Theory of the D_{13} Pion-Nucleon Amplitude*

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We calculate the π-N scattering amplitude in the D_{13} channel using our relativistic, unitary three-body theory extended to include a p-wave π-π interaction. The sharp π-N resonance and corresponding dip in the inelasticity are obtained with reasonable π-π parameters. ρ production seems to have little effect in the F_{15} state.

I. INTRODUCTION

THE principal feature of the low-energy π-N system in the D_{13} channel is a sharp resonance at 1525 MeV accompanied by a corresponding sharp dip (nearly to zero) in the inelasticity parameter η.[1] This resonance energy is very near the ρ-N threshold, and since the ρ may be produced in s wave in this state, many have suggested a close connection between the resonance and the opening of the ρ-N channel.[2–5] In this paper we apply our relativistic three-body formalism[6] to an investigation of this connection. We believe this is particularly appropriate since (a) the reaction is supposed to be dominated by a particular three-body mechanism (ρ production) and (b) the contributions from processes not easily included in our formalism (e.g., ρ and N^* exchange) are supposed to nearly cancel in this channel.[7] Our formalism is capable of treating the surviving mechanisms exactly without sacrificing relativistic invariance, two- and three-body unitarity, or ease of solution. Previous attempts to get the resonance have involved some very restrictive assumptions, or have been largely dynamically motivated parametrizations. Our three-body theory, having a more primary dynamical base, comes closer to being a dynamical calculation of the resonance. In particular, our input parameters are largely coupling constants, etc., which are obtained essentially from experiment. In that sense our parameters are highly constrained. Thus when our fits come out qualitatively we can be pleased, but at the same time we are not free to make gross changes in the input parameters to improve the fit.

In our previous work[6] we studied π-N scattering with only nucleon exchange as the driving mechanism. We now wish to include π-π interactions in the $T=1$, $J=1^-$ (ρ) channel. As we show, π-π interactions can be introduced without adding "channels" to our previous calculation, and in principle any covariant off-shell π-π amplitude may be used so long as it satisfies elastic unitarity and has the cluster decomposition property. (The results show that the D_{13} π-N phase shift is rather sensitive to the input π-π parameters and that our formalism, therefore, offers hope of choosing between proposed theories of π-π scattering.) In fact, we use a separable amplitude, as is appropriate to the resonance-dominated $T=1$, $J=1^-$ channel we consider. This also makes the technical problems of making partial-wave decompositions easier. The problems of higher spin do not trouble us since we use the techniques developed in AAY, which are nonrelativistic in spirit and complexity, but preserve relativistic invariance.

The π-N to ρ-N Born term involves π exchange. The three-body unitary theory of Blankenbecler and Sugar we apply to this term takes only the on-shell part of the π exchange.[8] From an analysis of Feynman diagrams one can show that the off-shell contributions of π exchange are as important as the on-shell—essentially because of the small π mass. Hence, including only the on-shell part underestimates the coupling of the π-N system to the ρ-N. We have found a scheme for including the off-shell π exchange to make up for this. The method is analogous to adding it in as a potential; therefore it generates no new right-hand cuts and hence preserves

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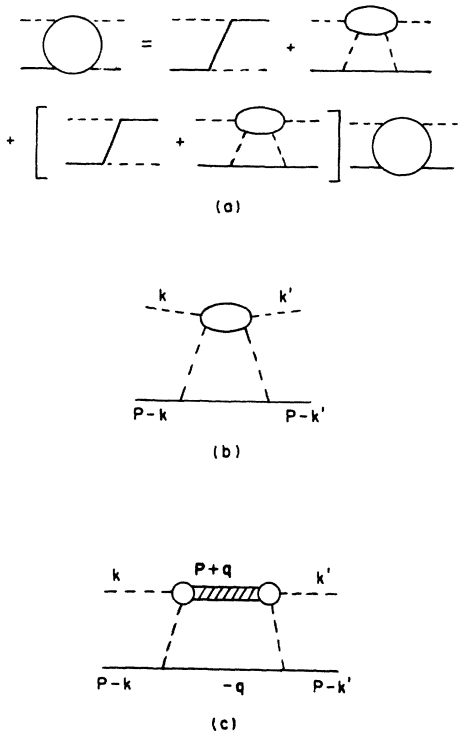

