, J. A. Smorodinsky, and M. Sheftel, Yadern. Fiz. **7**, 1325 (1968) [English transl.: Soviet J. Nucl. Phys. **7**, 785 (1968)], and Dubna report, 1968 (unpublished), and references contained therein.

mentum in all the channels and which is invariant under permutations of s, t, and u. The eigenvectors of $\mathcal{O}$ are the coordinates of the two-variable expansion. This operator approach, however, does not generalize easily[1c,2,3] when mass degeneracy is removed. An alternative method is therefore adopted in this paper. It was suggested by an idea due to Wilson[4] that regardless of the masses, the series should be defined by polynomials in the Mandelstam variables to preserve the finiteness of the angular momentum crossing relation. The next two sections contain its general theory. In Sec. II, we develop a set of rules for an *a priori* characterization of the basis. The design of the set leads to sum rules which involve a limited number of partial waves when the crossing symmetry of the collision amplitude is assumed. These sum rules (as well as a simple method of deriving results of this type without the construction of the two-variable expansion or of the basis) are also described here. In Sec. III, it is shown that such a basis almost always exists, and its construction is outlined. It is essentially unique, is diagonal in angular momentum, and exhibits the threshold and pseudothreshold behavior of partial waves. The expansion is valid on the Dalitz plot for a decay amplitude.[5]

Section IV illustrates the formalism for reactions of the type $a+a \to a+a$ and $a \to b+b+b$.[6] The results of Ref. 1(a) are recovered for the former. Systems with spin are also briefly studied by examining the specific instance of a process such as $\pi+N \to \pi+N$. The solution of the problem where the particles have arbitrary spins and masses will be treated in a future paper by W. Case, M. Modjtehedzadeh, and one of us (A. P. B.) some possible applications of the formalism are indicated.

In the final Section, Appendix A summarizes the general theory of orthogonal polynomials to the extent pertinent to the text. In Appendix B, the basis functions for reactions of the type $a+b \to a+b$ and their integral identities are expressed in terms of known special functions.

II. CHARACTERIZATION OF BASIS FUNCTIONS AND SUM RULES FOR PARTIAL WAVES

Let a, b, c, and d represent spinless particles with masses m_a, m_b, m_c, and m_d, respectively, and consider the reactions $a+b \to c+d$, $a+\bar{c} \to \bar{b}+d$, and $a+\bar{d} \to \bar{b}+c$. The bar identifies the antiparticle. If p_i is the four-momentum of particle i, the Mandelstam variables are

$$s=(p_c+p_d)^2, \quad t=(p_b-p_d)^2, \quad u=(p_b-p_c)^2, \quad (2.1)$$

where $s+t+u=m_a^2+m_b^2+m_c^2+m_d^2$. Let $P_{ab}(s)$ and $P_{cd}(s)$ denote the magnitudes of the incoming and outgoing center-of-mass momenta in the s channel, and let $P_{ac}(t)$, $P_{bd}(t)$, $P_{ad}(u)$, and $P_{bc}(u)$ denote the corresponding functions in the remaining channels. Then

$$[P_{ij}(x)]^2=[\Delta_{ij}(x)]^2/4x \quad (2.2)$$

if

$$[\Delta_{ij}(x)]^2=[x-(m_i+m_j)^2][x-(m_i-m_j)^2]. \quad (2.3)$$

The cosines of the center-of-mass scattering angles in the three channels are given by

$$z_s=\frac{s(t-u)+(m_a^2-m_b^2)(m_c^2-m_d^2)}{\Delta_{ab}(s)\Delta_{cd}(s)}, \quad (2.4a)$$

$$z_t=\frac{t(u-s)+(m_a^2-m_c^2)(m_d^2-m_b^2)}{\Delta_{ac}(t)\Delta_{bd}(t)}, \quad (2.4b)$$

$$z_u=\frac{u(s-t)+(m_a^2-m_d^2)(m_b^2-m_c^2)}{\Delta_{ad}(u)\Delta_{bc}(u)}. \quad (2.4c)$$

It is convenient to label the members of the basis with two discrete indices n, l (n, $l=0, 1, 2, \cdots$), in analogy with our previous work on the equal mass system.[1] [See, however, the discussion in Appendix B, Ref. 1(a).] Thus, the basis for the s-channel expansion is $\{\mathcal{S}_n^l(s,t)\}$, while those for the t- and u-channel expansions are $\{\mathcal{T}_n^l(s,t)\}$ and $\{\mathcal{U}_n^l(s,t)\}$.[7] We show below that it is sufficient to demand the following of the set $\{\mathcal{S}_n^l(s,t)\}$, for example, to replace the crossing symmetry equation by an equivalent set of partial-wave sum rules. Each of the latter will involve a finite number of angular momenta:

(*i*) *$\mathcal{S}_n^l$ must be diagonal in the s-channel angular momentum.* That is, it must have the form

$$\mathcal{S}_n^l(s,t)=\xi_l(s)S_n^l(s)P_l(z_s). \quad (2.5)$$

The notation with two factors ξ_l and S_n^l for the s-dependent part of $\mathcal{S}_n^l$ is convenient later. The factor ξ_l will be called the multiplier. It is clear that this condition is imposed so that the useful properties of the angular momentum basis are not lost in the new expansion.

(*ii*) *The partial-wave expansions of $\mathcal{S}_n^l$ in the t and u channels must terminate*[8]:

$$\mathcal{S}_n^l(s,t)=\sum_{L=0}^{L_t}(2L+1)\alpha_L(t)P_L(z_t) \quad (2.6a)$$

$$=\sum_{L=0}^{L_u}(2L+1)\beta_L(u)P_L(z_u). \quad (2.6b)$$

[2] J. M. Charap and B. M. Minton, J. Math. Phys. (to be published).

