Nonevasive Chew-Low Extrapolation for the Study of $\pi\pi$ Elastic Scattering

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A recent analysis of $(+-)$ dipion production indicates that the extrapolated production cross section does not vanish at $t=0$ as it would for one-pion exchange. We present a model for this phenomenon in which the spin and dipion mass dependences of the background are simply parametrized. This suggests a new prescription for performing Chew-Low extrapolation.

OF all the recipes for extracting $\pi\pi$ elastic phase shifts from peripheral production data,[1,2] the Chew-Low method[3] is distinguished by being, in principle, model-independent. In practice, the need to improve the "leverage" of the data in making the extrapolation has led to the custom of assuming that the extrapolated intensity function vanishes at $t=0$, as would be the case for elementary one-pion exchange (OPE). We shall term this procedure evasive Chew-Low extrapolation.[4] The purpose of this paper is to sketch alternative nonevasive extrapolation procedures. This problem has also been discussed by Kane and Ross[5] (KR), although with different assumptions.

The stimulus to do this comes from the data. Whereas for the reaction $\pi^-p\to\pi^0\pi^-p(0-)$, the evasive assumption appears to be pragmatically successful,[6] its recent application to the process $\pi^-p\to\pi^+\pi^-n(+-)$ by Marateck *et al.*[7] has yielded unreasonable extrapolations (for example, cross sections significantly below the unitarity limit at the ρ mass).

We consider the production of S-wave and P-wave dipion systems.[8] The elastic scattering parameters are to be determined by extrapolation to the pion pole in t, the momentum transfer to the nucleon vertex. In order to make a useful extrapolation with currently available statistics, one requires a model for the remaining contributions in the region of small $|t|$. There are a number of possible terms which could give non-vanishing contributions to the intensity at $t=0$. The form shown in Eqs. (1)–(4), in which, besides the OPE terms, only the most singular contributions are retained is suggested by experimental information on the dipion decay density matrices. This curtailment of the possible contributions is our first assumption.

Since we are interested in t-channel dynamics, it is convenient to use t-channel helicity amplitudes. These should exhibit relevant kinematic singularities and incorporate the appropriate constraints.[9,10] Our proposed formulas for the helicity amplitudes in the small-$|t|$ region (in a suitable normalization chosen to give the OPE terms a simple form, and using the phase conventions of Cohen-Tannoudji *et al.*[9]) are

$$f_{\frac{1}{2}\frac{1}{2};00}{}^{S,P}=A_{\pi\pi}{}^{S,P}\left[\frac{(-t)^{1/2}}{t-\mu^2}+\frac{\Gamma_0{}^{S,P}}{(-t)^{1/2}}\right],\qquad(1)$$

$$f_{\frac{1}{2}-\frac{1}{2};00}{}^{S,P}=A_{\pi\pi}{}^{S,P}\Gamma_0{}^{S,P}(t/t_{\min}-1)^{1/2}/(-t)^{1/2},\qquad(2)$$

$$f_{\frac{1}{2}\frac{1}{2};\pm10}{}^{P}=\pm A_{\pi\pi}{}^{P}\Gamma_1{}^{P}(t/t_{\min}-1)^{1/2}/(-t)^{1/2},\qquad(3)$$

$$f_{\frac{1}{2}-\frac{1}{2};\pm10}{}^{P}=A_{\pi\pi}{}^{P}\Gamma_1{}^{P}[(t/t_{\min})^{1/2}\mp1]/(-t)^{1/2}.\qquad(4)$$

Here all t dependence is explicitly shown and the quantities $A_{\pi\pi}{}^{S,P}$ for the $(+-)$ reaction are related to the $\pi\pi$ elastic phase shifts through the formulas

$$A_{\pi\pi}{}^{P}=3e^{i\delta_1{}^1}\sin\delta_1{}^1,\qquad(5)$$

$$A_{\pi\pi}{}^{S}=\tfrac{2}{3}e^{i\delta_0{}^0}\sin\delta_0{}^0+\tfrac{1}{3}e^{i\delta_0{}^2}\sin\delta_0{}^2.\qquad(6)$$

The intensity is given in terms of our t-channel helicity

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[1] W. D. Walker, J. Carroll, A. Garfinkel, and B. Oh, Phys. Rev. Letters **18**, 630 (1967); P. E. Schlein, *ibid.* **19**, 1052 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967); A. B. Clegg, Phys. Rev. **163**, 1664 (1967); L. J. Gutay *et al.*, Phys. Rev. Letters **18**, 142 (1967).

