Existence of the Covariant Time-Ordered Product of Currents

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The existence of the covariant T product of any number of currents is investigated without introducing any extraneous assumptions. We find that it does exist and can be constructed explicitly by an algebraic method.

I. INTRODUCTION

SOME time ago it was pointed out by Johnson[1] and by Bjorken[2] that the time-ordered product of two currents is in general not covariant. The construction of a covariant one has been studied by Brown[3] and others[4–6] within the framework of canonical filed theory. In this approach, one assumes that the currents under consideration are obtained from a Lagrangian through a gauge principle. By studying the response of the system to hypothetical external perturbations, one can show that there should exist a covariant version of the T product.

In this paper, we investigate the existence of the covariant T product for any number of currents without introducing the extraneous assumption that the currents have any connection with a gauge principle.[7] We assume only that the equal-time commutators of two time components are the usual ones and that the commutators of their time and space components contain terms which are no more singular than the first derivatives of a δ function. We assume that the Schwinger terms are operators plus possible infinite c numbers. However the infinite c-number Schwinger terms can always be ignored in considering the T products. The reason is that they can always be removed by subtracting the vacuum expectation values of the T products from the T products; that is, by considering only the connected diagrams. So only operator Schwinger terms need to be considered. We assume that these operator Schwinger terms are well defined so that Jacobi identities for currents, for example, are satisfied.

Only conserved currents are considered in this paper. For the case of nonconserved currents, if the T product is not so singular, we suppose that similar methods can be applied but would be rather complicated. It should be stressed that the covariant amplitude for nonconserved currents,[8] if it exists, would not be the same as that for the conserved currents discussed here.

The result we get is that there always exists a covariant T product for any number of currents, which can be constructed explicitly by an algebraic method. Furthermore, the divergences of this covariant T product are, as expected, those which one would obtain from an ordinary T product if Schwinger terms were consistently ignored.

For many applications, like soft-pion theorems, one only needs to know that such a covariant T product exists. In other cases, one needs to know the explicit form of the covariant T product. For example, the use of Bjorken's procedure to find the high-energy behavior of a covariant amplitude requires a knowledge of its explicit form. General operator expressions for the covariant T products are given here.

The paper is organized as follows: In Sec. II, we examine the case for two isovector currents to familiarize ourselves with the problem. In Sec. III, three and more currents are considered in the simple case when the Schwinger terms are Lorentz scalar operators. In Sec. IV, the general case is studied.

II. TWO CURRENTS

In this section we study the simple case of two currents. We first show that time-ordered product of two currents defined in the usual way is not a Lorentz covariant operator. Then the existence of the covariant one is demonstrated.[9] By studying the simple case first, one may gain some insight into the general case of n currents.

The currents under consideration, e.g., $j_\mu{}^a(x)$, $j_\nu{}^b(y)$, $j_\eta{}^c(z)$, etc., are assumed to be conserved, and they satisfy the following equal-time commutation relations:

$$[j_0{}^a(x),j_0{}^b(y)]\delta(x_0-y_0)=\epsilon_{abc}j_0{}^c(y)\delta(x-y)\,, \quad (2.1)$$

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$$[j_0{}^a(x),j_k{}^b(y)]\delta(x_0-y_0)=\epsilon_{abc}j_k{}^c(y)\delta(x-y)-S_{lk}{}^{ab}(y)\partial_l\delta(x-y)\,, \quad (2.2)$$

where a, b, c are isospin indices, and $S_{lk}{}^{ab}$ is the Schwinger term. We use the metric that $x_\mu=(x_1,x_2,x_3,ix_0)$. Henceforth, the Greek letters μ, ν, η, etc. are used as Lorentz indices, and the space indices are denoted by k, l, m, n, r, etc. Also, the repeated indices denote summation.

The time-ordered product $T_{\mu\nu}{}^{ab}$ of the currents $j_\mu{}^a(x)$ and $j_\nu{}^b(y)$ is defined in the usual way by

$$T_{\mu\nu}{}^{ab}=T(j_\mu{}^a(x)j_\nu{}^b(y))=j_\mu{}^a(x)j_\nu{}^b(y)\theta(x_0-y_0)+j_\nu{}^b(y)j_\mu{}^a(x)\theta(y_0-x_0)\,. \quad (2.3)$$

We notice that for an operator, say, $M_{\mu\nu}(x,y)$, to be Lorentz covariant, it has to satisfy the following commutation relation with K_r, the boosting operator of the Lorentz transformation in the r-axis direction:

$$[K_r,M_{\mu\nu}]=\delta_{\mu4}M_{r\nu}-\delta_{\mu r}M_{4\nu}+\delta_{\nu4}M_{\mu r}-\delta_{\nu r}M_{\mu4}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})M_{\mu\nu}\,, \quad (2.4)$$

where

$$L_{r4}{}^{(x)}=x_4(\partial/\partial x_r)-x_r(\partial/\partial x_4)\,. \quad (2.5)$$

So, by checking whether the commutator $[K_r,T_{\mu\nu}{}^{ab}]$ has the same form as above or not, one can tell immediately whether or not $T_{\mu\nu}{}^{ab}$ is covariant.

As a matter of fact, it is easily seen that $T_{\mu\nu}{}^{ab}$ is not covariant by noticing that

$$[K_r,T_{4r}{}^{ab}]\neq T_{rr}{}^{ab}-T_{44}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})T_{4r}{}^{ab}.$$

Instead, we have

$$\begin{aligned}[K_r,T_{4r}{}^{ab}]&=T([K_r,j_4{}^a(x)]j_r{}^b(y))+T(j_4{}^a(x)[K_r,j_r{}^b(y)])\\&=T_{rr}{}^{ab}-T_{44}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})T_{4r}{}^{ab}+(x_r-y_r)[j_0{}^a(x),j_r{}^b(y)]\delta(x_0-y_0)\end{aligned}$$

or

$$[K_r,T_{4r}{}^{ab}]=T_{rr}{}^{ab}-T_{44}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})T_{4r}{}^{ab}+S_{rr}{}^{ab}(y)\delta(x-y)\,, \quad (2.6)$$

where use was made of Eq. (2.2) and the fact that the current is covariant, i.e.,

$$[K_r,j_\mu{}^a(x)]=\delta_{\mu4}j_r{}^a-\delta_{\mu r}j_4{}^a+L_{r4}{}^{(x)}j_\mu{}^a. \quad (2.7)$$

We note that the presence of the Schwinger term $S_{rr}{}^{ab}$ in Eq. (2.6) causes the noncovariance of time-ordered product $T_{\mu\nu}{}^{ab}$.

To investigate further the origin of noncovariance, we compute the divergence of $T_{\mu\nu}{}^{ab}$:

$$\partial_\mu T_{\mu\nu}{}^{ab}=T(\partial_\mu j_\mu{}^a(x),j_\nu{}^b(y))+[j_0{}^a(x),j_\nu{}^b(y)]\delta(x_0-y_0)\,.$$

By using current conservation and Eqs. (2.1) and (2.2), we get

$$\partial_\mu T_{\mu4}{}^{ab}=\epsilon_{abc}j_4{}^c(y)\delta(x-y)\,, \quad (2.8a)$$

$$\partial_\mu T_{\mu r}{}^{ab}=\epsilon_{abc}j_r{}^c(y)\delta(x-y)-S_{lr}{}^{ab}(y)\partial_l\delta(x-y)\,. \quad (2.8b)$$

Again, due to the presence of the Schwinger term $S_{lr}{}^{ab}$ in Eq. (2.8), we see that the divergence of $T_{\mu\nu}{}^{ab}$ is not covariant either.

Realizing that the noncovariance of $T_{\mu\nu}{}^{ab}$ and its divergence $\partial_\mu T_{\mu\nu}{}^{ab}$ comes from the same source, the Schwinger terms, we now try to find a way to fix things up. Since it is very easy to find an operator, say $\tilde{T}_{\mu\nu}{}^{ab}$, which has a covariant divergence, we ask ourselves whether this operator which has the right divergence is also covariant. The answer is yes as we soon see.

It is easily seen with Eqs. (2.7) and (2.8) that the operator $\tilde{T}_{\mu\nu}{}^{ab}$ defined by

$$\tilde{T}_{4\nu}{}^{ab}=T_{4\nu}{}^{ab}\,,\quad \tilde{T}_{\mu4}{}^{ab}=T_{\mu4}{}^{ab} \quad (2.9)$$

and

$$\tilde{T}_{lk}{}^{ab}=T_{lk}{}^{ab}+S_{lk}{}^{ab}(y)\delta(x-y) \quad (2.10)$$

has a covariant divergence, i.e.,

$$\partial_\mu\tilde{T}_{\mu\nu}{}^{ab}=\epsilon_{abc}j_\nu{}^c(y)\delta(x-y)\,. \quad (2.11)$$

To see that the operator $\tilde{T}_{\mu\nu}{}^{ab}$ is Lorentz covariant, it is sufficient to show its commutation relation with boosting operator K_r has the same form as Eq. (2.4).