[3] It may be shown that there is no second-order differential operator which commutes with angular momentum even in two channels if the masses are arbitrary.

[4] K. Wilson (private communication).

[5] There is a fair amount of literature which deals with harmonic analysis on the Dalitz plot. Reference 1(a) contains a partial list. See also the recent work of K. E. Eriksson, Göteborg reports, 1968 (unpublished).

[6] Particles of the same mass are denoted by a common symbol. They are not necessarily identical.

[7] We have changed the notation somewhat from Refs. 1(a)–1(d).

[8] For equal masses, $\mathcal{S}_n^l$ is a finite linear combination of $\mathcal{T}_n^l$ and $\mathcal{U}_n^l$.[1] This result is false for general masses since $\alpha_L(t)$ and $\beta_L(u)$ are singular at the origin unlike the P_l projections $(2l+1)^{-1}\eta_l(t)\times T_n^l(t)$, $(2l+1)^{-1}\zeta_l(u)U_n^l(u)$ of $\mathcal{T}_n^l$ and $\mathcal{U}_n^l$ in the t and u channels (cf. Sec. III).

Here L_t and L_u are finite integer-valued functions of n, l. This condition is important for displaying the crossing properties of partial waves in a simple form.

(iii) $\{\mathcal{S}_n^l\}$ *must form a complete pairwise orthogonal system on a scalar product* $(\cdot,\cdot)$ *where* $(\cdot,\cdot)$ *fulfills the following identities:*

$$(f,g)=\int_R\int ds\,dt\,\rho(s,t)f^*(s,t)g(s,t) \tag{2.7a}$$

$$=\frac{1}{2}\int_{si}^{sf} ds\,\rho_s(s)\int_{-1}^{+1} dz_s f^*(s,t)g(s,t) \tag{2.7b}$$

$$=\frac{1}{2}\int_{ti}^{tf} dt\,\rho_t(t)\int_{-1}^{+1} dz_t f^*(s,t)g(s,t) \tag{2.7c}$$

$$=\frac{1}{2}\int_{ui}^{uf} du\,\rho_u(u)\int_{-1}^{+1} dz_u f^*(s,t)g(s,t)\,. \tag{2.7d}$$

Here R is the integration domain. We comment briefly on the conditions (2.7b)–(2.7d). In the variables s and z_s, the measure $ds\,dt\rho(s,t)$ reads, in general, $\frac{1}{2}ds\,dz_s\rho_s(s,z_s)$. But since we know that the Legendre polynomials are orthogonal for the measure dz_s and the interval $[-1,+1]$, we require $\rho_s(s,z_s)$ to be independent of z_s and R to be bounded by $z_s^2=1$ for each fixed s. The importance of the symmetry of the measure and of R implied by (2.7c) and (2.7d) will soon become evident. Thus, R is the domain where $-1\leqslant z_s\leqslant+1$ (for fixed s), $-1\leqslant z_t\leqslant+1$ (for fixed t), and $-1\leqslant z_u\leqslant+1$ (for fixed u).

The existence of the set $\{\mathcal{S}_n^l\}$ which fulfills (i)–(iii) as well as the existence of the domain R with the requisite geometry will be studied in Sec. III. Analogous conditions are also to be imposed on the sets $\{\mathcal{T}_n^l\}$ and $\{\mathcal{U}_n^l\}$.

The square of the norm of $\mathcal{S}_n^l$ will be denoted by $(2l+1)^{-1}N_n^l$:

$$(\mathcal{S}_n^l,\mathcal{S}_N^L)=(2l+1)^{-1}N_n^l\delta_{nN}\delta_{lL}\,. \tag{2.8}$$

If $F(s,t)$ is the total amplitude and $(F,F)<\infty$, we may write

$$F(s,t)=\sum_{n,l=0}^{\infty}(2l+1)a_n^l\mathcal{S}_n^l(s,t)\,. \tag{2.9a}$$

The partial-wave expansion of F is

$$F(s,t)=\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(z_s) \tag{2.9b}$$

if $f_l(s)$ is the lth partial wave in the s channel. It follows that

$$f_l(s)=\xi_l(s)\sum_{n=0}^{\infty}a_n^lS_n^l(s)\,, \tag{2.9c}$$

where, due to the normalization (2.8), a_n^l is given by

$$N_n^la_n^l=(\mathcal{S}_n^l,F)$$
$$=\int_{si}^{sf} ds\,\rho_s(s)\xi_l(s)S_n^l(s)f_l(s)\,. \tag{2.10}$$

Now suppose that because of s-t crossing symmetry,

$$F(s,t)=G(t,s)\,. \tag{2.11}$$

Let $g_L(t)$ be the t-channel partial wave. The scalar product of (2.11) with $\mathcal{S}_n^l$ yields

$$\int_{si}^{sf} ds\,\rho_s(s)\xi_l(s)S_n^l(s)f_l(s)$$
$$=\frac{1}{2}\int_{ti}^{tf} dt\,\rho_t(t)\int_{-1}^{+1} dz_t\Big[\sum_{L=0}^{L_t}(2L+1)\alpha_L(t)P_L(z_t)\Big]G(t,s)$$
$$=\int_{ti}^{tf} dt\,\rho_t(t)\Big[\sum_{L=0}^{L_t}(2L+1)\alpha_L(t)g_L(t)\Big], \tag{2.12}$$

where we used (2.7c), (2.6a), and the definition of g_L. This is a crossing relation for partial waves which is *finite dimensional* in the s- and t-channel angular momenta. The set of all such relations when n,l span their range is equivalent to (2.11) because of the completeness of $\mathcal{S}_n^l$. It is shown in Sec. III that $L_{t,u}=n+2l$ if there are no mass degeneracies. In any event, $L_{t,u}\leqslant n+2l$.