FIG. 1. (a) Graphical representation of our π-N linear integral equation of the Blandenbecler-Sugar type. The driving terms are nucleon exchange and the π-π interactions box. Nucleons are represented by solid lines and pions by dashed lines. (b) π-π interaction box with the "blob" representing an arbitrary π-π t matrix. (c) π-π interaction box with a separable π-π interaction. Momentum labels are those appropriate to Eqs. (4)–(6).

unitarity. The method gives hope of providing a scheme for adding left-hand cut contributions in other processes where these are important in the three-body theory without spoiling any of its features—unitarity, covariance, or solubility. The price of doing this is that one introduces more parameters, as one might expect.

In Sec. II we outline the equations, stressing the way of introducing the π-π interactions and also of including the off-shell π exchange. In Sec. III we present our results for the D_{13} phase shift and inelasticity parameter η for various choices of π-π parameters. Section III also contains some discussion, particularly of the results and the effects of other mechanisms on them as well as a discussion of our future plans.

II. EQUATIONS

The three-body formalism of Blankenbecler and Sugar[8] and its application to π-N scattering has been explained in detail previously.[6] Here we only describe the addition of π-π interactions. The π-N off-shell three-body equations including the π-π interactions take the form represented graphically in Fig. 1(a). We see that the π-π interaction produces an additional Born or driving term, but does not add a coupled "ρ"-N channel amplitude since a "π-π"-N to "π-π"-N Born term would correspond to at least a four-body intermediate state. In order to associate an equation with Fig. 1(a) we must specify the π-π box diagram of Fig. 1(b). The blob represents π-π scattering, and as mentioned above, in principle, any covariant off-shell form satisfying the cluster property may be used for it. In fact, it is convenient and appropriate to use a separable π-π amplitude that emphasizes the ρ-resonance dominance. In this case the $J=1^-$ π-π amplitude in the π-π c.m. system takes the form

$$(p|t(\sigma)|p')=v(p)\mathbf{p}\cdot\mathbf{p}'v(p')D_\rho^{-1}(\sigma), \tag{1}$$

with a propagator

$$D_\rho(\sigma)=\sigma-M_0^2-\frac{\Gamma_{\rho\pi\pi}}{(2\pi)^3}\int\frac{d^3q\,q^2v^2(q)}{\omega_q(\sigma-4\omega_q^2)}. \tag{2}$$

For $v(p)$ we take the form

$$v(p)=\beta_\rho^2/(p^2+\beta_\rho^2). \tag{3}$$

The parameters M_0^2, $\Gamma_{\rho\pi\pi}^2$, and β_ρ^2, in Eqs. (2) and (3), are chosen to give the correct ρ mass and width as well as the π-π p-wave scattering length. The separable form (1) allows us to factorize the π-π box as shown in Fig. 1(c). This corresponds to writing

$$(k,S|\,\mathrm{Box}(s)\,|k',S')=\sum_{i,S''}\frac{C^{(T)}}{(2\pi)^4}\int d^4q(k,S|B(s)|q,S'',i)\times G(s,q)(q,S'',i|B(s)|k',S'), \tag{4}$$

where S, S', and S'' are nucleon spin projections, i is the ρ spin projection, and $C^{(T)}$ is an isotopic spin factor which is $\frac{2}{3}$ when $T=\frac{1}{2}$, and $\frac{1}{6}$ when $T=\frac{3}{2}$. In terms of the ρ propagator (2), the three-body propagator G can be written

$$G(s,q)=2\pi(2M)\delta(q^2-M^2)/D_\rho(s-2s^{1/2}E_q+M^2),\qquad E_q=(\mathbf{q}^2+M^2)^{1/2},\quad s=P^2, \tag{5}$$