Now with the definition (2.9) and Eq. (2.1), it is automatic that

$$[K_r,\tilde{T}_{44}{}^{ab}]=\tilde{T}_{4r}{}^{ab}+\tilde{T}_{r4}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})\tilde{T}_{44}{}^{ab}. \quad (2.12)$$

By making use of Eqs. (2.6), (2.9), and (2.10), one can easily see that

$$[K_r,\tilde{T}_{4r}{}^{ab}]=\tilde{T}_{rr}{}^{ab}-\tilde{T}_{44}{}^{ab}+(L_{r4}{}^{(x)}+L_{4r}{}^{(y)})\tilde{T}_{4r}{}^{ab}. \quad (2.13)$$

So to prove the covariance of $\tilde{T}_{\mu\nu}{}^{ab}$, it remains to show that the commutation of K_r with its space-space components takes the same form as Eq. (2.4). By using Eq. (2.7), we see that

$$[K_r,T_{lk}{}^{ab}]=-\delta_{lr}T_{4k}{}^{ab}-\delta_{kr}T_{l4}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})T_{lk}{}^{ab}+(1/i)(x_r-y_r)[j_l{}^a(x),j_k{}^b(y)]\delta(x_0-y_0)\,. \quad (2.14)$$

So if we want to show that the commutator of the boosting operator K_r with the space-space component $\tilde{T}_{lk}{}^{ab}$ takes the same form as Eq. (2.4), i.e.,

$$[K_r,\tilde{T}_{lk}{}^{ab}]=-\delta_{lr}\tilde{T}_{4k}{}^{ab}-\delta_{kr}\tilde{T}_{l4}{}^{ab}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)})\tilde{T}_{lk}{}^{ab},$$

then with Eqs. (2.10) and (2.14), this would require an identity among the Schwinger terms in $[j_l{}^a(x),j_k{}^b(y)]\times\delta(x_0-y_0)$ and $[K_r,S_{lk}{}^{ab}(y)]\delta(x-y)$, i.e.,

$$\begin{aligned}(1/i)(x_r-y_r)[j_l{}^a(x),j_k{}^b(y)]\delta(x_0-y_0)&=-[K_r,S_{lk}{}^{ab}(y)]\delta(x-y)\\&\quad+(L_{r4}{}^{(y)}S_{lk}{}^{ab}(y))\delta(x-y)\,. \quad (2.15)\end{aligned}$$

To prove that the above identity holds, we commute K_r with Eq. (2.2) and then set $y=0$:

$$[K_r,[j_0{}^a(x),j_k{}^b(0)]\delta(x_0)]=\epsilon_{abc}[K_r,j_k{}^c(0)]\delta(x)-[K_r,S_{lk}{}^{ab}]\partial_l\delta(x)\,.$$

Using Eq. (2.7) then yields

$$[j_r^a(x)-x_r(\partial/\partial x_4)j_4^a(x),j_k^b(0)]\delta(x_0) = -i[K_r,S_{lk}^{ab}(0)]\partial_l\delta(x).$$

Upon using current conservation $\partial_\mu j_\mu^a(x)=0$, the above identity becomes

$$\partial_l\{x_r[j_l^a(x),j_k^b(0)]\delta(x_0)\} = -i[K_r,S_{lk}^{ab}(0)]\partial_l\delta(x). \quad (2.16)$$

We observe that if we can remove the derivative in the above equation, then we have actually shown the validity of (2.15) with $y=0$. To see how the spatial divergence can be removed, we assume that the equal-time commutator of the spatial components of two currents takes the following most general form:

$$[j_l^a(x),j_k^b(0)]\delta(x_0)=f_{lk}^{ab}(0)\delta(x)+f_{lkm}^{ab}(0)\partial_m\delta(x) + f_{lkmn}^{ab}(0)\partial_m\partial_n\delta(x)+f_{lkmnp}^{ab}(0)\partial_m\partial_n\partial_p\delta(x) + \cdots + \text{higher terms}. \quad (2.17)$$

We show that with Eq. (2.16), all the singular terms involving more than one derivative of the δ function actually vanish. First, we notice that since the differential operators, say ∂_m and ∂_n, etc., commute with each other and the indices m, n, p, etc. are dummy summation indices, we can set f_{lkmn}^{ab}, f_{lkmnp}^{ab}, etc. to be symmetric in these dummy indices, i.e.,

$$f_{lkmn}^{ab}=f_{lknm}^{ab}, \quad f_{lkmnp}^{ab}=f_{lknmp}^{ab}=f_{lkmpn}^{ab}, \quad \text{etc.} \quad (2.18)$$

By inserting (2.17) into (2.16), we find

$$f_{lkr}^{ab}(0)=i[K_r,S_{lk}^{ab}(0)], \quad (2.19)$$

$$f_{lkmn}^{ab}=-f_{mkln}^{ab}, \quad f_{lkmnp}^{ab}=-f_{mklnp}^{ab}, \quad \text{etc.} \quad (2.20)$$

By using (2.18) and (2.20), we can see that $f_{lkmn}^{ab}=0$, because

$$\begin{aligned} f_{lkmn}^{ab} &= -f_{mkln}^{ab} \\ &= -f_{mknl}^{ab} \\ &= +f_{nkml}^{ab} \\ &= +f_{nklm}^{ab} \\ &= -f_{lknm}^{ab} \\ &= -f_{lkmn}^{ab}. \end{aligned}$$

Similarly, one can show that $f_{lkmnp}^{ab}=0$ and that all higher terms vanish. In short, we conclude that

$$[j_l^a(x),j_k^b(0)]\delta(x_0)=f_{lk}^{ab}(0)\delta(x)+f_{lkm}^{ab}(0)\partial_m\delta(x).$$

From this and (2.19), we find immediately that

$$(1/i)x_r[j_l^a(x),j_k^b(0)]\delta(x_0)=-[K_r,S_{lk}^{ab}(0)]\delta(x). \quad (2.21)$$

By replacing x by $x-y$ in (2.21), we see that

$$(1/i)(x_r-y_r)[j_l^a(x),j_k^b(y)]\delta(x_0-y_0) = -e^{ip\cdot y}[K_r,S_{lk}^{ab}(0)]e^{-ip\cdot y}\delta(x-y), \quad (2.22)$$

where $e^{ip\cdot y}$ is the translational operator. But since we have

$$[K_r,e^{ip\cdot y}]=iy_\mu[K_r,p_\mu]e^{ip\cdot y} = i(y_4p_r-y_rp_4)e^{ip\cdot y},$$

Eq. (2.22) then becomes

$$(1/i)(x_r-y_r)[j_l^a(x),j_k^b(y)]\delta(x_0-y_0) = -\{[K_r,S_{lk}^{ab}(y)]-L_{r4}^{(y)}S_{lk}^{ab}(y)\}\delta(x-y), \quad (2.23)$$

which is just what we want.

We summarize what we have learned in this section: (1) The noncovariance and the wrong divergence of the time-ordered product for two current is due to the presence of Schwinger term in the equal-time commutator of time and space components of the currents; (2) we have an algebraic method to show the existence of a covariant T product, which can be generalized; and (3) the way to guess the form of the covariant T product is to require it to have the right divergence.

III. THREE AND MORE CURRENTS

When there are more than two currents, the situation becomes very much involved. In order to have a better idea of what is going on, we study the problem in this section under two restricted assumptions:

(a) All the currents are the same.

(b) The Schwinger term is a Lorentz scalar. When we come to the discussion of the general case in Sec. IV, these assumptions are removed.

With these assumptions, the problems are very much simplified. Not only can the general form for the covariant time-ordered product of n currents be written down explicitly, but also the proofs for its covariance are shortened considerably. Furthermore, we can use the results that we get in this section as a guide for treating the general case in Sec. IV.

Under these assumptions, the equal-time commutator for time-time components and time-space components of the current now reads, respectively,

$$[j_0(x),j_0(y)]\delta(x_0-y_0)=0, \quad (3.1)$$

$$[j_0(x),j_k(y)]\delta(x_0-y_0)=-S(y)\partial_k\delta(x-y), \quad (3.2)$$

where $S(y)$ is the Schwinger term.

By introducing $a_{\mu\nu}$ defined by

$$a_{\mu\nu}=g_{\mu\nu}+\delta_{\mu 4}\delta_{\nu 4}, \quad (3.3)$$

where

$$g_{\mu\nu}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},$$

we can rewrite Eqs. (3.1) and (3.2) as a single equation, i.e.,

$$[j_0(x),j_\nu(y)]\delta(x_0-y_0)=-a_{\mu\nu}S(y)\partial_\mu\delta(x-y). \quad (3.4)$$

By using Eqs. (3.3) and (3.4) together with the assumption of the Schwinger term being a Lorentz scalar, we obtain from Eq. (2.23) the identity

$$(1/i)(x_r-y_r)[j_\mu(x),j_\nu(y)]\delta(x_0-y_0) \\ =\delta_{\mu 4}a_{r\nu}S(y)\delta(x-y)+\delta_{\nu 4}a_{\mu r}S(y)\delta(x-y). \quad (3.5)$$

In order to pave the way for the discussion of the problem of n currents, we first study the case of three and four currents. We have learned from Sec. II that a covariant time-ordered product for two currents can be constructed from the usual time-ordered product by requiring it to have the right divergence. We show that this is also true for three and four currents.

Under the assumption that all currents are of the same kind, the right divergence condition becomes the divergenceless condition. For the case of three currents, the operator $\tilde{T}_{\mu\nu\eta}$ defined by

$$\tilde{T}_{\mu\nu\eta}=T_{\mu\nu\eta}+\rho_{\mu\nu\eta} \quad (3.6)$$

with

$$T_{\mu\nu\eta}=T(j_\mu(x)j_\nu(y)j_\eta(z)), \quad (3.7)$$

$$\rho_{\mu\nu\eta}=a_{\mu\nu}T(S(y)j_\eta(z))\delta(x-y)+a_{\nu\eta}T(S(z)j_\mu(x))\delta(y-z) \\ +a_{\eta\mu}T(S(z)j_\nu(y))\delta(z-x), \quad (3.8)$$

has the divergence

$$\partial\tilde{T}_{\mu\nu\eta}/\partial x_\mu=a_{\nu\eta}\delta(y-z)[j_0(x),S(y)]\delta(x_0-y_0). \quad (3.9)$$

So it will be divergenceless if we can show that the equal-time commutator for the time component of the current with the Schwinger term actually vanishes. To see that this is really the case, we use the Jacobi identity:

$$[j_0(x),[j_0(y),j_k(0)]]\delta(x_0)\delta(y_0) \\ =[j_0(y),[j_0(x),j_k(0)]]\delta(x_0)\delta(y_0) \\ +[[j_0(x),j_0(y)],j_k(0)]\delta(x_0)\delta(y_0). \quad (3.10)$$

With Eqs. (3.1) and (3.2), this yields

$$[j_0(x),S(0)]\delta(x_0)(\partial/\partial y_k)\delta(y) \\ =[j_0(y),S(0)]\delta(y_0)(\partial/\partial x_k)\delta(x),$$

so the only solution is that

$$[j_0(x),S(0)]\delta(x_0)=0. \quad (3.11)$$

By replacing x by $x-y$ and then applying the translational operator $e^{ip\cdot y}$ to the above equation, we obtain

$$[j_0(x),S(y)]\delta(x_0-y_0)=0, \quad (3.12)$$

which shows that $\tilde{T}_{\mu\nu\eta}$ defined by (3.6) is divergenceless.