When the primary interest is in sum rules of the type (2.12), it may not be necessary to construct either the two-variable expansion or the basis $\{\mathcal{S}_n^l\}$. Instead, we may take the scalar product of (2.11) with a suitable polynomial $P(s,t)$ in s and t, expand it in turn as a (finite) sum of partial waves in the s and t channels and use the symmetry of the measure. The completeness of the set of all polynomials in s and t in the scalar product $(\cdot,\cdot)$ implies the equivalence of these sum rules and (2.12).

III. CONSTRUCTION OF THE BASIS

We first take up Eqs. (2.5) and (2.6). As the variable z_s is linear in t for fixed s, the function $\mathcal{S}_n^l$ must be a polynomial in t for fixed s because of (2.5). But, for a similar reason, (2.6a) requires it to be a polynomial in s for fixed t, so that $\mathcal{S}_n^l$ *must be a polynomial in s and t.* Conversely, if $\mathcal{S}_n^l$ is of the form (2.5) and is a polynomial in s and t, it fulfills (2.6a). Because a polynomial in s and t is a polynomial in s and u as well, (2.6b) leads to nothing new. Such results may have been anticipated from the remarks at the end of Sec. II.

The Legendre polynomial which appears in (2.5) has singularities in s for each t because of the singularities of z_s at the zeros of the Δ functions in (2.4a). But, $P_l(z_s)$ contains only even (odd) powers of z_s when l is even (odd) so that $[\Delta_{ab}(s)\Delta_{cd}(s)]^lP_l(z_s)$ is a polynomial in s and t. We therefore set[9]

$$\xi_l(s)=[\Delta_{ab}(s)\Delta_{cd}(s)]^l\,, \qquad (3.1a)$$

$$\mathcal{S}_n{}^l(s,t)=[\Delta_{ab}(s)\Delta_{cd}(s)]^l S_n{}^l(s)P_l(z_s)\,, \qquad (3.1b)$$

where $S_n{}^l(s)$ is a polynomial of degree n in s.

In general, the basis function $\mathcal{S}_n{}^l$ is a polynomial of degree l in t and $n+2l$ in s. As a result, the crossing relations of Sec. II relate a partial wave with angular momentum l in the s channel to those with $L\leqslant L_{t,u}=n+2l$ in the t and u channels. (If there are mass degeneracies, it is possible that $L_{t,u}<n+2l$.[9])

The expansion of a scattering amplitude in a series of $\mathcal{S}_n{}^l$ will display the threshold and pseudothreshold behavior of the partial waves because of the multiplier ξ_l. We recall that any partial wave which is derived from a polynomial in s and t is characterized by these "centrifugal zeros."

Next consider Eqs. (2.7). Locally they imply the equalities $ds\;\rho_s(s)dz_s=dt\;\rho_t(t)dz_t=du\;\rho_u(u)dz_u$, or in terms of s, t, and u,

$$ds\,dt\frac{s\rho_s(s)}{\Delta_{ab}(s)\Delta_{cd}(s)}=ds\,dt\frac{t\rho_t(t)}{\Delta_{ac}(t)\Delta_{bd}(t)}=dsdu\frac{u\rho_u(u)}{\Delta_{ad}(u)\Delta_{bc}(u)}\,, \qquad (3.2)$$

where, in the end, all the integrals are supposed to be in the increasing direction of the variables. Thus, $\rho(s,t)\equiv 1$ if we choose a possible over-all constant to be 1, and[10]

$$\rho_s(s)=\Delta_{ab}(s)\Delta_{cd}(s)/s\,, \qquad (3.3a)$$

$$\rho_t(t)=\Delta_{ac}(t)\Delta_{bd}(t)/t\,, \qquad (3.3b)$$

$$\rho_u(u)=\Delta_{ad}(u)\Delta_{bc}(u)/u\,. \qquad (3.3c)$$

We have yet to specify the region R of integration. It will, of course, be such that the ρ's are real and of constant signs. In terms of the variables s and z_s, R must be the direct product of two intervals $[s_i,s_f]$ and $[-1,+1]$ while it must have a similar decomposition in the remaining pairs t,z_t and u,z_u. We distingush two possibilities for R:

(*a*) *Decay process.* Suppose that one of the particles, say, a, can decay into b, c, and d. The momentum of b is then $-p_b$, while P_{cd} for instance denotes the magnitude of the three-momentum of c or d in the c-d center-of-mass frame. The appearance of $P_l(z_s)$ in the decay amplitude signals an angular momentum state l for the c-d system.

The Dalitz plot serves as the region R. In the variables s, z_s,

$$R=[(m_c+m_d)^2,(m_a-m_b)^2]\otimes[-1,\,+1]\,. \qquad (3.4)$$

The form of R in t, z_t or u, z_u is obtained by permutation. This result is rather ancient.