[2] A useful survey of peripheral analysis has been given by L. J. Gutay *et al.*, Nucl. Phys. **B12**, 31 (1969).

[3] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

[4] We extend the nomenclature of E. Leader (see Ref. 14) from the concept of evasive trajectory couplings to that of evasive t-channel helicity amplitudes. Nonevasive means cooperatively singular, e.g., conspiratorial.

[5] G. L. Kane and M. Ross, Phys. Rev. **177**, 2353 (1969). These authors propose that the evasive assumption be abandoned only for the production of transverse ρ's—in our notation of Eqs. (1)–(4), $\Gamma_1{}^P\neq0$. They make no assumption on the dependence of this background on dipion mass.

[6] J. P. Baton, G. Laurens, and J. Reignier, Nucl. Phys. **B3**, 349 (1967); Phys. Letters **26B**, 471 (1968); **25B**, 419 (1967); J. P. Baton and G. Laurens, Phys. Rev. **176**, 1574 (1968).

[7] S. Marateck *et al.*, Phys. Rev. Letters **21**, 1613 (1968).

[8] D-wave dipions are commonly ignored in peripheral analysis below and around the ρ region. See, however, G. Wolf, Phys. Letters **19**, 328 (1965). It seems likely that the $I=0$ D-wave phase shift in the vicinity of the ρ mass is of the order of 2° [D. Morgan and G. Shaw, Nucl Phys. **B10**, 261 (1969)]. As statistics improve, its effects should be observable and inferences on the $I=0$ S wave consequently modified. It could readily be included in the present formalism at the price of more parameters.

[9] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N.Y.) **46**, 239 (1968); G. C. Fox, Ph.D. thesis, Cambridge University, 1967 (unpublished).

[10] J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968).

amplitudes $f^{S,P}$ by the formula

$$\frac{\partial\sigma}{\partial t\partial\omega^2\partial\cos\theta\partial\phi}=\tfrac{1}{2}C\sum_{\lambda\lambda'}|f_{\lambda\lambda'}{}^S+\sum_{\nu}f_{\lambda\lambda';\nu 0}{}^P d_{\nu 0}{}^1(\theta)e^{i\nu\phi}|^2. \quad (7)$$

The quantities θ and ϕ are the standard angles describing the dipion decay in the Jackson frame,[2] ω is the dipion mass, and λ, λ', and ν are the helicities of the nucleon, antinucleon, and ρ. The kinematic constant C is taken to be[7]

$$C=\frac{[\omega/(\frac{1}{4}\omega^2-\mu^2)^{1/2}]f^2}{2\pi k_{\mathrm{lab}}{}^2\mu^2}. \quad (8)$$

The quantity $t_{\min}$ is the minimum momentum transfer in the s channel, given by

$$4st_{\min}=(\omega^2-\mu^2)^2-\{([s-(m+\mu)^2][s-(m-\mu)^2])^{1/2}-([s-(m+\omega)^2][s-(m-\omega)^2])^{1/2}\}^2, \quad (9)$$

with m and μ the nucleon and pion mass and $\sqrt{s}$ the total c.m. energy.

Our formulas can be derived starting from the expressions given by Diu and LeBellac[11] for the required amplitudes in terms of invariant amplitudes. The kinematic singularities at $t=0$ and $t_{\min}$ enter their formulas explicitly through the quantities $\sqrt{t}$, $k_{\rho\pi}{}^t$, the t-channel c.m. momentum, and θ_t, the scattering angle for the process $\pi\rho(\sigma)\rightarrow\bar{N}N$. For small $|t|$, these kinematic quantities take the approximate forms

$$k_{\rho\pi}{}^t\approx(\omega^2-\mu^2)/2(t)^{1/2}, \quad (10)$$

$$\sin\theta_t\approx[1-t/t_{\min}]^{1/2}. \quad (11)$$

Substituting these relations and keeping only leading terms in the vicinity of $t=0$, we obtain our formulas (1)–(4).