Another fact about the commutators of j_k and S which is useful in proving the covariance of $\tilde{T}_{\mu\nu\eta}$ is that

$$(1/i)(x_r-y_r)[j_k(x),S(y)]\delta(x_0-y_0)=0.$$

To prove this, we use Eq. (3.11) to get

$$[K_r,[j_0(x),S(0)]]\delta(x_0)=0.$$

By using current conservation and remembering that $S(0)$ is a Lorentz scalar, we get

$$\partial_k\{x_r[j_k(x),S(0)]\}\delta(x_0)=0.$$

Using an argument similar to that which leads from (2.16) to (2.21) allows us to remove the spatial divergence in the above equation to obtain

$$x_r[j_k(x),S(0)]\delta(x_0)=0. \quad (3.13)$$

Replacing x by $x-y$ and applying the translational operator $e^{ip\cdot y}$ to (3.13) yields

$$(1/i)(x_r-y_r)[j_k(x),S(y)]\delta(x_0-y_0)=0. \quad (3.14)$$

We are now in a position to show that $\tilde{T}_{\mu\nu\eta}$, which is divergenceless, is also Lorentz covariant. First, we note from the definition of $a_{\mu\nu}$ in (3.3) that

$$a_{\mu 4}=a_{4\nu}=0.$$

So from (3.8), we have

$$\rho_{4\nu\eta}=a_{\nu\eta}T(S(z)j_4(x))\delta(y-z). \quad (3.15)$$

From the definition of $\rho_{\mu\nu\eta}$ in (3.8) and by using (3.12), (3.14), and (3.15), it is easy to see that

$$[K_r,\rho_{\mu\nu\eta}]=\delta_{\mu 4}a_{\nu\eta}T(S(z)j_r(x))\delta(y-z)-\delta_{\mu r}\rho_{4\nu\eta} \\ +\delta_{\nu 4}a_{\eta\mu}T(S(x)j_r(y))\delta(z-x)-\delta_{\nu r}\rho_{\mu 4\eta} \\ +\delta_{\eta 4}a_{\mu\nu}T(S(y)j_r(z))\delta(x-y)-\delta_{\eta r}\rho_{\mu\nu 4} \\ +(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)})\rho_{\mu\nu\eta}. \quad (3.16)$$

Upon using (3.5), we find that

$$[K_r,T_{\mu\nu\eta}]=\delta_{\mu 4}T_{r\nu\eta}-\delta_{\mu r}T_{4\nu\eta}+\delta_{\nu 4}T_{\mu r\eta}-\delta_{\nu r}T_{\mu 4\eta} \\ +\delta_{\eta 4}T_{\mu\nu r}-\delta_{\eta r}T_{\mu\nu 4}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)})T_{\mu\nu\eta} \\ +(\delta_{\mu 4}a_{r\nu}+\delta_{\nu 4}a_{\mu r})T(S(y)j_\eta(z))\delta(x-y) \\ +(\delta_{\nu 4}a_{r\eta}+\delta_{\eta 4}a_{\nu r})T(S(z)j_\mu(x))\delta(y-z) \\ +(\delta_{\eta 4}a_{\mu r}+\delta_{\mu 4}a_{r\eta})T(S(x)j_\nu(y))\delta(z-x). \quad (3.17)$$

Now combining (3.16) and (3.17) together with (3.8) yields

$$[K_r,\tilde{T}_{\mu\nu\eta}]=\delta_{\mu 4}\tilde{T}_{r\nu\eta}-\delta_{\mu r}\tilde{T}_{4\nu\eta}+\delta_{\nu 4}\tilde{T}_{\mu r\eta}-\delta_{\nu r}\tilde{T}_{\mu 4\eta}+\delta_{\eta 4}\tilde{T}_{\mu\nu r} \\ -\delta_{\eta r}\tilde{T}_{\mu\nu 4}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)})\tilde{T}_{\mu\nu\eta}, \quad (3.18)$$

which shows that $\tilde{T}_{\mu\nu\eta}$ is Lorentz covariant.

So we have seen that the covariant time-ordered product for three currents can be obtained from the usual T product by requiring it to have zero divergence. We would like to see whether or not this rule is also applicable to the case of four currents and finally to the general case of n currents.

According to this rule, we can construct the covariant time-ordered product $\tilde{T}_{\mu\nu\eta\xi}$ for four currents from the T product by requiring it to have zero divergence. So to find $\tilde{T}_{\mu\nu\eta\xi}$, we set

$$\tilde{T}_{\mu\nu\eta\xi}=T_{\mu\nu\eta\xi}+\rho_{\mu\nu\eta\xi},$$

where $T_{\mu\nu\eta\xi}$ is the time-ordered product for four currents i.e.,

$$T_{\mu\nu\eta\xi}=T(j_\mu(x)j_\nu(y)j_\eta(z)j_\xi(w)).$$

To find $\rho_{\mu\nu\eta\xi}$, we require $\tilde{T}_{\mu\nu\eta\xi}$ to satisfy the divergenceless conditions, i.e.,

$$\partial_\mu{}^{(x)}\tilde{T}_{\mu\nu\eta\xi}=\partial_\nu{}^{(y)}\tilde{T}_{\mu\nu\eta\xi}=\partial_\eta{}^{(z)}\tilde{T}_{\mu\nu\eta\xi}=\partial_\xi{}^{(w)}\tilde{T}_{\mu\nu\eta\xi}=0\,, \quad (3.19)$$

where

$$\partial_\mu{}^{(x)}=(\partial/\partial x_\mu)\,,\quad \partial_\nu{}^{(y)}=(\partial/\partial y_\nu)\,,\quad \text{etc.}$$

Thus we find that

$$\rho_{\mu\nu\eta\xi}=\rho_{\mu\nu\eta\xi}{}^{(1)}+\rho_{\mu\nu\eta\xi}{}^{(2)}$$

with

$$\begin{aligned}\rho_{\mu\nu\eta\xi}{}^{(1)}=&\,a_{\mu\nu}\delta(x-y)T(S(y)j_\eta(z)j_\xi(w))\\&+a_{\mu\eta}\delta(z-x)T(S(z)j_\nu(y)j_\xi(w))\\&+a_{\mu\xi}\delta(x-w)T(S(w)j_\nu(y)j_\eta(z))\\&+a_{\nu\eta}\delta(y-z)T(S(z)j_\mu(x)j_\xi(w))\\&+a_{\nu\xi}\delta(y-w)T(S(w)j_\mu(x)j_\eta(z))\\&+a_{\eta\xi}\delta(z-w)T(S(w)j_\mu(x)j_\nu(y))\end{aligned}\quad (3.20)$$

and

$$\begin{aligned}\rho_{\mu\nu\eta\xi}{}^{(2)}=&\,a_{\mu\nu}a_{\eta\xi}\delta(x-y)\delta(z-w)T(S(y)S(w))\\&+a_{\mu\eta}a_{\nu\xi}\delta(z-x)\delta(y-w)T(S(z)S(w))\\&+a_{\mu\xi}a_{\nu\eta}\delta(x-w)\delta(y-z)T(S(x)S(y))\end{aligned}\quad (3.21)$$

makes $\tilde{T}_{\mu\nu\eta\xi}$ divergenceless.

To show that the rule also works for the case of four currents, we have to prove that $\tilde{T}_{\mu\nu\eta\xi}$ obtained in this way is Lorentz covariant. Due to the terms of the type $T(S(x)S(y))$ in $\tilde{T}_{\mu\nu\eta\xi}$, we need to show that

$$(x_r-y_r)[S(x),S(y)]\delta(x_0-y_0)=0 \quad (3.22)$$

in order to prove the covariance of $\tilde{T}_{\mu\nu\eta\xi}$. To see that Eq. (3.22) actually holds, we again use the Jacobi identity

$$\begin{aligned}[j_0(y),[j_k(x),S(0)]]\delta(x_0)\delta(y_0)&=[j_k(x),[j_0(y),S(0)]]\delta(x_0)\delta(y_0)\\&\quad+[[j_0(y),j_k(x)],S(0)]\delta(x_0)\delta(y_0)\,.\end{aligned}$$

Multiplying the above identity by x_r and using (3.11), (3.13), and (3.2) yields

$$x_r[S(x),S(0)]\delta(x_0)\partial_k{}^{(y)}\delta(y-x)=0\,.$$

So we get

$$x_r[S(x),S(0)]\delta(x_0)=0\,.$$

By replacing x by $x-y$ and applying the translational operator to it, we obtain

$$(x_r-y_r)[S(x),S(y)]\delta(x_0-y_0)=0\,. \quad (3.22)$$

We refer to Appendix A for the rest of the proof of the covariance of $\tilde{T}_{\mu\nu\eta\xi}$.