(*b*) *Scattering process.* Here, it is assumed that each of the particles is stable against decay into the remaining three. We claim that R is the so-called Euclidean region where the momenta $P_{ij}(x)$ $(x=s,\,t,\,u)$ are imaginary or zero. The proof is deferred to the end of this section. Thus, in terms of s and z_s,

$$R=[\max\{(m_a-m_b)^2,(m_c-m_d)^2\},\; \min\{(m_a+m_b)^2,(m_c+m_d)^2\}]\otimes[-1,\,+1]\,. \qquad (3.5)$$

There are similar expressions for R in the t- and u-channel variables. The factor $[s_i,s_f]$ in (3.5) is the intersection of the intervals where P_{ab} and P_{cd} are imaginary or zero and requires $s_i<s_f$. But if, for example,

$$(m_c-m_d)^2>(m_a-m_b)^2\,,\quad (m_a+m_b)^2<(m_c+m_d)^2\,,$$

and

$$(m_c-m_d)^2\geqslant(m_a+m_b)^2\,,$$

then c or d is unstable, which is contrary to our hypothesis.[11]

The construction of the polynomials $S_n{}^l$ proceeds as follows: From

$$\frac{1}{2}\int_{s_i}^{s_f}ds\;\rho_s(s)\int_{-1}^{+1}dz_s\mathcal{S}_n{}^l(s,t)\mathcal{S}_N{}^L(s,t)=(2l+1)^{-1}\mathcal{N}_n{}^l\delta_{nN}\delta_{lL}$$

and from (3.1) and (3.3), we have

$$\int_{s_i}^{s_f}ds\frac{[\Delta_{ab}(s)\Delta_{cd}(s)]^{2l+1}}{s}S_n{}^l(s)S_N{}^l(s)=\mathcal{N}_n{}^l\delta_{nN}\,. \qquad (3.6)$$

The set $\{S_n{}^l\}$ is determined by the Gram-Schmidt orthogonalization of polynomials. The solution is described in Appendix A.[12]

[9] There is an ambiguity here. $\mathcal{S}_n{}^l$ and ξ_l can be multiplied by any polynomial in s which depends only on l and which has no zeros in $[s_i,s_f]$ (cf. Ref. 12). However: (i) Such a polynomial increases the degree of $\mathcal{S}_n{}^l$ in s for fixed t, and hence $L_{t,u}$; (ii) by displaying possibly nonexistent zeros in partial waves, it tends to decrease the rate of convergence of the series (2.9). The choice (3.1) thus seems most appropriate. It should also be remarked that if $m_a=m^q$ and/or $m_c=m_d$, then z_s is less singular than $[\Delta_{ab}(s)\Delta_{cd}(s)]^{-1}$, and the most economical form of the multiplier is not the one suggested by (3.1) [cf. Sec. IV]. The choice of ξ_l which leads to an $\mathcal{S}_n{}^l$ with the least power in s for fixed t may be called the principle of the minimal multiplier. One sees that once this principle is used, the functions $\{\mathcal{S}_n{}^l\}$ given by (3.1) do not reduce to those for the degenerate problems ($m_a=m_b$ and/or $m_c=m_d$) under the appropriate limits on the masses. We list the exceptional minimal multipliers: (i) $\xi_l=(s-4m_a{}^2)^{l/2}(s-4m_c{}^2)^{l/2}$ if $m_a=m_b$, $m_c=m_d$ [cf. Eq. (4.2)]; (ii) $\xi_{2l}=[\Delta_{ab}(s)]^{2l}(s-4m_c{}^2)^l$ and $\xi_{2l+1}=[\Delta_{ab}(s)]^{2l+1}(s-4m_c{}^2)^{l+1/2}\times s^{-1/2}$ if $m_a\neq m_b$, $m_c=m_d$ [cf. Eq. (4.6)]. There are similar expressions for ξ_l when $m_a=m_b$, $m_c\neq m_d$.

[10] We choose that determination of the square roots which leads to a non-negative ρ_x over R.

[11] The domain of integration R disappears if (a) the mass of one of the particles is equal to the sum of the other three, or (b) at least one of the masses, say, m_d, is zero and none of the remaining masses is greater than the sum of the others [that is, if $(m_a-m_b)^2\leqslant m_c{}^2\leqslant(m_a+m_b)^2$]. Our analysis does not apply in these situations.

[12] We sketch a proof of completeness of $\{\mathcal{S}_n{}^l\}$. The set of all polynomials $P(s,t)$ in s and t is dense in the Hilbert space with the scalar product $(\cdot,\cdot)$. $P(s,t)$ has a terminating expansion in $P_l(z_s)$. The corresponding partial wave $p_l(s)$ displays the correct threshold behavior and so $p_l(s)/\xi_l(s)$ is L^2 for the measure $ds[\Delta_{ab}(s)\Delta_{cd}(s)]^{2l+1}/s$ and the interval $[s_i,s_f]$. However, the set of all polynomials in s and hence $\{S_n{}^l\}$ are complete for such L^2 functions.

We indicate a proof that R has product decompositions of the form (3.5) in the three channels. The scattering process alone is considered since the result is known for the Dalitz plot. Further, the proof, in reality, is valid for the latter if the group SO(4) is replaced by SO(3,1). It is adequate to show, for instance, that when s,z_s span a dense domain in $[s_i,s_f]\otimes[-1,+1]$ [that is, a domain whose complement in R is a set of zero measure for the scalar product $(\cdot,\cdot)$], then t,z_t are mapped onto a dense domain in $[t_i,t_f]\otimes[-1,+1]$ as our concern is in integrals over R. Let, $s,z_s\in[s_i,s_f]\otimes[-1,+1]$. The class $\mathcal{C}_s=\{p_a,p_b,p_c,p_d\}$ of the s-channel center-of-mass momenta which generate the given s,z_s are of the form

$$p_a=((s+m_a^2-m_b^2)/2s^{1/2},\mathbf{P}_{ab}(s)),\quad |\mathbf{P}_{ab}(s)|=|P_{ab}(s)| \tag{3.7}$$

$$p_c=((s+m_c^2-m_a^2)/2s^{1/2},\mathbf{P}_{cd}(s)),\quad |\mathbf{P}_{cd}(s)|=|P_{cd}(s)|,\quad \text{etc.},$$

where

$$\mathbf{P}_{ab}(s)\cdot\mathbf{P}_{cd}(s)=P_{ab}(s)P_{cd}(s)z_s. \tag{3.8}$$

Since P_{ab}, P_{cd} are imaginary or zero and the time components of p_i are real, we can factor an i out of the $\mathbf{P}_{ab}$, $\mathbf{P}_{cd}$ and treat the four-momenta and their scalar products as Euclidean. This is understood hereafter in this section.