Terms omitted from Eqs. (1)–(4) which could contribute at $t=0$ are, first, linear terms in the expansion of the $\Gamma_i{}^{S,P}$ regarded as functions of t,[12] and, secondly, contributions to the $f_{\frac{1}{2}\pm\frac{1}{2};\pm 10}$ which do not have a branch point at $t=0$,[13] e.g., A_2 exchange. We emphasize in our model the $\Gamma_0{}^{S,P}$ (regarded once again as constants) because they lead to cross terms with the dominant OPE terms, and the $\Gamma_1{}^P$ in order to reproduce the substantial ρ_{10} and small ρ_{1-1} [Eq. (13)] which are observed at small $|t|$. Nonvanishing $\Gamma_i{}^{S,P}$ at $t=0$ corresponds to what would, in Regge analysis, be termed a conspiratorial situation.[14] The individual helicity amplitudes are singular at $t=0$, but in a correlated fashion, so that the intensity function is nonsingular.[10,15]

Note that in allowing the $\Gamma_0{}^{S,P}$ to be nonzero, we differ from Kane and Ross.[5] Nonzero $\Gamma_0{}^P$ seem definitely to be required to understand the cross-section extrapolation of Marateck *et al.*[7]

The evasive extrapolation procedure corresponds to assuming that the $\Gamma_i{}^{S,P}$ vanishes at $t=0$. In replacing it, we have to supply a model for the $\Gamma_i{}^{S,P}$ ($t=0$) as functions of ω^2. We desire a form which is physically reasonable and statistically economical. Our second assumption is therefore that the dependence of $[A_{\pi\pi}{}^{S,P}\Gamma_i{}^{S,P}]$ on ω^2 is predominantly given by the Watson factor[16] $e^{i\delta_{\pi\pi}}\sin\delta_{\pi\pi}$. Thus, the $\Gamma_i{}^{S,P}$ are to be taken as complex constants (as the notation anticipated) or, if statistics permit, as slowly varying functions or ω^2. They will, of course, be functions of beam momentum. Here again we differ from Kane and Ross[5] who make no assumption on the ω^2 dependence of their background.

The Watson-factor assumption seems not unreasonable; for example, the cross section for $\pi^+\pi^-$ obtained by extrapolating evasively[7] lies systematically below the unitarity bound and appears to have the right shape but with the wrong normalization. For $\pi^-\pi^0$, where the evasive extrapolation actually seems to work well, so that the $\Gamma_0{}^{S,P}$ would be essentially zero, the linear term in the expansion away from $t=\mu^2$ has a similar shape to the extrapolated cross section,[17] suggesting that the Watson factor dominates this term as well; however, the same does not seem to go for the corresponding term in the $P_2(\cos\theta)$ coefficient.[17] In fact, the procedure we suggest does not necessitate any assumption on these terms. We would expect that the Watson assumption worked best for very small t.

The formula for the intensity is obtained on substituting the above expressions for the helicity amplitudes into Eq. (7). For making the Chew-Low extrapolation, we work with the function

$$F=C^{-1}(t-\mu^2)^2\frac{\partial\sigma}{\partial t\partial\omega^2\partial\cos\theta\partial\phi}, \quad (12)$$

which takes the form

$$F=a_0+a_1\cos\theta+a_2\cos^2\theta-(a_0+\tfrac{1}{3}a_2)\times[3\sqrt{2}\,\mathrm{Re}\rho_{10}\sin 2\theta\cos\phi+3\rho_{1-1}\sin^2\theta\cos 2\phi+2(\sqrt{6})\,\mathrm{Re}\rho_{10}{}^{\mathrm{int}}\sin\theta\cos\phi]. \quad (13)$$

[11] B. Diu and M. LeBellac, Nuovo Cimento **53A**, 158 (1968).

[12] The linear coefficients of $\Gamma_i{}^{S,P}$ in Eqs. (1) and (2) or in Eqs. (3) and (4) need not, of course, be the same.

[13] Strictly after the $\cos\theta_t$ and $\sin\theta_t$ factors have been removed.

[14] E. Leader, Phys. Rev. **166**, 1599 (1968).