where M is the nucleon mass. In order that three-body unitarity be satisfied, the π-exchange Born terms B must contain the on-shell three-body term, constructed according to the Blankenbecler-Sugar prescription,[8] which has the form[6]

$$(k,S|B_{BS}(s)|q,S'',i)=V_i\bar{u}_S(k)\gamma_5u_{S''}(q)\gamma_{NN\pi}\Gamma_{\rho\pi\pi}f((\tfrac{1}{2}k-q)^2)v((\tfrac{1}{2}q-k)^2)\times\frac{\omega_{k-q}+\omega_k+E_q}{\omega_{k-q}[s-(\omega_{k-q}+\omega_k+E_q)^2]}. \tag{6}$$

u_ν is a nucleon spinor, and $\omega_q^2=q^2+\mu^2$ with μ the pion mass. f is the $NN\pi$ vertex defined in AAY. The vector $\mathbf{V}$ is defined in Eq. (39) of AAY. It has the property that $\mathbf{V}\cdot\mathbf{V}'$ is a Lorentz scalar that reduces to the dot

product of the relative momenta in the π-π center-of-mass system. Its use makes for great technical simplification.

Since the pion has a small mass, off-mass-shell π exchange is important in the π-N to ρ-N Born term. The Blankenbecler-Sugar[8] procedure (which puts all particles on their mass shells), while giving the correct unitarity cuts, does not necessarily give the correct potential strength. In particular, the effective Feynman residue at the pion pole ($t=\mu^2$) is too small. We can correct this problem by adding to B_{BS} a term which contributes nothing to the unitarity cuts and may thus be interpreted as a potential background. Such a term is provided by taking a fraction of the Feynman contribution to π exchange

$$(k,S|B_{\mathrm{F}}(s)|q,S'',i)=V_i\cdots v((\tfrac{1}{2}q-k)^2)1/[(k-q)^2-\mu^2]. \tag{7}$$

This is just of the form of (6) with the Blankenbecler-Sugar propagator replaced by the Feynman propagator. The mass-shell δ functions which always occur in our formalism will ensure that the Feynman denominator can never vanish and therefore that B_{F} will introduce no new right-hand s cuts. Adding this term therefore improves our left-hand cut without affecting the right. In the *static limit* and *on the energy shell* B_{BS} actually equals[9] $\frac{1}{2}B_{\mathrm{F}}$, i.e., in the limit $M\to\infty$ we note that $E_q\to M$, $s^{1/2}\to M+\omega_k$, and therefore that

$$\frac{1}{(k-q)^2-\mu^2}\to-\frac{1}{\omega_{k-q}{}^2},\quad \frac{\omega_{k-q}+\omega_k+E_q}{\omega_{k-q}[s-(\omega_{k-q}+\omega_k+E_q)^2]}\to-\frac{1}{2\omega_{k-q}{}^2}. \tag{8}$$

Away from this limit the fraction of B_{F} that should be added to best mock up the left-hand cut depends on s, and other parameters, with $\frac{1}{2}$ as the upper limit; however, we are never far from the static limit for the energy range we consider, and thus we take the fraction to be $\frac{1}{2}$. This choice presumably overestimates the π-exchange potential by a small amount. It should be noted that the separation into backward and forward going graphs is not really equivalent to this separation since B_{BS} and B_{F} are separately covariant. This method of adding "potential" terms either as generated from Feynman graphs, or of more empirical origin, to the driving term makes it possible to contemplate applying this formalism to other problems where three-body unitarity is important, but where the dynamical mechanism is more complicated, or less well known. Such a semi-empirical approach may go some way toward curing the formalism of its greatest shortcoming—the poor treatment of crossing and left-hand cuts—as well as providing a means for including other dynamical mechanisms.