The variables t, z_t can be expressed as functions of the four-dimensional scalar products of p_i through (2.1) and (2.4b). Let us change variables from p_i to

$$q_{i\mu}=\Lambda_{\mu\nu}p_{i\nu}, \tag{3.9}$$

where $\Lambda\in SO(4)$ and for each member of $\mathcal{C}_s$ choose a Λ so as to reach the t-channel center-of-mass system. The momenta q_i are Euclidean and fulfill the mass shell and energy-momentum conservation constraints, while t,z_t are the same functions of the q_i as they were of the p_i. We may therefore regard z_t as the cosine of the scattering angle and t as the square of the total energy of the Euclidean reaction $q_a-q_c\to -q_b+q_d$ in this system, and so $t,z_t\in[t_i,t_f]\otimes[-1,+1]$. Thus, each $s,z_s\in[s_i,s_f]\otimes[-1,+1]$ denotes a $t,z_t\in[t_i,t_f]\otimes[-1,+1]$. Since the converse is also true, the result follows.

There is a fault in the reasoning when any one of the momenta $P_{ij}(x)$ vanishes and the various transformations become singular. But such singularities occur only when s, t, or u assumes one of its extremal values in R (that is, on the boundary of R), and such surfaces of lower dimensionality than R itself may be ignored in proving integral identities over R.

IV. EXAMPLES

(*i*) *The process* $a+a\to a+a$. When the masses are equal, the functions z_x are defined by

$$z_s=\frac{t-u}{s-4m_a^2},\quad z_t=\frac{u-s}{t-4m_a^2},\quad z_u=\frac{s-t}{u-4m_a^2}. \tag{4.1}$$

The multiplier $[\Delta_{aa}(s)]^{2l}$ of (3.1) contains an excessive number of powers of s [9] since we may write

$$\mathcal{S}_n{}^l(s,t)=(s-4m_a^2)^l S_n{}^l(s)P_l(z_s), \tag{4.2}$$

where $S_n{}^l$ is a polynomial of degree n in s. The domain R is the Mandelstam triangle, the measure ρ_s is $(4m_a^2-s)$ (there are similar expressions for ρ_t and ρ_u), and the defining equation (3.6) for $S_n{}^l$ reduces to

$$\int_0^{4m_a^2} ds(4m_a^2-s)^{2l+1}S_n{}^l(s)S_N{}^l(s)=N_n{}^l\delta_{nN}. \tag{4.3a}$$

Therefore,[13]

$$S_n{}^l(s)\propto P_n{}^{(2l+1,0)}[(s-2m_a^2)/2m_a^2], \tag{4.3b}$$

which is the fundamental result of Ref. 1(a) for the s channel. $\mathcal{S}_n{}^l$ is a polynomial of degree $n+l$ in s for fixed t or u, so that $L_{t,u}=n+l$ in the crossing relations.

The t and u channels are treated by symmetry.

(*ii*) *The process* $a\to b+b+b$. The significance of the variables was explained in Sec. III. In terms of the momenta [cf. (2.2)]

$$P_{bb}(x)=\tfrac{1}{2}(x-4m_b^2)^{1/2}. \tag{4.4}$$

The z_x's are given by

$$z_s=s^{1/2}(t-u)/2P_{bb}(s)\Delta_{ab}(s),\quad \text{etc.} \tag{4.5}$$

The multiplier is different when l is even and odd[9]:

$$\mathcal{S}_n{}^l(s,t)=\{[P_{bb}(s)\Delta_{ab}(s)]^l/s^{\epsilon_l/2}\}S_n{}^l(s)P_l(z_s), \tag{4.6a}$$

where

$$\begin{aligned}\epsilon_l&=0,\quad l\text{ even}\\ &=1,\quad l\text{ odd.}\end{aligned} \tag{4.6b}$$

The domain of expansion is the Dalitz plot and[10]

$$\rho_s(s)=P_{bb}(s)\Delta_{ab}(s)/s^{1/2}. \tag{4.7}$$

The polynomials $S_n{}^l$ are fixed by requiring that they should be of precise degree n and fulfill

$$\int_{4m_b^2}^{(m_a-m_b)^2} ds\frac{[P_{bb}(s)\Delta_{ab}(s)]^{2l+1}}{s^{1/2+\epsilon_l}}S_n{}^l(s)S_N{}^l(s)=N_n{}^l\delta_{nN}. \tag{4.8}$$

The integers $L_{t,u}$ of (2.6) are determined by the degree of $\mathcal{S}_n{}^l$ in s for fixed t or u:

$$L_{t,u}=n+l+\tfrac{1}{2}(l-\epsilon_l). \tag{4.9}$$

(*iii*) *The process* $\pi+N\to\pi+N$. This is an example of a process with spin. The crossing relations are simple in terms of the A and B amplitudes,[14] since

$$A^{\pm}(s,u)=\pm A^{\pm}(u,s),\quad B^{\pm}(s,u)=\mp B^{\pm}(u,s), \tag{4.10}$$

[13] The Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. II, p. 168.