[15] G. Cohen-Tannoudji, A. Kotanski, and Ph. Salin, Phys. Letters **27B**, 42 (1968).

[16] K. M. Watson, Phys. Rev. **88**, 1163 (1952); **95**, 228 (1954); D. Morgan, *ibid.* **166**, 1731 (1967) (see discussion in Sec. III). The "factorization" assumption is often made on the whole background; e.g., see J. H. Scharenguivel *et al.*, in Proceedings of the Conference on $\pi\pi$ and πK Interactions, Argonne National Laboratory, 1969 (unpublished).

[17] J. P. Baton, G. Laurens, and J. Reignier, Nucl. Phys. **B3**, 349 (1967); see Figs. 3 and 4. Note that the evasive extrapolation corresponds to taking our α_i's zero, so that the linear terms referred to correspond essentially to our β_i's. Their behavior has no direct bearing on our assumption but reinforces the philosophy.

Here, the coefficients a_i and ρ_{ij} are given by

$$a_0=|A_{\pi\pi}{}^S|^2[-t+2(t-\mu^2)\ \mathrm{Re}\Gamma_0{}^S+(t-\mu^2)^2|\Gamma_0{}^S|^2/(-t_{\min})]+2|A_{\pi\pi}{}^P|^2(t-\mu^2)^2|\Gamma_1{}^P|^2/(-t_{\min}), \quad (14)$$

$$a_1=2\ \mathrm{Re}\{A_{\pi\pi}{}^S A_{\pi\pi}{}^{P*}[-t+(t-\mu^2)(\Gamma_0{}^S+\Gamma_0{}^{P*})+(t-\mu^2)^2\Gamma_0{}^S\Gamma_0{}^{P*}/(-t_{\min})]\}, \quad (15)$$

$$a_2=|A_{\pi\pi}{}^P|^2[-t+2(t-\mu^2)\ \mathrm{Re}\Gamma_0{}^P+(t-\mu^2)^2(|\Gamma_0{}^P|^2-2|\Gamma_1{}^P|^2)/(-t_{\min})], \quad (16)$$

$$(a_0+\tfrac{1}{3}a_2)\ \mathrm{Re}\rho_{10}=\tfrac{1}{3}|A_{\pi\pi}{}^P|^2(t-\mu^2)\ \mathrm{Re}\Gamma_1{}^P(t/t_{\min}-1)^{1/2}, \quad (17)$$

$$(a_0+\tfrac{1}{3}a_2)\rho_{1-1}=0, \quad (18)$$

$$(a_0+\tfrac{1}{3}a_2)\ \mathrm{Re}\rho_{10}{}^{\mathrm{int}}=(1/\sqrt{3})\ \mathrm{Re}[A_{\pi\pi}{}^S A_{\pi\pi}{}^{P*}(t-\mu^2)\Gamma_1{}^{P*}(t/t_{\min}-1)^{1/2}]. \quad (19)$$

These formulas are to be employed for very small $|t|$. Our proposal is that they should be used to parametrize the values at $t=0$ of the quantities a_0, a_1, and a_2. Thus, isolating the pion pole terms $a_i{}^{\mathrm{pole}}$, we would suggest that data on the $a_i(t)$ be actually fitted to the formulas

$$a_i=-a_i{}^{\mathrm{pole}}t+(t-\mu^2)(\alpha_i+\beta_i t+\cdots), \quad (20)$$

with the α_i constrained to the forms obtained from Eqs. (14)–(16), e.g.,

$$\alpha_0=|A_{\pi\pi}{}^S|^2[2\ \mathrm{Re}\Gamma_0{}^S-\mu^2|\Gamma_0{}^S|^2/(-t_{\min})]-2\mu^2|A_{\pi\pi}{}^P|^2|\Gamma_1{}^P|^2/(-t_{\min}), \quad (21)$$

and with the β_i and any higher coefficients of t as free parameters for each dipion mass bin. If, for example, one omitted higher terms after β_i, one would have six free parameters for each mass interval, in addition to the $\Gamma_i{}^{S,P}$ which would be common to all dipion masses. These latter would be the only extra parameters as compared to the evasive approach. All mass intervals would be fitted together and the fitted values of $a_i{}^{\mathrm{pole}}$ would of course be the result of the extrapolation.