Having learned how to construct the covariant time-ordered product for two, three, and four currents, we now try to generalize it to the case of n currents. We claim that given n conserved currents $j_{\mu_1}(x_1)$, $j_{\mu_2}(x_2)\cdots j_{\mu_n}(x_n)$ which satisfy the equal-time commutation relations (3.1) and (3.2), then there exists a covariant time-ordered product $\tilde{T}_{\mu_1\cdots\mu_n}$ obtained from the usual time-ordered product by requiring it to have zero divergences, i.e.,

$$\partial_{\mu_1}{}^{(x_1)}\tilde{T}_{\mu_1\cdots\mu_n}=\partial_{\mu_2}{}^{(x_2)}\tilde{T}_{\mu_2\cdots\mu_n}=\cdots=\partial_{\mu_n}{}^{(x_n)}\tilde{T}_{\mu_1\cdots\mu_n}=0\,.$$

It takes the form

$$\tilde{T}_{\mu_1\mu_2\cdots\mu_n}=T_{\mu_1\cdots\mu_n}+\sum_{l=1}^{m}\rho_{\mu_1\mu_2\cdots\mu_n}{}^{(l)}\,, \quad (3.23)$$

where $m=\frac{1}{2}n$ if n is even or $m=\frac{1}{2}(n-1)$ if n is odd. $T_{\mu_1\mu_2\cdots\mu_n}$ is the usual time-ordered product, i.e.,

$$T_{\mu_1\cdots\mu_n}=T(j_{\mu_1}(x_1)j_{\mu_2}(x_2)\cdots j_{\mu_n}(x_n))$$

and $\rho_{\mu_1,\mu_2\cdots\mu_n}{}^{(l)}$ is the sum of all possible distinct terms which contain the T product of l Schwinger terms with $(n-2l)$ currents, that is, the sum of all possible distinct terms similar to

$$\begin{aligned}&a_{\mu_1\mu_2}a_{\mu_3\mu_4}\cdots a_{\mu_{2l-1}\mu_{2l}}\delta(x_1-x_2)\delta(x_3-x_4)\cdots\delta(x_{2l-1}-x_{2l})\\&\quad\times T(S(x_2)S(x_4)\cdots S(x_{2l})j_{\mu_{2l+1}}(x_{2l+1})\cdots j_{\mu_n}(x_n))\,.\end{aligned}$$

The proofs for its covariance under Lorentz transformation are given in Appendix A.

IV. GENERAL CASE

We now come to the discussion of the general case by removing those two restricted conditions assumed in Sec. III. With the Schwinger terms not being Lorentz scalar, the new complications arise due to the fact that the time component of the current no longer commutes with the Schwinger term at equal time. In fact, a new singular term involving the first derivatives of the δ function shows up in the commutator $[j_0{}^a(x),S_{lk}{}^{bc}(0)]\times\delta(x_0)$. As we see later, if this term were zero, then the covariant T product of three isovector currents would have the same structure as the one obtained in Sec. III. So in order to investigate the existence of the covariant T product of three isovector currents, we need to know the structure of the commutator $[j_0{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)$.

To find the structure of this equal-time commutator, we use the Jacobi identity

$$\begin{aligned}[j_0{}^a(x),[j_0{}^b(y),j_k{}^c(0)]]\delta(x_0)\delta(y_0)&=[[j_0{}^a(x),j_0{}^b(y)],j_k{}^c(0)]\delta(x_0)\delta(y_0)\\&\quad+[j_0{}^b(y),[j_0{}^a(x),j_k{}^c(0)]]\delta(x_0)\delta(y_0)\,.\end{aligned}$$

By multiplying the above identity by y_l and using (2.1) and (2.2), we find that

$$\begin{aligned}[j_0{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)\delta(y)=&\,\epsilon_{abd}S_{lk}{}^{dc}(0)\delta(x)\delta(y)\\&+\epsilon_{acd}S_{lk}{}^{bd}(0)\delta(x)\delta(y)\\&-y_l[j_0{}^b(y),S_{mk}{}^{ac}(0)]\delta(y_0)\partial_m\delta(x)\,.\end{aligned}\quad (4.1)$$

This assures the existence of a new (4.1) term $S_{mlk}{}^{abc}(0)$ defined by

$$S_{mlk}{}^{abc}(0)\delta(x)=x_m[j_0{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)\,. \quad (4.2)$$

By inserting (4.1) into (4.2), we see that $S_{mlk}{}^{abc}$ is completely symmetric in the pairs of indices $({}_m{}^a)$, $({}_l{}^b)$, and $({}_k{}^c)$. So we have

$$
\begin{aligned}
[j_0{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)
&=\{\epsilon_{abd}S_{lk}{}^{dc}(0)+\epsilon_{acd}S_{lk}{}^{bd}(0)\}\delta(x)\\
&\quad -S_{mlk}{}^{abc}(0)\partial_m\delta(x)\,. \qquad (4.3)
\end{aligned}
$$

By replacing x by $x-y$ in (4.3) and applying the translational operator $e^{ip\cdot y}$ to it, we get

$$
\begin{aligned}
[j_0{}^a(x)S_{lk}{}^{bc}(y)]\delta(x_0-y_0)
&=\{\epsilon_{abd}S_{lk}{}^{dc}(y)+\epsilon_{acd}S_{lk}{}^{bd}(y)\}\delta(x-y)\\
&\quad -S_{mlk}{}^{abc}(y)\partial_m\delta(x-y)\,. \qquad (4.4)
\end{aligned}
$$

For convenience we introduce the operators $R_{\mu\nu}{}^{ab}(y)$, $A_{\mu\nu\eta}{}^{abc}(y)$ defined, respectively, by

$$
\begin{aligned}
&R_{lk}{}^{ab}(y)\equiv S_{lk}{}^{ab}(y)\,,\\
&R_{4\nu}{}^{ab}(y)=R_{\mu4}{}^{ab}(y)\equiv 0
\end{aligned}
\qquad (4.5)
$$

and

$$
\begin{aligned}
&A_{mlk}{}^{abc}(y)\equiv S_{mlk}{}^{abc}(y)\,,\\
&A_{4\nu\eta}{}^{abc}(y)=A_{\mu4\eta}{}^{abc}(y)=A_{\mu\nu4}{}^{abc}\equiv 0\,.
\end{aligned}
\qquad (4.6)
$$

In terms of them, Eq. (4.4) now reads

$$
\begin{aligned}
[j_0{}^a(x),R_{\nu\eta}{}^{bc}(y)]\delta(x_0-y_0)
&=\{\epsilon_{abd}R_{\nu\eta}{}^{dc}(y)+\epsilon_{acd}R_{\eta\nu}{}^{bd}(y)\}\delta(x-y)\\
&\quad -A_{\mu\nu\eta}{}^{abc}(y)\partial_\mu\delta(x-y)\,. \qquad (4.7)
\end{aligned}
$$

We are now in a position to find the covariant time-ordered product for three currents. According to the rule of what we have got in Secs. II and III, we require it to have the right divergence which in this case is equal to the sum of covariant time-ordered product of two currents shown in Sec. II. By using the result obtained in Sec. III for the case of three currents as a guide, we try to see whether or not the operator $\bar{T}_{\mu\nu\eta}{}^{abc}$ defined by

$$
\bar{T}_{\mu\nu\eta}{}^{abc}=T_{\mu\nu\eta}{}^{abc}+\rho_{\mu\nu\eta}{}^{abc} \qquad (4.8)
$$

has the right divergence, where

$$
T_{\mu\nu\eta}{}^{abc}=T(j_\mu{}^a(x)j_\nu{}^b(y)j_\eta{}^c(z)) \qquad (4.9)
$$

and

$$
\begin{aligned}
\rho_{\mu\nu\eta}{}^{abc}=T(R_{\mu\nu}{}^{ab}(y)j_\eta{}^c(z))\delta(x-y)&\\
+T(R_{\nu\eta}{}^{bc}(z)j_\mu{}^a(x))\delta(y-z)&\\
+T(R_{\eta\mu}{}^{ca}(x)j_\nu{}^b(y))\delta(x-y)&\,. \qquad (4.10)
\end{aligned}
$$

With (2.1), (2.2), and (4.7), we find that

$$
\begin{aligned}
\partial_\mu{}^{(x)}\bar{T}_{\mu\nu\eta}{}^{abc}
&=\epsilon_{abd}\tilde{T}_{\nu\eta}{}^{dc}\delta(x-y)+\epsilon_{acd}\tilde{T}_{\nu\eta}{}^{bd}\delta(x-z)\\
&\quad -A_{\mu\nu\eta}{}^{abc}(z)\delta(y-z)\partial_\mu\delta(x-z)\,, \qquad (4.11)
\end{aligned}
$$

where $\tilde{T}_{\nu\eta}{}^{dc}$ is the covariant time-ordered product for two currents found in Sec. II, that is,

$$
\tilde{T}_{\nu\eta}{}^{dc}=T(j_\nu{}^d(y)j_\eta{}^c(z))+R_{\nu\eta}{}^{dc}(z)\delta(y-z)\,.
$$

Because of the presence of $A_{\mu\nu\eta}{}^{abc}(z)$, we see that $\bar{T}_{\mu\nu\eta}{}^{abc}$ has the wrong divergence. Instead, it is easily seen from (4.11) that $\tilde{T}_{\mu\nu\eta}{}^{abc}$ defined by

$$
\tilde{T}_{\mu\nu\eta}{}^{abc}=\bar{T}_{\mu\nu\eta}{}^{abc}+A_{\mu\nu\eta}{}^{abc}(z)\delta(x-z)\delta(x-z) \qquad (4.12)
$$

has the right divergence, that is,

$$
\partial_\mu{}^{(x)}\tilde{T}_{\mu\nu\eta}{}^{abc}=\epsilon_{acd}\tilde{T}_{\nu\eta}{}^{bd}\delta(x-z)+\epsilon_{abd}\tilde{T}_{\nu\eta}{}^{dc}\delta(x-y)\,.
$$

It is easy to check that we also have

$$
\partial_\eta{}^{(y)}\tilde{T}_{\mu\nu\eta}{}^{abc}=\epsilon_{bad}\tilde{T}_{\mu\eta}{}^{dc}\delta(y-z)+\epsilon_{bcd}\tilde{T}_{\mu\eta}{}^{ad}\delta(y-z)
$$

and

$$
\partial_\eta{}^{(z)}\tilde{T}_{\mu\nu\eta}{}^{abc}=\epsilon_{cad}\tilde{T}_{\mu\nu}{}^{db}\delta(z-x)+\epsilon_{cbd}\tilde{T}_{\mu\nu}{}^{ad}\delta(z-y)\,.
$$

Thus we have found that $\tilde{T}_{\mu\nu\eta}{}^{abc}$ has the right divergence. To see that it is also covariant, we refer to the Appendix B for the proofs.