[14] The notation is standard. See, for example, J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

etc. We shall therefore develop a suitable basis for the expansion of $A^{\pm}$, say.[15] For brevity, only the s-u crossing relation (4.10) will be treated.

The projection

$$h_l^{\pm}(s)=\int_{-1}^{+1} dz_s(1-z_s^2)P_l'(z_s)A^{\pm}(s,u) \quad (4.11)$$

consists of a finite number of partial waves:

$$h_l^{\pm}(s)=8\pi\frac{l(l+1)}{2l+1}\left(\frac{W+m}{E+m}(f_{(l-1)+}{}^{\pm}-f_{(l+1)-}{}^{\pm}) - \frac{W-m}{E-m}(f_{l-}{}^{\pm}-f_{l+}{}^{\pm})\right). \quad (4.12)$$

[This is shown by expanding $A^{\pm}$ as a series in $f_{l\pm}{}^{\pm}$ and using (4.16).] It is thus adequate to construct a basis $\{\mathcal{S}_n{}^l\}$ which can relate a finite number of $h_l^{\pm}$ when (4.10) is assumed. We set

$$\mathcal{S}_n{}^l(s,t)=\xi_l(s)S_n{}^l(s)P_l'(z_s)\,,\qquad n=0,1,2,\cdots;\quad l=1,2,3,\cdots. \quad (4.13)$$

Experience with spinless systems suggests the hypothesis

$$\mathcal{S}_n{}^l(s,u)=\sum_{L=1}^{L_u}\frac{2L+1}{2L(L+1)}\beta_L(u)P_L'(z_u),\quad L_u<\infty \quad (4.14)$$

while for the scalar product $(\cdot,\cdot)$ on which $\mathcal{S}_n{}^l$ are orthogonal,

$$\begin{aligned}(f,g)&=\int_R\int ds\,du\,\rho(s,u)f^*(s,u)g(s,u)\\ &=\int_{s_i}^{s_f} ds\,\rho_s(s)\int_{-1}^{+1}dz_s(1-z_s^2)f^*(s,u)g(s,u)\\ &=\int_{u_i}^{u_f} du\,\rho_u(u)\int_{-1}^{+1}dz_u(1-z_u^2)f^*(s,u)g(s,u)\end{aligned} \quad (4.15)$$

since $\{P_l'(z)\}$ is an orthogonal system on the measure $dz(1-z^2)$ and the interval $[-1,+1]$:

$$\int_{-1}^{+1}dz(1-z^2)P_l'(z)P_L'(z)=\frac{2l(l+1)}{2l+1}\delta_{lL}. \quad (4.16)$$

These properties of $\mathcal{S}_n{}^l$ are sufficient to derive finite-dimensional crossing relations for $h_l^{\pm}$. The calculation which led to (2.12) now shows that

$$\int_{s_i}^{s_f} ds\,\rho_s(s)\xi_l(s)S_n{}^l(s)h_l^{\pm}(s) =\pm\int_{u_i}^{u_f}du\,\rho_u(u)\left(\sum_{L=1}^{L_u}\frac{2L+1}{2L(L+1)}\beta_L(u)h_L^{\pm}(u)\right). \quad (4.17)$$

[15] For the B amplitudes, the nucleon pole terms must first be subtracted before the analysis is applied.

We proceed with the construction of $\mathcal{S}_n{}^l$. It must be a polynomial in s and u because of (4.13) and (4.14). Since

$$z_s=[s(t-u)+(m^2-\mu^2)^2]/\Delta_{\pi N}(s)^2\,, \quad (4.18a)$$

$$[\Delta_{\pi N}(s)]^2=[s-(m+\mu)^2][s-(m-\mu)^2]\,, \quad (4.18b)$$

where μ and m are pion and nucleon masses, the multiplier is given by

$$\xi_l(s)=\{[\Delta_{\pi N}(s)]^2\}^{(l-1)}\,, \quad (4.19)$$

and $S_n{}^l$ is a polynomial of degree n. The domain R is the Euclidean region:

$$\begin{aligned}R&=\{s\,|\,s\in[(m-\mu)^2,(m+\mu)^2]\}\otimes\{z_s|z_s\in[-1,+1]\} \quad (4.20a)\\ &=\{u\,|\,u\in[(m-\mu)^2,(m+\mu)^2]\}\otimes\{z_u|z_u\in[-1,+1]\}\,. \quad (4.20b)\end{aligned}$$

It remains to find ρ_s and ρ_u. If $S=p_a+p_b$, $T=p_a-p_c$, $U=p_a-p_d$ in the notation of Sec. II, their Gram determinant

$$\Phi=\det\begin{vmatrix}S^2 & S\cdot T & S\cdot U\\ T\cdot S & T^2 & T\cdot U\\ U\cdot S & U\cdot T & U^2\end{vmatrix} \quad (4.21)$$

is the Kibble function.[16] The surface $\Phi=0$ bounds the physical regions, and it is known that

$$\Phi=4s[P(s)]^4(1-z_s^2) \quad (4.22a)$$

$$=4u[P(u)]^4(1-z_u^2)\,, \quad (4.22b)$$

where

$$[P(x)]^2=[\Delta_{\pi N}(x)]^2/4x. \quad (4.22c)$$

Thus

$$\rho(s,u)=8\Phi \quad (4.23a)$$

and

$$\rho_s(x)=\rho_u(x)=\{-[\Delta_{\pi N}(x)]^2\}^3/x^2. \quad (4.23b)$$

The integrals are supposed to be in the positive sense. The equations

$$\int_{(m-\mu)^2}^{(m+\mu)^2}ds\frac{\{-[\Delta_{\pi N}(s)]^2\}^{2l+1}}{s^2}S_n{}^l(s)S_N{}^l(s) =N_n{}^l\delta_{nN} \quad (4.24)$$

determine the polynomials $S_n{}^l$ of degree n uniquely up to a normalization.