There should, in principle, be enough information to constrain the $\Gamma_i{}^{S,P}$ parameters using the ϕ-dependent density matrices as well as the a_i's. [Probably, statistics would necessitate fitting data at small $|t|$ directly to Eqs. (17)–(19) without higher terms.] The appearance of a complex coefficient in the expression for a_1 allows the interesting possibility of distortions in the forward-backward asymmetry at physical momentum transfer.

One can get an idea of the sort of Γ's that might be required for the data sample listed by Marateck *et al.*[7] by fixing on a particular beam momentum from their sample, namely, 2.7 GeV/c, for which the density matrices have been published.[18] The quantity $t_{\min}$ comes out (for $\omega=m_\rho=760$ MeV) as

$$t_{\min}=-0.645\mu^2.$$

Alternative Regge-pole fits have been made to a body of pion-production data including that at 2.7 GeV/c.[19] A markedly better fit to the small-$|t|$ density matrices was achieved in the conspiratorial alternative with the pion and pion conspirator trajectories as dominant contributions. The A_2 term came out as small [see the remark shortly after Eq. (11)]. From this fit, we can extract an estimate of $\Gamma_1{}^P$, $\Gamma_1{}^P\sim0.06$. Note that $\Gamma_1{}^P$ is real in this model, which seems to be borne out by the large values of $\mathrm{Re}\rho_{10}$ which are observed.[20]

The evasive extrapolation of Marateck *et al.*[7] gives cross sections and a_2, the coefficient of $\cos^2\theta$, which are systematically too low in magnitude. This suggests that the assumption of zero intercept at $t=0$ will have to be corrected downwards and it appears that $\Gamma_0{}^P\sim0.1$ would give the desired type of effect. The reported extrapolations of $a_0(t)$ indicate that $\Gamma_0{}^S$ would also have to be appreciable but with the opposite sign. For the quantities a_0 and $a_0+\frac{1}{3}a_2$, the extent to which the intercept at $t=0$ can go negative is limited by the requirement that they be positive in the physical region. This constraint is, of course, built in to the full formulas of Eqs. (14)–(19), but is lost if one merely extracts the zero intercept as in Eq. (20).

The success of evasive extrapolation for (0−) production and its apparent failure for (+ −) production constitutes something of a puzzle. The difference does not seem to be accounted for by special features of $I=0$ dipion production, since the a_2 coefficient extrapolates to too low a value. In our notation, the quantity $\mathrm{Re}\Gamma_0{}^P$ has to be significant for (+ −) and can be neglected for (0−). The former is pure $I=1$ exchange, the latter a mixture of $I=0$ and $I=1$. On the Regge-exchange model, one would require a contributor to $I=1$ exchange which was singular and had an appreciable real part at $t=0$ (the A_1 together with its daughter readily fulfills the first but not the second requirement). One would have further to postulate a singular $I=0$ contribution[21,22] which canceled this real part in (0−) production. One has grown accustomed to the notion of correlated trajectories as in exchange degeneracy, daughters, and conspirators and this would furnish yet another instance. The crucial test would come in ρ^+ production at small $|t|$ for which the relevant real parts would add. Unfortunately, the data are rather sparse.

[18] D. H. Miller *et al.*, Phys. Rev. **153**, 423 (1967).

[19] G. V. Dass and C. D. Froggatt, Nucl. Phys. **B8**, 661 (1968).

[20] See Ref. 19. An interesting difference between the ρ_{1-1} for (+ −) and (0−) production comes out in this analysis. The latter process admits $I=0$ natural-parity exchange (i.e., ω exchange) and this is found to give a much larger contribution than does A_2 exchange. In the conspiracy fit, one finds at 2.7 GeV/c for $t=-5\mu^2$, $\rho_{1-1}{}^0\approx0.01$ and $\rho_{1-1}{}^-\approx0.05$. [We emphasize that Eq. (18) refers to the Jackson frame.]

[21] The $I=0$ companion of the B trajectory, when discovered [the "H meson?" (see Ref. 22)], would be a suitable candidate.

[22] Particle Data Group (N. Barash-Schmidt *et al.*), Rev. Mod. Phys. **41**, 109 (1969).