Now we come to the case of four currents. Again the complications arise due to the fact that another new singular term shows up in the commutator $[j_0{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)$. For the same reason as stated before for the case of three currents, we need to know the structure of this commutator. To find it, we again use the Jacobi identity

$$
\begin{aligned}
[j_0{}^a(x),[j_0{}^b(y),S_{lk}{}^{cd}(0)]]\delta(x_0)\delta(y_0)
&=[[j_0{}^a(x),j_0{}^b(y)],S_{lk}{}^{cd}(0)]\delta(x_0)\delta(y_0)\\
&\quad +[j_0{}^b(y),[j_0{}^a(x),S_{lk}{}^{cd}(0)]]\delta(x_0)\delta(y_0)\,.
\end{aligned}
$$

Multiplying the above identity by y_m and using (2.1) and (4.3) yields

$$
\begin{aligned}
[j_0{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)\delta(y)&=C_{mlk}{}^{abcd}\delta(x)\delta(y)\\
&\quad -y_m[j_0{}^b(y),S_{nlk}{}^{abc}(0)]\delta(y_0)\partial_n\delta(x)\,, \qquad (4.13)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{mlk}{}^{abcd}\equiv \epsilon_{abe}S_{mlk}{}^{ecd}(0)+\epsilon_{ace}S_{mlk}{}^{bed}(0)&\\
+\epsilon_{ade}S_{mlk}{}^{bce}(0)&\,. \qquad (4.14)
\end{aligned}
$$

Again Eq. (4.13) assures the existence of another new term $S_{nmlk}{}^{abcd}$ defined by

$$
S_{nmlk}{}^{abcd}(0)\delta(x)=x_n[j_0{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)\,. \qquad (4.15)
$$

By inserting (4.13) into (4.15), we see that $S_{nmlk}{}^{abcd}$ is completely symmetric in the pairs of indices $({}_n{}^a)$, $({}_m{}^b)$, $({}_e{}^c)$, and $({}_k{}^d)$. So (4.13) now reads

$$
\begin{aligned}
[j_0{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)&=C_{mlk}{}^{abcd}(0)\delta(x)\\
&\quad -S_{nmlk}{}^{abcd}(0)\partial_n\delta(x)\,. \qquad (4.16)
\end{aligned}
$$

With this we see that the covariant time-ordered product $\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$ for four-currents is given by

$$
\begin{aligned}
\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}=T_{\mu\nu\eta\xi}{}^{abcd}+\rho^{(1)}{}_{\mu\nu\eta\xi}{}^{abcd}+\rho^{(2)}{}_{\mu\nu\eta\xi}{}^{abcd}&\\
+R_{\mu\nu\eta\xi}{}^{abcd}&\,, \qquad (4.17)
\end{aligned}
$$

with

$$
T_{\mu\nu\eta\xi}{}^{abcd}=T(j_\mu{}^a(x)j_\nu{}^b(y)j_\eta{}^c(z)j_\xi{}^d(w))\,, \qquad (4.18)
$$

$$
\begin{aligned}
\rho^{(1)}{}_{\mu\nu\eta\xi}{}^{abcd}=\delta(x-y)T(R_{\mu\nu}{}^{ab}j_\eta{}^c(z)j_\xi{}^d(w))&\\
+\delta(x-z)T(R_{\mu\eta}{}^{ac}(z)j_\nu{}^b(y)j_\xi{}^d(w))&\\
+\delta(x-w)T(R_{\mu\xi}{}^{ad}(w)j_\nu{}^b(y)j_\eta{}^c(z))&\\
+\delta(y-z)T(R_{\nu\eta}{}^{bc}(z)j_\mu{}^a(x)j_\xi{}^d(w))&\\
+\delta(y-w)T(R_{\nu\xi}{}^{bd}(w)j_\mu{}^a(x)j_\eta{}^c(z))&\\
+\delta(z-w)T(R_{\eta\xi}{}^{cd}(w)j_\mu{}^a(x)j_\nu{}^b(y))&\,, \qquad (4.19)
\end{aligned}
$$

$$\rho^{(2)}{}_{\mu\nu\eta\xi}{}^{abcd}=\delta(x-y)\delta(z-w)T(R_{\mu\nu}{}^{ab}(y)R_{\eta\xi}{}^{cd}(w))$$
$$+\delta(y-z)\delta(x-w)T(R_{\mu\xi}{}^{ad}(w)R_{\nu\eta}{}^{bc}(z))$$
$$+\delta(x-z)\delta(y-w)T(R_{\mu\eta}{}^{ac}(z)R_{\nu\xi}{}^{bd}(w)), \tag{4.20}$$

$$R_{\mu\nu\eta\xi}{}^{abcd}=\delta(x-y)\delta(x-z)T(A_{\mu\nu\eta}{}^{abc}(z)j_{\xi}{}^{d}(w))$$
$$+\delta(z-x)\delta(x-w)T(A_{\mu\eta\xi}{}^{abd}(w)j_{\nu}{}^{b}(y))$$
$$+\delta(x-y)\delta(x-w)T(A_{\mu\nu\xi}{}^{abd}(w)j_{\eta}{}^{c}(z))$$
$$+\delta(y-z)\delta(z-w)T(A_{\nu\eta\xi}{}^{bcd}(w)j_{\mu}{}^{a}(x))$$
$$+D_{\mu\nu\eta\xi}{}^{abcd}(w)\delta(x-w)\delta(y-w)\delta(z-w), \tag{4.21}$$

where we have introduced $D_{\mu\nu\eta\xi}{}^{abcd}$ defined by

$$D_{nmlk}{}^{abcd}\equiv S_{nmlk}{}^{abcd},$$
$$D_{4\nu\eta\xi}{}^{abcd}=D_{\mu4\eta\xi}{}^{abcd}=D_{\mu\nu4\xi}{}^{abcd}=D_{\mu\nu\eta4}{}^{abcd}\equiv 0.$$

With (4.16), it is not hard to see that $\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$ has the right divergences, that is,

$$\partial_{\mu}{}^{(x)}\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}=\epsilon_{abe}\tilde{T}_{\nu\eta\xi}{}^{ecd}\delta(x-y)+\epsilon_{ace}\tilde{T}_{\nu\eta\xi}{}^{bed}\delta(x-z)$$
$$+\epsilon_{ade}\tilde{T}_{\nu\eta\xi}{}^{bce}\delta(x-w),$$

and similar expressions for $\partial_{\nu}{}^{(y)}\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$, etc. The proofs for its covariance is quite complicated. It is given in Appendix B.

If we compare the covariant time-ordered product for the case of three or four currents in this section with the corresponding one obtained in Sec. III, we find they almost have the same structure except for the additional term which involves the new terms $S_{mlk}{}^{abc}$ and $S_{nmlk}{}^{abcd}$.

When we come to the general case of n currents, a series of new terms $S_{mlk}{}^{abc}$, $S_{nmlk}{}^{abcd}$, $\cdots$ shows up. So things get quite complicated. The explicit form for covariant T product is too involved to be written down, and the proofs of its covariance are too messy to be given. But we emphasize that the answer does exist. We note that if all these new terms were zero, then it would have the same structure as the one we have obtained in Sec. III for the case of n currents.

APPENDIX A

We show in this Appendix that $\tilde{T}_{\mu_1\mu_2\cdots\mu_n}$ defined by (3.23) is covariant under Lorentz transformation. As we have seen earlier in Secs. II and III, an operator $M_{\mu_1\mu_2\cdots\mu_n}$ is covariant if and only if it has the following commutation relation with the boosting operator K_r:

$$[K_r,M_{\mu_1\mu_2\cdots\mu_n}]=\sum_{i=1}^{n}\delta_{\mu_i4}M_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}$$
$$-\sum_{i=1}^{n}\delta_{\mu_ir}M_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}$$
$$+(\sum_{i=1}^{n}L_{r4}{}^{(x_i)})M_{\mu_1\mu_2\cdots\mu_n}. \tag{A1}$$

We observe that for any index, say μ_i, in $\rho_{\mu_1\cdots\mu_n}{}^{(l)}$ it is either attached to the a's as a subscript or attached to the current as a subscript. For any index μ_i, we define $\rho_{\mu_1\cdots\mu_i\cdots\mu_n}{}^{l[\mu_i,\mathrm{in}]}$ to be the sum of terms in $\rho_{\mu_1\cdots\mu_n}{}^{(l)}$ with μ_i attached to the current. The number of terms in $\rho_{\mu_1\cdots\mu_n}{}^{l[\mu_i,\mathrm{in}]}$ is $N_{\mu_i,\mathrm{in}}{}^{(l)}$ with

$$N_{\mu_i,\mathrm{in}}{}^{(l)}=C_2{}^{n-1}C_2{}^{n-3}C_2{}^{n-5}\cdots C_2{}^{n-(2l-1)}/l!, \tag{A2}$$

where the C's are the binomial coefficients, i.e.,

$$C_2{}^n=\binom{n}{2}=\frac{n(n-1)}{2!}.$$

Denote $\rho_{\mu_1\cdots\mu_i\cdots\mu_n}{}^{l[\mu_i,\mathrm{out}]}$ to be the sum of terms in $\rho_{\mu_1\cdots\mu_n}{}^{(l)}$ with μ_i attached to the a's. The number of terms in $\rho_{\mu_1-\mu_i\mu_n}{}^{l[\mu_i,\mathrm{out}]}$ is $N_{\mu_i,\mathrm{out}}{}^{(l)}$ with

$$N_{\mu,i\mathrm{out}}{}^{(l)}=C_1{}^{n-1}C_2{}^{n-2}C_2{}^{n-4}\cdots C_2{}^{n-2(l-1)}/(l-1)! \tag{A3}$$