With the B of Eq. (4) taken as $B_{\mathrm{BS}}+\frac{1}{2}B_{\mathrm{F}}$, from (6) and (7) we can construct the box matrix element in (4). In fact, since the integral equation represented by Fig. 1(a) is solved one partial wave at a time, it is more convenient to make a partial-wave decomposition of the B's before forming (4). Given (4), the integral equation of Fig. 1(a) is easily constructed as in AAY, and we obtain

$$(k,S|T^{(T)}(s)|k',S')=(k,S|B_N{}^{(T)}(s)|k',S')+(k,S|\mathrm{Box}^{(T)}(s)|k',S')+\sum_t\frac{1}{(2\pi)^3}\int\frac{d^3q}{2\omega_q}\frac{[(k,S|B_N{}^{(T)}(s)|q,S'')+(k,S|\mathrm{Box}^{(T)}(s)|qS'')](q,S''|T^{(T)}(s)|k',S')}{D_N(\sigma_q)}, \tag{9}$$

where $(k,S|\mathrm{Box}^{(T)}(s)|k',S')$ is given earlier in Eq. (4), and

$$(k,S|B_N{}^{(T)}(s)|k',S')=\sum_{S''}\bar{C}^{(T)}\gamma^2 f((P-k-2k')^2)\bar{u}_S(P-k)\gamma_5 u_{S''}(P-k-k')\times J(k,k,s)\bar{u}_{S''}(P-k-k')\gamma_5 u_{S'}(P-k')f((P-k'-2k)^2), \tag{10}$$

with

$$J(k,q,s)=2Mi(E_{k-q}+\omega_k+\omega_q)/E_{k-q}[s-(E_{k-q}+\omega_k+\omega_q)^2], \tag{11}$$

and

$$D_N(\sigma)=\frac{\sigma-M^2}{2Mi}\times\left[1-(\sigma-M^2)\frac{M^2\gamma^2}{2\pi^2}\int_0^\infty\frac{dk\ \rho(k)}{(\sigma-x)(M^2-x)^2}\right], \tag{12}$$

with

$$x=(E_k+\omega_k)^2,$$
$$\rho(k)=k^4f^2x^{1/2}/E_k\omega_k(E_k+M).$$

In the above equations the superscript T is the isotopic spin, that can be $\frac{3}{2}$ or $\frac{1}{2}$. The remaining quantities enter-

[9] A similar difficulty arises in the relativistic treatments of the three-pion systems such as that of Basdevant and Kreps [Phys. Rev. 141, 1398 (1966)] and was not remedied in their work. Also, their two-body $\pi\pi$ amplitude corresponds to that of a repulsive potential (i.e., the $\pi\pi$ phase shift first falls and then rises back through $\frac{1}{2}\pi$). Both these considerations account, at least in part, for the fact that they do not obtain a 3π (ω) resonance as one might have hoped.