It is possible to develop simple formulas for $S_n{}^l$ in terms of known special functions. The corresponding polynomials for the spinless reaction $a+b\to a+b$ are expressed as linear combinations of Jacobi polynomials in Appendix B. The same method leads to an identity between the π-N and the a-b functions.

V. CONCLUDING REMARKS

We shall finally indicate some possible applications of the preceding formalism. The uses of suitable two-

[16] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

variable expansions for the storage of data on the Dalitz plot have been extensively discussed by many authors.[5] The basis we have developed has many good physical properties, since it is diagonal in the two-body angular momentum and displays the threshold and crossing properties of partial waves. It should therefore be effective in this context. For a scattering amplitude, the expansion is valid in an unphysical region R. If at least the first few partial waves can therefore be parametrized in a form which is approximately valid inside R as well as at low scattering energies, the subset of crossing relations in (2.12) which involve only these waves will lead to constraints on the parameters and to verifiable predictions. The difficulty seems to be in devising satisfactory parametrizations. An alternative and more rigorous approach would be to try to use these crossing relations in conjunction with the positivity properties of the scattering amplitude perhaps along the lines due to Martin.[17] We understand that such work is in progress by Roskies[1g] and by Wanders and co-workers.

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APPENDIX A: PROPERTIES OF ORTHOGONAL POLYNOMIALS

We summarize some general results from the theory of orthogonal polymonials of one real variable in this appendix.[18]

Given an interval $[a,b]$ and a weight function $\omega(x)$ non-negative on $[a,b]$,[10] let

$$(f,g)=\int_a^b dx\,\omega(x)f^*(x)g(x). \qquad \text{(A1)}$$

A polynomial $q_n(x)$ of precise degree n is an orthogonal polynomial of degree n with respect to ω if

$$\int_a^b dx\,\omega(x)x^\nu q_n(x)=0\,, \qquad \nu=0,1,2,\cdots,n-1;\quad n=1,2,3,\cdots \qquad \text{(A2)}$$

[17] A. Martin, CERN report No. TH. 1008-CERN, 1969 (unpublished).

[18] The standard reference is G. Szëgo, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939). See also Ref. 13, p. 153.

or

$$(q_n,q_m)\propto\delta_{nm},\quad n,m=0,1,2,\cdots. \qquad \text{(A3)}$$

The orthogonal polynomials q_n are unique up to a normalization and can be determined by the Gram-Schmidt orthogonalization of polynomials. The following explicit construction is known: Let

$$\mu_\nu=\int_a^b dx\,\omega(x)x^\nu \qquad \text{(A4)}$$

be the moments of ω. Then,

$$q_0(x)=\lambda_0\,, \qquad \text{(A5a)}$$

$$q_n(x)=\lambda_n\det\begin{vmatrix}1 & x & x^2 & \cdots & x^n\\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n\\ & & \cdots & & \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1}\end{vmatrix}, \qquad n=1,2,3,\cdots \qquad \text{(A5b)}$$

where $\lambda_n(\neq 0)$ are constants. The proof is as follows. The determinant defines a polynomial of precise degree n since it may be shown that the coefficient of $(-x)^n$ is positive for all $n\geqslant 1$ (unless ω is a finite sum of δ functions). It also fulfills (A2) because of the coincidence of its rows. By uniqueness, it may be identified with q_n modulo a multiplicative constant.

The norm of q_n is given by

$$(q_n,q_n)=\lambda_n{}^2\Delta_n(n)\Delta_{n+1}(n+1)\,, \qquad \text{(A6)}$$

where

$$\Delta_p(n)=\det\begin{vmatrix}\mu_0 & \mu_1 & \cdots & \mu_{p-1} & \mu_{p+1} & \cdots & \mu_n\\ \mu_1 & \mu_2 & \cdots & \mu_p & \mu_{p+2} & \cdots & \mu_{n+1}\\ & & & \cdots & & & \\ \mu_{n-1} & \mu_n & \cdots & \mu_{n+p-2} & \mu_{n+p} & \cdots & \mu_{2n-1}\end{vmatrix} \qquad \text{(A7)}$$

and the λ_n's are taken to be real. To show this, expand one of the q_n's in a power series in x using (A5b), and invoke (A2).

The numerical solution of q_n is often simplified by the recurrence relation[19]

$$q_{n+1}(x)=\frac{\lambda_{n+1}}{\lambda_n}\frac{\Delta_{n+1}(n+1)}{\Delta_n(n)}\times\left(x+\frac{\Delta_{n-1}(n)}{\Delta_n(n)}-\frac{\Delta_n(n+1)}{\Delta_{n+1}(n+1)}\right)\cdot q_n(x)-\frac{\lambda_{n+1}}{\lambda_{n-1}}\left(\frac{\Delta_{n+1}(n+1)}{\Delta_n(n)}\right)^2 q_{n-1}(x),\quad n=1,2,3,\cdots. \qquad \text{(A8)}$$

[19] Reference 13, p. 158.