As a check we see

$$N_{\mu_i,\mathrm{in}}{}^{(l)}+N_{\mu_i,\mathrm{out}}{}^{(l)}$$
$$=C_1{}^{n-1}C_2{}^{n-2}C_2{}^{n-4}\cdots C^{n-2(l-1)}/(l-1)!$$
$$+C_2{}^{n-1}C_2{}^{n-3}C_2{}^{n-5}\cdots C^{n-(2l-1)}/l!$$
$$=[C_2{}^{n-2}C_2{}^{n-4}\cdots C^{n-2(l-1)}/l!]$$
$$\times[2l+n-2(l-1)-1-1]\tfrac{1}{2}(n-1)$$
$$=C_2{}^nC_2{}^{n-2}\cdots C_2{}^{n-2(l-1)}/l!$$
$$=N^{(l)}.$$

It is just equal to the total number of terms in $\rho_{\mu_i\cdots\mu_n}{}^{(l)}$ or, in other words,

$$\rho_{\mu_1\cdots\mu_i\cdots\mu_n}{}^{(l)}=\rho_{\mu_1\cdots\mu_n}{}^{l[\mu_i,\mathrm{in}]}+\rho_{\mu_1\cdots\mu_n}{}^{l[\mu_i,\mathrm{out}]}. \tag{A4}$$

From (3.5) we get

$$(1/i)(x_i-x_j)_r[j_{\mu_i}(x_i),j_{\mu_j}(x_j)]\delta(x_{i0}-x_{j0})$$
$$=\delta_{\mu_i4}a_{r\mu_j}\delta(x_i-x_j)S(x_j)+\delta_{\mu_j4}a_{\mu_ir}\delta(x_i-x_j)S(x_i). \tag{A5}$$

By taking into account (3.12), (3.14), (3.22), and (A5) together with the definitions of $\rho_{\mu_1\cdots\mu_i\cdots\mu_n}{}^{l[\mu_i,\mathrm{in}]}$ and $\rho_{\mu_1\cdots\mu_i\cdots\mu_n}{}^{l[\mu_i,\mathrm{out}]}$, we get

$$[K_r,T_{\mu_1\mu_2\cdots\mu_n}]=\sum_{i=1}^{n}\delta_{\mu_i4}T_{\mu_1\mu_2\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}$$
$$-\sum_{i=1}^{n}\delta_{\mu_ir}T_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}$$
$$+\sum_{i=1}^{n}\delta_{\mu_i4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l[r,\mathrm{out}]}$$
$$+(\sum_{i=1}^{n}L_{r4}{}^{(x_i)})T_{\mu_1\mu_2\cdots\mu_n} \tag{A6}$$

and

$$[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(l)}]=\sum_{i=1}^{n}\delta_{\mu_i4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l[r,\text{in}]}-\sum_{i=1}^{n}\delta_{\mu_i,r}\rho_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}{}^{l[4,\text{in}]}+\sum_{i=1}^{n}\delta_{\mu_i,4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l+1[r,\text{out}]}+\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\rho_{\mu_1\cdots\mu_n}{}^{(l)}\quad\text{(A7)}$$

for $l<\frac{1}{2}n$ if n is even or $\frac{1}{2}(n-1)$ if n is odd.

The terms $\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l+1[r,\text{out}]}$ are obtained as a result of making use of (A5). As a check, we see the number of terms having the coefficient $\delta_{\mu_i,4}$, coming from applying (A5) in $[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(l)}]$ is

$$\begin{aligned}&N_{\mu_i,\text{in}}{}^{(l)}C_1{}^{n-(2l+1)}\\&=C_2{}^{n-1}C_2{}^{n-3}C_2{}^{n-5}\cdots C_2{}^{n-(2l-1)}(n-2l-1)/l!\\&=\tfrac{1}{2}(n-1)(n-2)[\tfrac{1}{2}(n-3)(n-4)]\cdots\\&\qquad\times[\tfrac{1}{2}(n-2l+1)(n-2l)](n-2l-1)/l!\\&=C_1{}^{n-1}C_2{}^{n-2}\cdots C_2{}^{n-2}/l!\\&=N_{\mu_i,\text{out}}{}^{(l+1)},\end{aligned}$$

where use was made of (A3), which is just the number of terms in $\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l+1[r,\text{out}]}$ which appeared in (A7). Recall that $a_{4\mu}=a_{\mu4}=0$. So it is easily seen that

$$\rho_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}{}^{l[4,\text{out}]}=0\ \text{ for all } i.\qquad\text{(A8)}$$

We can therefore rewrite (A7) as follows by using (A4) and (A8): For $l<\frac{1}{2}n$ if n is even, or $\frac{1}{2}(n-1)$ if n is odd,

$$[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(l)}]=\sum_{i=1}^{n}\delta_{\mu_i4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l[r,\text{in}]}-\sum_{i=1}^{n}\delta_{\mu_ir}\rho_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}{}^{(l)}+\sum_{i=1}^{n}\delta_{\mu_i4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l+1[r,\text{out}]}+\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\rho_{\mu_1\cdots\mu_n}{}^{(l)}.\quad\text{(A9)}$$

Now for even n, $\rho_{\mu_1\cdots\mu_n}{}^{(n/2)}$ contains only time-ordered products of $\frac{1}{2}n$ Schwinger terms, so we use (3.22) to get

$$[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(n/2)}]=\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\rho_{\mu_1\cdots\mu_n}{}^{(n/2)}.\quad\text{(A10)}$$

For odd n, $\rho_{\mu_1\cdots\mu_n}{}^{(n-1)/2}$ contains only T products of $\frac{1}{2}(n-1)$ Schwinger terms with only one current, so by using (3.12), (3.14), and (3.22) we get

$$[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(n-1)/2}]=\sum_{i=1}^{n}\delta_{\mu_i4}\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{(n-1)/2\,[r,\text{in}]}-\sum_{i=1}^{n}\delta_{\mu_ir}\rho_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}{}^{(n-1)/2}+\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\rho_{\mu_1\cdots\mu_n}{}^{(n-1)/2}.\quad\text{(A11)}$$

From (3.24),

$$[K_r,\tilde{T}_{\mu_1\cdots\mu_n}]=[K_r,T_{\mu_1\mu_2\cdots\mu_n}]+\sum_{l=1}^{m}[K_r,\rho_{\mu_1\cdots\mu_n}{}^{(l)}],$$

where

$$\begin{aligned}m&=\tfrac{1}{2}n\text{ for even }n\\&=\tfrac{1}{2}(n-1)\text{ for odd }n.\end{aligned}$$

Using (A6), (A9), and (A11) then yields

$$\begin{aligned}[K_r,\tilde{T}_{\mu_1\cdots\mu_n}]=&\sum_{i=1}^{n}\delta_{\mu_i4}\tilde{T}_{\mu_1\cdots\mu_{i-1}r\mu_{i+n}\cdots\mu_n}\\&-\sum_{i=1}^{n}\delta_{\mu_ir}\Big(T_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}+\sum_{i=1}^{m}\rho_{\mu_i\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}{}^{(l)}\Big)\\&+\sum_{i=1}^{n}\delta_{\mu_i4}\sum_{l=1}^{m}\Big(\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l[r,\text{in}]}+\rho_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}{}^{l[r,\text{out}]}\Big)\\&+\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\Big(T_{\mu_1\cdots\mu_n}+\sum_{i=1}^{m}\rho_{\mu_1\cdots\mu_n}{}^{(l)}\Big)\end{aligned}$$

or

$$[K_r,\tilde{T}_{\mu_1\cdots\mu_n}]=\sum_{i=1}^{n}\delta_{\mu_i4}\tilde{T}_{\mu_1\cdots\mu_{i-1}r\mu_{i+1}\cdots\mu_n}-\sum_{i=1}^{n}\delta_{\mu_ir}\tilde{T}_{\mu_1\cdots\mu_{i-1}4\mu_{i+1}\cdots\mu_n}+\Big(\sum_{i=1}^{n}L_{r4}{}^{(x_i)}\Big)\tilde{T}_{\mu_1\cdots\mu_n},\quad\text{(A12)}$$

which is exactly the same form as (A1) with $\tilde{T}_{\mu_1\cdots\mu_n}$ in place of $M_{\mu_1\cdots\mu_n}$. Therefore, $\tilde{T}_{\mu_1\cdots\mu_n}$ so defined by (3.23) is indeed covariant under Lorentz transformation.