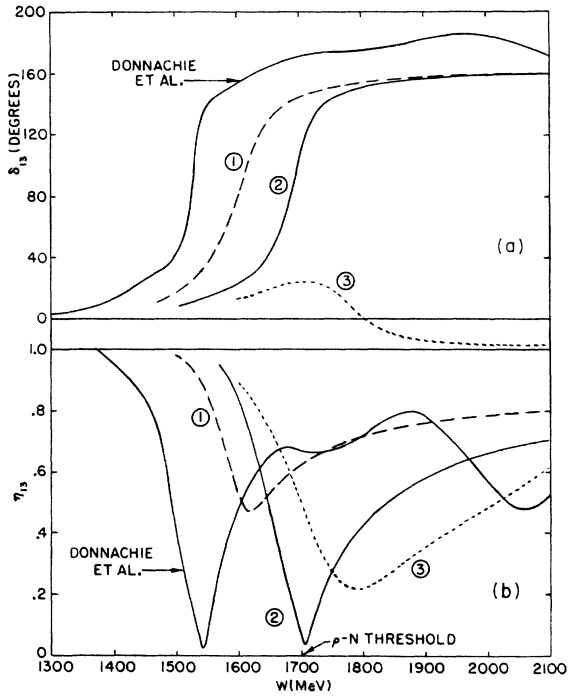

FIG. 2. Plot of our calculated D_{13} phase shift δ_{13} (a) and inelasticity parameter η_{13} (b) versus total center-of-mass energy W for three sets of p-wave π-π parameters. Also shown is the phenomenological analysis of Donnachie *et al.* The calculated set 1 corresponds to a p-wave scattering length of $a_1=0.0313m_\pi^{-1}$, a ρ width of $\Gamma_\rho=129$ MeV, and a ρ mass of $M_\rho=760$ MeV. Set 2 corresponds to $a_1=0.03m_\pi^{-1}$, $\Gamma_\rho=118$ MeV, and $M_\rho=760$ MeV, and set 3 to $a_1=0.04m_\pi^{-1}$, $\Gamma_\rho=140$ MeV, and $M_\rho=760$ MeV.

ing our equations, such as $\bar{C}^{(T)}$, γ^2, and the π-N vertex parameters appearing in f, are chosen as in AAY with a Gaussian cutoff parameter of $\beta^2=160$. This gives the 3-3 resonance correctly. There are no further free parameters in the theory except the p-wave π-π parameters, and these we express in terms of a ρ position and width and a p-wave π-π scattering length. Having constructed the linear integral equation, we make a partial-wave decomposition to turn it into a one-dimensional equation and solve it on a high-speed computer by approximating the integrals by sums. We use the method of contour deformation to avoid the difficulties associated with the kernel's singularities.

III. RESULTS AND DISCUSSION

The results obtained for the phase shift and inelasticity parameter η for the D_{13} channel are shown in Fig. 2 for various choices of the p-wave π-π parameters compared with the recent phenomenological analysis of Donnachie *et al.*[1] We see that the dominant experimental features—the sharp resonance and sharp dip in η—are well reproduced, although the calculated resonance comes out at a higher energy. The fits labeled 1 and 2 in Fig. 2 are much better than that labeled 3, and it is reassuring that the associated ρ parameters are consistent with the predictions of recent experimental and theoretical analysis.[10] It is perhaps surprising that our results are so sensitive to the input parameters, but this should not be viewed as a sensitivity to some arbitrary cutoff, but rather as a reflection of the D_{13} resonance mechanism and its dependence on the parameters of the particles driving it.

It is not difficult to find missing attractive effects that will pull the D_{13} resonance to lower energy and hence improve agreement with experiment, although it is difficult to do this with our mechanisms and still produce the accompanying sharp dip in η. It is true that any sharp dip in η must have a resonance associated with it,[11] but of course the converse is not true. As we see from Fig. 2, the rapid variation in η at lower energies is easier to produce than the value of η at the minimum. In the context of a calculation such as ours the constraints on the input parameters are fairly strong, and it does not seem appropriate to attempt to vary these parameters over a wide range in order to fit the data. In that sense the calculation is not as well off as a more empirical one, which can fit the data better by exploiting its phenomenological latitude. However, if one believes that our formalism and input are reasonable, one must believe that there are missing *attractive* and *inelastic* mechanisms around which push the resonance and η dip to lower energy even though ρ production is the dominant cause in the first place. Among the most obvious mechanisms are π-π interactions in the $I=0$, $J=0$ state, and π-$N^*(1236)$ production. The former can be easily included in the same way as the ρ. Its effect on the D_{13} channel turns out to be negligible. π-N^* intermediate states are technically more difficult to include. Estimates indicate that they are important and go in the right direction, but that they will not alter our qualitative conclusions. We are presently working to include them. There are many other mechanisms which we have neglected. Presumably introducing them could improve detailed agreement with experiment, but that would not change the conclusion about the basic ρ-production mechanism for the resonance and would probably involve us in considerable parameter adjusting. The only arbitrary mechanism we do include is the off-shell part of π exchange, and the general agreement with experiment indicates that our method of including it is reasonable.