APPENDIX B: POLYNOMIALS S_n^l FOR THE PROCESS $a+b\rightarrow a+b$

The polynomials S_n^l for the spinless process $a+b\rightarrow a+b$ which are of precise degree n and fulfill

$$\int_{(m_a-m_b)^2}^{(m_a+m_b)^2} ds\frac{\{-[\Delta_{ab}(s)]^2\}^{2l+1}}{s}s^\nu S_n^l(s)=0,$$
$$\nu=0,1,2,\cdots,n-1;\quad n=1,2,3,\cdots \quad \text{(B1)}$$

have a simple representation in terms of Jacobi polynomials. We prove the following: Let

$$\tau_0=(m_a^2+m_b^2)/2m_am_b, \quad \text{(B2a)}$$
$$\tau=(s-m_a^2-m_b^2)/2m_am_b. \quad \text{(B2b)}$$

Then, with a choice of normalization,

(i)
$$S_0^l(s)\equiv 1, \quad \text{(B3a)}$$
$$S_n^l(s)=[1/2^{2l+1}(n+1)_{2l}][Q_{n-1}^{(2l+1,2l+1)}(\tau_0)P_n^{(2l+1,2l+1)}(\tau)+Q_n^{(2l+1,2l+1)}(\tau_0)P_{n-1}^{(2l+1,2l+1)}(\tau)],$$
$$n=1,2,3,\cdots \quad \text{(B3b)}$$

(ii)
$$\int_{(m_a-m_b)^2}^{(m_a+m_b)^2} ds\frac{\{-[\Delta_{ab}(s)]^2\}^{2l+1}}{s}[S_n^l(s)]^2$$
$$=2(2m_am_b)^{4l+2}(\tau_0^2-1)^{2l+1}Q_0^{(2l+1,2l+1)}(\tau_0),\quad n=0 \quad \text{(B4a)}$$
$$=2(2m_am_b)^{4l+2}\frac{n+l+1}{(n+1)_{4l+2}}Q_{n-1}^{(2l+1,2l+1)}(\tau_0)\times Q_n^{(2l+1,2l+1)}(\tau_0),\quad n=1,2,3,\cdots \quad \text{(B4b)}$$

where

$$(a)_\nu\equiv\Gamma(a+\nu)/\Gamma(\nu).$$

To show (B3), we have to prove (B1). The latter follows for $\nu=1,2,3,\cdots,(n-1)$ since $P_n^{(2l+1,2l+1)}(\tau)$ is orthogonal to all polynomials of degree less than n with respect to the measure $d\tau(1-\tau^2)^{2l+1}$ over the interval $[-1,+1]$. It also follows for $\nu=0$ from the integral representation[13]

$$Q_n^{(2l+1,2l+1)}(x)=\frac{1}{2(x^2-1)^{2l+1}}\times\int_{-1}^{+1}dy\frac{(1-y^2)^{2l+1}P_n^{(2l+1,2l+1)}(y)}{x-y} \quad \text{(B5a)}$$

and the symmetry property[13]

$$Q_n^{(2l+1,2l+1)}(-x)=(-1)^{n+1}Q_n^{(2l+1,2l+1)}(x). \quad \text{(B5b)}$$

The demonstration of (B4) is not much harder. (B4a) is obtained from (B5). If the variable τ is used, (B4b) becomes

$$\int_{-1}^{+1}d\tau\frac{(1-\tau^2)^{2l+1}}{\tau+\tau_0}\{S_n^l(s)\}^2=2\frac{(n+2l+1)}{(n+1)_{4l+2}}Q_{n-1}^{(2l+1,2l+1)}(\tau_0)Q_n^{(2l+1,2l+1)}(\tau_0). \quad \text{(B6)}$$

Now, if λ_n is the coefficient of τ^n in S_n^l, then because of (B1), we have

$$\int_{-1}^{+1}d\tau\frac{(1-\tau^2)^{2l+1}}{\tau+\tau_0}\{S_n^l(s)\}^2=\lambda_n\int_{-1}^{+1}d\tau\frac{(1-\tau^2)^{2l+1}}{\tau+\tau_0}\tau^nS_n^l(s). \quad \text{(B7)}$$

But from (B3b) and the known properties of Jacobi polynomials,[13] we have

$$\lambda_n=\frac{1}{2^{2l+1}(n+1)_{2l}}Q_{n-1}^{(2l+1,2l+1)}(\tau_0)\frac{(n+4l+3)_n}{2^nn!}. \quad \text{(B8)}$$

It remains to compute the nth moment of (B3b). For $1\leqslant\nu\leqslant n$, we insert the identity $\tau^\nu=\tau^{\nu-1}(\tau+\tau_0)-\tau_0\tau^{\nu-1}$ repeatedly to find

$$\int_{-1}^{+1}d\tau\frac{(1-\tau^2)^{2l+1}}{\tau+\tau_0}\tau^nP_n^{(2l+1,2l+1)}(\tau)=2\tau_0^n(\tau_0^2-1)^{2l+1}Q_n^{(2l+1,2l+1)}(\tau_0), \quad \text{(B9)}$$

where (B5) was also made use of. By the same method,

$$\int_{-1}^{+1}d\tau\frac{(1-\tau^2)^{2l+1}}{\tau+\tau_0}\tau^nP_{n-1}^{(2l+1,2l+1)}(\tau)=\int_{-1}^{+1}d\tau(1-\tau^2)^{2l+1}\tau^{n-1}P_{n-1}^{(2l+1,2l+1)}(\tau)-2\tau_0^n(\tau_0^2-1)^{2l+1}Q_{n-1}^{(2l+1,2l+1)}(\tau_0). \quad \text{(B10)}$$

Since[13]

$$\int_{-1}^{+1}d\tau(1-\tau^2)^{2l+1}\tau^{n-1}P_{n-1}^{(2l+1,2l+1)}(\tau)=2^{n+4l+2}\frac{[(n+2l)!]^2}{(2n+4l+1)!}, \quad \text{(B11)}$$

(B6) and therefore (B4b) are also proved.

We remark that the method outlined in this appendix is generally useful for the inversion of Christoffel's formula.[18]