APPENDIX B

We will show in this appendix that $\tilde{T}_{\mu\nu\eta}{}^{abc}$ and $\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$, defined, respectively, by (4.8) and (4.17), are covariant under Lorentz transformation. Applying the booster K_r to Eq. (4.3) yields

$$[K_r,[j_0{}^a(x),S_{lk}{}^{bc}(0)]]\delta(x_0)=[K_r,B_{lk}{}^{abc}(0)]\delta(x)-[K_r,S_{mlk}{}^{abc}(0)]\partial_m\delta(x)$$

or

$$(1/i)(\partial/\partial x_m)x_r[j_m{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)+[j_0{}^a(x),[K_r,S_{lk}{}^{bc}(0)]]\delta(x_0)=[K_r,B_{lk}{}^{abc}(0)]\delta(x)-[K_r,S_{mlk}{}^{abc}(0)]\partial_m\delta(x)\,, \quad \text{(B1)}$$

where $B_{lk}{}^{abc}(0)\equiv\epsilon_{abd}S_{lk}{}^{dc}(0)+\epsilon_{acd}S_{lk}{}^{bd}(0)$ and we have used current conservation. Now from (2.21) and the Jacobi identity,

$$\begin{aligned}
[j_0{}^a(x),[K_r,S_{lk}{}^{bc}(0)]]\delta(x_0)\delta(y)&=-(1/i)[j_0{}^a(x),y_r[j_l{}^b(y),j_k{}^c(0)]]\delta(x_0)\delta(y_0)\\
&=-(y_r/i)[[j_0{}^a(x),j_l{}^b(y)],j_k{}^c(0)]\delta(x_0)\delta(y_0)-(y_r/i)[j_l{}^b(y),[j_0{}^a(x),j_k{}^c(0)]]\delta(x_0)\delta(y_0)\\
&=-(y_r/i)\epsilon_{abd}[j_l{}^d(y),j_k{}^c(0)]\delta(y_0)\delta(x-y)-(y_r/i)\epsilon_{acd}[j_l{}^b(y),j_k{}^d(0)]\delta(y_0)\delta(x)\\
&\quad+(y_r/i)[S_{ml}{}^{ab}(y),j_k{}^c(0)]\partial_m\delta(x-y)\delta(y_0)+(y_r/i)[j_l{}^b(y),S_{mk}{}^{ac}(0)]\partial_m\delta(x-y)\delta(y_0)\\
&=[K_r,B_{lk}{}^{abc}(0)]+(y_r/i)[S_{ml}{}^{ab}(y),j_k{}^c(0)]\partial_m\delta(x-y)\delta(y_0)\\
&\quad+(y_r/i)[j_l{}^b(y),S_{mk}{}^{ac}(0)]\partial_m\delta(x)\delta(y_0)\,,
\end{aligned}$$

so that Eq. (B1) becomes

$$\partial_m\{x_r[j_m{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)\}\delta(y)+y_r[S_{ml}{}^{ab}(y),j_k{}^c(0)]\partial_m\delta(x-y)\delta(y_0)+y_r[j_l{}^b(y),S_{mk}{}^{ac}(0)]\partial_m\delta(x)\delta(y_0)=-i[K_r,S_{mlk}{}^{abc}(0)]\partial_m\delta(x)\delta(y)\,. \quad \text{(B2)}$$

By using the argument similar to that which leads from (2.16) to (2.21), we get

$$x_r[j_m{}^a(x),S_{lk}{}^{bc}(0)]\delta(x_0)\delta(y)+y_r[S_{ml}{}^{ab}(y),j_k{}^c(0)]\delta(x-y)\delta(y_0)+y_r[j_l{}^b(y),S_{mk}{}^{ac}(0)]\delta(y_0)\delta(x)=-i[K_r,S_{mlk}{}^{abc}(0)]\delta(x)\delta(y)\,. \quad \text{(B3)}$$

In terms of $R_{\nu\eta}{}^{bc}(z)$, $R_{\mu\eta}{}^{ac}(z)$, and $A_{\mu\nu\eta}{}^{abc}(z)$, the above equation becomes

$$\begin{aligned}
&[(x_r-z_r)/i][j_\mu{}^a(x),R_{\nu\eta}{}^{bc}(z)]\delta(x_0-z_0)\delta(y-z)+[(y_r-z_r)/i][R_{\mu\nu}{}^{ab}(y)j_\eta{}^c(z)]\delta(y_0-z_0)\delta(x-y)\\
&\quad+[(y_r-z_r)/i][j_\nu{}^b(y),R_{\mu\eta}{}^{ac}(z)]\delta(y_0-z_0)\delta(x-z)\\
&=[\delta_{\mu4}A_{r\nu\eta}{}^{abc}(z)+\delta_{\nu4}A_{\mu r\eta}{}^{abc}(z)+\delta_{\eta4}A_{\mu\nu r}{}^{abc}(z)]\delta(x-y)\delta(y-z)\\
&\quad-\{[K_r,A_{\mu\nu\eta}{}^{abc}(z)]-L_{r4}{}^{(z)}A_{\mu\nu\eta}{}^{abc}(z)\}\delta(z-y)\delta(y-z)\,. \quad \text{(B4)}
\end{aligned}$$

Note that

$$\begin{aligned}
[K_r,T_{\mu\nu\eta}{}^{abc}]&=\delta_{\mu4}T_{r\nu\eta}{}^{abc}-\delta_{\mu r}T_{4\nu\eta}{}^{abc}+\delta_{\nu4}T_{\mu r\eta}{}^{abc}-\delta_{\nu r}T_{\mu4\eta}{}^{abc}+\delta_{\eta4}T_{\mu\nu r}{}^{abc}-\delta_{\eta r}T_{\mu\nu4}{}^{abc}\\
&\quad+[(x_r-y_r)/i]T([j_\mu{}^a(x),j_\nu{}^b(y)]\delta(x_0-y_0)j_\eta{}^c(z))+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)})T_{\mu\nu\eta}{}^{abc}\\
&\quad+[(y_r-z_r)/i]T([j_\nu{}^b(y),j_\eta{}^c(z)]\delta(y_0-z_0)j_\mu{}^a(x))+[(z_r-x_r)/i]T([j_\eta{}^c(z),j_\mu{}^a(x)]\delta(z_0-x_0)j_\nu{}^b(y)) \quad \text{(B5)}
\end{aligned}$$

and

$$\begin{aligned}
[K_r,T(R_{\mu\nu}{}^{ab}(y),j_\eta{}^c(z))]&=T\{([K_r,R_{\mu\nu}{}^{ab}(y)]-L_{r4}{}^{(y)}R_{\mu\nu}{}^{ab})j_\eta{}^c(z)\}+\delta_{\mu4}T(R_{\mu\nu}{}^{ab}(y)j_r{}^c(z))-\delta_{\eta r}T(R_{\mu\nu}{}^{ab}(y)j_4{}^c(z))\\
&\quad+[(x_r-y_r)/i][R_{\mu\nu}{}^{ab}(y),j_\eta{}^c(z)]\delta(y_0-z_0)+(L_{r4}{}^{(y)}+L_{r4}{}^{(z)})T(R_{\mu\nu}{}^{ab}(y)j_\eta{}^c(z))\,. \quad \text{(B6)}
\end{aligned}$$

Using the relation

$$[(x_r-y_r)/i][j_\mu{}^a(x),j_\nu{}^b(y)]\delta(x_0-y_0)=\delta_{\mu4}R_{r\nu}{}^{ab}\delta(x-y)+\delta_{\nu4}R_{\mu r}{}^{ab}(y)\delta(x-y)-\{[K_r,R_{\mu\nu}{}^{ab}(y)]-L_{r4}{}^{(y)}R_{\mu\nu}{}^{ab}(y)]\delta(x-y)\}\,,$$

together with (4.8), (4.10), and (B4), yields

$$\begin{aligned}
[K_r,\tilde{T}_{\mu\nu\eta}{}^{abc}]&=[K_r,T_{\mu\nu\eta}{}^{abc}]+[K_r,\rho_{\mu\nu\eta}{}^{abc}]+[K_r,A_{\mu\nu\eta}{}^{abc}(z)]\delta(x-y)\delta(y-z)\\
&=\delta_{\mu4}\tilde{T}_{r\nu\eta}{}^{abc}+\delta_{\nu4}\tilde{T}_{\mu r\eta}{}^{abc}+\delta_{\eta4}\tilde{T}_{\mu\nu r}{}^{abc}-\delta_{\mu r}[T_{4\nu\eta}{}^{abc}+T(R_{\nu\eta}{}^{bc}(z)j_4{}^a(x))\delta(y-z)]\\
&\quad-\delta_{\nu r}[T_{\mu4\eta}{}^{abc}+T(R_{\eta\mu}{}^{ca}(z)j_4{}^b(y))\delta(x-z)]-\delta_{\eta r}[T_{\mu\nu4}{}^{abc}+T(R_{\mu\nu}{}^{ab}(y)j_4{}^c(z))\delta(x-y)]\\
&\quad+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)})\tilde{T}_{\mu\nu\eta}{}^{abc}\,. \quad \text{(B7)}
\end{aligned}$$

From the definition of $A_{\mu\nu\eta}{}^{abc}$ and $R_{\mu\nu}{}^{ab}$, we recall

$$A_{4\nu\eta}{}^{abc}=A_{\mu4\eta}{}^{abc}=A_{\mu\nu4}{}^{abc}=0 \qquad \text{and} \qquad R_{4\nu}{}^{ab}=R_{\mu4}{}^{ab}=0\,.$$

So (B7) now reads

$$\begin{aligned}
[K_r,\tilde{T}_{\mu\nu\eta}{}^{abc}]&=\delta_{\mu4}\tilde{T}_{r\nu\eta}{}^{abc}-\delta_{\mu r}\tilde{T}_{4\nu\eta}{}^{abc}+\delta_{\nu4}\tilde{T}_{\mu r\eta}{}^{abc}-\delta_{\nu r}\tilde{T}_{\mu4\eta}{}^{abc}+\delta_{\eta4}\tilde{T}_{\mu\nu r}{}^{abc}-\delta_{\eta r}\tilde{T}_{\mu\nu4}{}^{abc}\\
&\quad+[L_{r4}{}^{(x)}+(L_{r4}{}^{(y)}+L_{r4}{}^{(z)})\tilde{T}_{\mu\nu\eta}{}^{abc}]\,. \quad \text{(B8)}
\end{aligned}$$

Therefore, we see that $\tilde{T}_{\mu\nu\eta}{}^{abc}$ defined by (4.8) is covariant under Lorentz transformation. Now we go on to show that $\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$ defined by (4.17) is Lorentz covariant.