The fact that our results for the D_{13} amplitude agree so closely with experiment indicates that virtual ρ production is the dominant mechanism over a wide range of energy and that our formalism is a sensible way to treat it over that range. Agreement seems to depend on using sensible ρ parameters, i.e., the width, position, and p-wave scattering length. We have also found it

[10] For example, hard-pion calculations give $a_1=0.033m_\pi^{-1}$ for the p-wave scattering length [R. Arnowitt (private communication)] and it is generally agreed that the width of the ρ is somewhere between 100 and 130 MeV.

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essential to include off-shell as well as on-shell π exchange. Our formalism, which treats unitarity correctly at the expense of some left-hand-cut analyticity, is probably at its best in the D_{13} channel, because of the dominant three-body mechanism and the very small potential or left-hand-cut contribution from other processes.[7]

Finally, we have also calculated the F_{15} πN phase shift. The effect of the pion-exchange mechanism producing ρ-N intermediate states is extremely small in this state, contradicting several previous analyses.[2,3] Carruthers[7] has shown that many other mechanism are large in the F_{15} state and probably capable of producing a resonance. For example, the left-hand cuts, which cancel in the D_{13} state, add to give a large effect in the F_{15} state. Hence the resonance in this channel is driven largely by the left-hand cuts or potentials rather than by three-body production.

In the future we hope to treat other partial waves, but presumably some phenomenological treatment of the neglected exchange processes and other left-hand cuts will be necessary. These can be included as a "potential" background, without sacrificing two- and three-body unitarity just as was done with the extra part of the π exchange in this work. One of the goals of treating all waves is to calculate π production and in particular to study overlapping resonances in $\pi\pi N$ final states. To do this we must of course include πN^* final states as well as N-ρ, and we are currently working toward that.

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Numerical Solutions of the Bethe-Salpeter Equation*

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After summarizing the application of the Rayleigh-Ritz and Schwinger variational principles to the unequal-mass ϕ^3 Bethe-Salpeter equation, we present, in graphical and tabular form, the solution of the bound-state problem. The dependence of the coupling-parameter eigenvalue on the exchange mass, external mass ratio, and binding energy is examined in detail for s and p ground states. Mixing of excited levels leading to complex solutions is briefly studied, and some Regge trajectories are also calculated. Scattering phase shifts for unequal-mass scattering have been calculated and representative examples are given. The fact that certain levels do not appear to contribute to Levinson's theorem is also examined. Finally, the foregoing methods are generalized to two-channel systems, and channel phase shifts and inelasticities are computed.

I. INTRODUCTION

PERTURBATION theory has been the single most powerful tool in analyzing electromagnetic scattering processes from the point of view of field theory. On the other hand, in strong-interaction processes, the coupling strength of the interaction Hamiltonian may be large enough that bound states occur, in which case the perturbation series diverges and this approach fails. One solution to this problem is to reduce the coupling strength until the perturbation series converges, express the series in closed form, and analytically continue the result as a function of the coupling constant to the desired value. In terms of potential theory this corresponds to expressing the Born series (inside its domain of convergence) in the form of an (Schrödinger) integral equation. The integral equation can then be continued in coupling strength to values for which the Born series diverges. This trick can be used to sum the perturbation series of field theory, and leads to the Bethe-Salpeter (BS) equation,[1] a completely relativistic wave equation describing two interacting particles.

In a classic paper,[2] Wick reduced the ladder approximation of the BS equation to Fredholm form and showed that it possessed a discrete coupling-parameter (λ) spectrum at fixed energy below the elastic threshold. Even in the ladder approximation the equation still has two nontrivial independent variables, and hence was considered somewhat intractible. Upon setting the exchange quantum mass M equal to zero, Wick[2] and Cutkosky[3] were able to reduce the equation to a one-dimensional form, from which they obtained a clear picture of the spectrum. Using a form of the Rayleigh-

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