Apply booster K_r to (4.16):

$$[K_r,[j_0{}^a(x),S_{mlk}{}^{bcd}(0)]]\delta(x_0)=[K_r,C_{mlk}{}^{abcd}(0)]\delta(x)-[K_r,S_{nmlk}{}^{abcd}(0)]\partial_n\delta(x)$$

or

$$(1/i)(\partial/\partial x_n)x_r[j_n{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)+[j_0{}^a(x),[K_r,S_{mlk}{}^{bcd}(0)]]\delta(x_0)$$
$$=[K_r,C_{mlk}{}^{abcd}(0)]\delta(x)-[K_r,S_{nmlk}{}^{abcd}(0)]\partial_n\delta(x)\,. \quad \text{(B9)}$$

Replacing x by y, y by z, a by b, b by c, and c by d in (B3), then making use of it, we find

$$-i[j_0{}^a(x),[K_r,S_{mlk}{}^{bcd}(0)]]\delta(x_0)\delta(y)\delta(z)=y_r[j_0{}^a(x),[j_m{}^b(y),S_{lk}{}^{cd}(0)]]\delta(x_0)\delta(z)\delta(y_0)$$
$$+z_r[j_0{}^a(x),[S_{ml}{}^{bc}(z),j_k{}^d(0)]]\delta(x_0)\delta(y-z)\delta(z_0)+z_r[j_0{}^a(x),[j_l{}^c(z),S_{mk}{}^{bd}(0)]]\delta(x_0)\delta(y)\delta(z_0)\,. \quad \text{(B10)}$$

Now by the Jacobi identity, we see that

$$y_r[j_0{}^a(x),[j_m{}^b(y),S_{lk}{}^{cd}(0)]]\delta(x_0)\delta(z)\delta(y_0)$$
$$=y_r[\epsilon_{abe}j_m{}^e(y)\delta(x-y)-S_{nm}{}^{ab}(y)\partial_n\delta(x-y),S_{lk}{}^{cd}(0)]\delta(x_0)\delta(y_0)\delta(z)+y_r[j_m{}^b(y),[j_0{}^a(x),S_{lk}{}^{cd}(0)]]\delta(x_0)\delta(z)\delta(y_0)$$
$$=\epsilon_{abe}y_r[j_m{}^e(y),S_{lk}{}^{cd}(0)]\delta(x-y)\delta(y_0)\delta(z)+\epsilon_{ace}y_r[j_m{}^b(y),S_{lk}{}^{ed}(0)]\delta(y_0)\delta(x)\delta(z)$$
$$+\epsilon_{ade}y_r[j_m{}^b(y),S_{lk}{}^{ce}(0)]\delta(y_0)\delta(x)\delta(z)-y_r[S_{nm}{}^{ab}(y),S_{lk}{}^{cd}(0)]\delta(y_0)\delta(z)\partial_n\delta(x-y)$$
$$-y_r[j_m{}^b(y),S_{nlk}{}^{acd}(0)]\delta(y_0)\delta(z)\partial_n\delta(x)\,.$$

Using the above relation together with (B3) and (4.14), Eq. (B10) becomes

$$i[j_0{}^a(x),[K_r,S_{mlk}{}^{bcd}(0)]]\delta(x_0)\delta(y)\delta(z)$$
$$=i[K_r,C_{mlk}{}^{abcd}(0)]\delta(x)\delta(y)\delta(z)+y_r[S_{nm}{}^{ab}(y),S_{lk}{}^{cd}(0)]\delta(y_0)\delta(z)\partial_n\delta(x-y)$$
$$+z_r[S_{ml}{}^{bc}(z),S_{nk}{}^{ad}(0)]\delta(z_0)\delta(y-z)\partial_n\delta(x)+z_r[S_{nl}{}^{ac}(z),S_{mk}{}^{bd}(0)]\delta(z_0)\delta(y)\partial_n\delta(x-y)$$
$$+y_r[j_m{}^b(y),S_{nlk}{}^{acd}(0)]\delta(y_0)\delta(z)\partial_n\delta(x)+z_r[S_{nml}{}^{abc}(z),j_k{}^d(0)]\delta(z_0)\delta(y-z)\partial_n\delta(x-z)$$
$$+z_r[j_l{}^c(z),S_{nmk}{}^{abd}(0)]\delta(z_0)\delta(y)\partial_n\delta(x)\,. \quad \text{(B11)}$$

By inserting (B11) into (B9) and using the argument similar to that which leads from (2.16) to (2.21), we obtain

$$y_r[S_{nm}{}^{ab}(y),S_{lk}{}^{cd}(0)]\delta(y_0)\delta(z)\delta(x-y)+z_r[S_{ml}{}^{bc}(z),S_{nk}{}^{ad}(0)]\delta(z_0)\delta(x)\delta(y-z)+z_r[S_{nl}{}^{ac}(z),S_{mk}{}^{bd}(0)]\delta(z_0)\delta(y)\delta(x-z)$$
$$+y_r[j_m{}^b(y),S_{nlk}{}^{acd}(0)]\delta(y_0)\delta(x)\delta(z)+z_r[S_{nml}{}^{abc}(z),j_k{}^d(0)]\delta(z_0)\delta(x-y)\delta(y-z)$$
$$+z_r[j_l{}^c(z),S_{nmk}{}^{abd}(0)]\delta(z_0)\delta(x)\delta(y)+x_r[j_n{}^a(x),S_{mlk}{}^{bcd}(0)]\delta(x_0)\delta(y)\delta(z)$$
$$=-[K_r,S_{nmlk}{}^{abcd}(0)]\delta(x)\delta(y)\delta(z)\,. \quad \text{(B12)}$$

Replacing x, y, and z by $x-\omega$, $y-\omega$, $z-\omega$, respectively, in (B12), and applying the translational operator $e^{iP\omega}$, then rewriting it in terms of $D_{\mu\nu\eta\xi}{}^{abcd}$, $A_{\mu\nu\eta}{}^{abc}$, $R_{\mu\nu}{}^{ab}$, etc., we get

$$(y_r-\omega_r)[R_{\mu\nu}{}^{ab}(y),R_{\eta\xi}{}^{cd}(\omega)]\delta(y_0-\omega_0)\delta(z-\omega)\delta(x-y)+(z_r-\omega_r)[R_{\nu\eta}{}^{bc}(z),R_{\mu\xi}{}^{ad}(\omega)]\delta(z_0-\omega_0)\delta(x-\omega)\delta(y-z)$$
$$+(z_r-\omega_r)[R_{\mu\eta}{}^{ac}(z),R_{\nu\xi}{}^{bd}(\omega)]\delta(z_0-\omega_0)\delta(y-\omega)\delta(x-z)+(y_r-\omega_r)[j_\nu{}^b(y),A_{\mu\eta\xi}{}^{acd}(\omega)]\delta(y_0-\omega_0)\delta(x-\omega)\delta(z-\omega)$$
$$+(x_r-\omega_r)[A_{\mu\nu\eta}{}^{abc}(x),j_\xi{}^d(\omega)]\delta(z_0-\omega_0)\delta(x-z)\delta(y-z)+(z_r-\omega_r)[j_\eta{}^c(z),A_{\mu\nu\xi}{}^{abd}(\omega)]\epsilon(z_0-\omega_0)\delta(z-\omega)\delta(y-\omega)$$
$$+(x_r-\omega_r)[j_\mu{}^a(x),A_{\nu\eta\xi}{}^{bcd}(\omega)]\delta(x_0-\omega_0)\delta(y-\omega)\delta(z-\omega)=[\delta_{\mu4}D_{r\nu\eta\xi}{}^{abcd}(\omega)+\delta_{\nu4}D_{\mu r\eta\xi}{}^{abcd}(\omega)+\delta_{\eta4}D_{\mu\nu r\xi}{}^{abcd}(\omega)$$
$$+\delta_{4\xi}D_{\mu\nu\eta r}{}^{abcd}(\omega)]\delta(x-\omega)\delta(y-\omega)\delta(z-\omega)-\{[K_r,D_{\mu\nu\eta\xi}{}^{abcd}(\omega)]-L_{r4}{}^{(\omega)}D_{\mu\nu\eta\xi}{}^{abcd}(\omega)\}\delta(x-\omega)\delta(y-\omega)\delta(z-\omega)\,.$$
$$\text{(B13)}$$

Using the method similar to that employed in Sec. III to prove the existence of the covariant time-ordered product of 4 currents together with (B13), which takes care of the additional term $R_{\mu\nu\eta\xi}{}^{abcd}$ under Lorentz transformation, one gets

$$[K_r,\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}]=\delta_{\mu4}\tilde{T}_{r\nu\eta\xi}{}^{abcd}-\delta_{\mu r}\tilde{T}_{4\nu\eta\xi}{}^{abcd}+\delta_{\nu4}\tilde{T}_{\mu r\eta\xi}{}^{abcd}-\delta_{\nu r}\tilde{T}_{\mu4\eta\xi}{}^{abcd}+\delta_{\eta4}\tilde{T}_{\mu\nu r\xi}{}^{abcd}-\delta_{\eta r}\tilde{T}_{\mu\nu4\xi}{}^{abcd}$$
$$+\delta_{\xi4}\tilde{T}_{\mu\nu\eta r}{}^{abcd}-\delta_{\xi r}\tilde{T}_{\mu\nu\eta4}{}^{abcd}+(L_{r4}{}^{(x)}+L_{r4}{}^{(y)}+L_{r4}{}^{(z)}+L_{r4}{}^{(\omega)})\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}\,. \quad \text{(B14)}$$

This assures that $\tilde{T}_{\mu\nu\eta\xi}{}^{abcd}$, as defined by (4.17), is Lorentz covariant.