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# Infinities of Nonlinear and Lagrangian Theories

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The technique for summing perturbation contributions introduced by Efimov and Fradkin is extended and applied to nonlinear (chiral) Lagrangian theories. It is shown that the only likely infinites in these theories are those associated with self-mass and self-charge.

## I. INTRODUCTION

ONE of the significant recent advances in particle theory has been the formulation of chirally invariant Lagrangian theories.[1] These theories have so far been used with reasonable success for predicting low-energy (soft-meson) amplitudes in the following way: The interaction Lagrangian—an exponential or rational function of the spin-zero meson fields $\varphi^i$—is expanded as an infinite power series in $\varphi^i$ and then used to evaluate tree-diagram[2] contributions to the amplitudes. Clearly, at the next level of sophistication one is interested in the closed-loop contributions, at which stage two related problems arise:

(i) Since the Lagrangian itself is expressed as an infinite power series, $\mathcal{L}_{\text{int}}=\sum_n a_n g^n \varphi^n(\partial\varphi)^2$, the *number* of perturbation diagrams in each order $n$ increases (typically) as fast or faster than $n!$. On any reasonable estimate, the perturbation expansion must be a *divergent series*. For respectable theories like quantum electrodynamics, with Lagrangians which are polynomials in the field variables, one has always suspected[3] that the perturbation expansion provides an *asymptotic series* in $e^2/\hbar c$; here, with Lagrangians which are themselves infinite series, this behavior appears to be a virtual certainty.

(ii) Each of the terms in the expansion of the Lagrangian [terms like $\varphi^n(\partial\varphi)^2$; $n\geqslant 1$] represents a nonrenormalizable interaction in the conventional sense. The ultraviolet infinities of the perturbation expansion therefore get progressively more virulent. On the face of it, this is rather surprising, since it is well known that every nonlinear theory can be reformulated as a theory of linear group representations[4] with polynomial Lagrangians together with a certain number of constraints on the fields $\varphi^i$. Before the imposition of the constraint, the theories are renormalizable; if any nonrenormalizability occurs, it must arise through the imposition of the constraint.

In this paper, we argue that *both* difficulties (i) and (ii) stem from the same circumstance, namely, the expansion of the Lagrangian in a power series of the field variables, and that a summation,[5] or even a partial summation, of the divergent perturbation series is likely at the same time to reduce the problem of ultraviolet infinities.[6]

An advance was made towards the (partial) summation of the perturbation series arising from rational and exponential Lagrangians in a series of papers by Efimov and Fradkin[7,8] during 1963. Like all summation methods for divergent series, the problem of uniqueness of the sum remains unresolved in their technique. Efimov, however, has claimed that besides satisfying the usual analyticity requirements, the Efimov-Fradkin (EF) summation method meets the demand of *consistency with Landau-Cutkosky unitarity* at least for the self-energy and vertex functions. In this paper, we wish to apply the EF method to summing the perturbation series of nonlinear Lagrangians of the chiral variety.[9] We wish to show that the infinities in such theories appear to be no worse after summation than those encountered in conventionally renormalizable theories. Central to our discussion is the result which states that the degree of ultraviolet infinity of EF sums depends on the growth of $\mathcal{L}_{\text{int}}(\varphi)$ as $\varphi\to\infty$ for nonlinear theories just as for the usual linear theories. To be more specific, the result (extended below to include derivative cou-

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[1] These effective Lagrangians were originated by F. Gürsey, Nuovo Cimento **16**, 230 (1960); S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); J. Schwinger, Phys. Letters **24B**, 473 (1967). Two useful reviews are F. Gürsey and N. Chang, Phys. Rev. **164**, 1752 (1967); and S. Gasiorowicz and D. Geffen, Argonne National Laboratory Report No. ANL/HEP 6801 (unpublished).

[2] B. W. Lee and M. Nieh, Phys. Rev. **166**, 1507 (1968); Y. Nambu, Phys. Letters **26B**, 626 (1968); L. Prokhorov, Nuovo Cimento **57A**, 245 (1968).

[3] C. A. Hurst, Phys. Rev. **85**, 920L (1952); F. J. Dyson, *ibid.* **85**, 631 (1952).

[4] Abdus Salam and J. Strathdee, Phys. Rev. (to be published); C. Isham, Nuovo Cimento **59A**, 356 (1969).

[5] We list here some of the papers where summation of perturbation diagrams of infinite parts of such diagrams has been carried out using widely different techniques. R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955); G. Feinberg and A. Pais, *ibid.* **131**, 2724 (1963); T. D. Lee and C. N. Yang, *ibid.* **128**, 885 (1962); Abdus Salam, *ibid.* **130**, 1287 (1963); Abdus Salam and R. Delbourgo, *ibid.* **135**, B1398 (1964); M. Baker, K. Johnson, and R. Willey, *ibid.* **136**, B1111 (1964); T. D. Lee, Nuovo Cimento **59A**, 579 (1969).

[6] Throughout this paper we use the terms "divergent" for *series* and "ultraviolet infinite" for *integrals*.

[7] G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **44**, 2107 (1963) [English transl.: Soviet Phys.—JETP **17**, 1417 (1963)].

[8] E. S. Fradkin, Nucl. Phys. **49**, 624 (1963). See also subsequent work by G. V. Efimov, Nuovo Cimento **32**, 1046 (1964); Nucl. Phys. **74**, 657 (1965).

[9] After completion of this paper we became aware of the work of H. M. Fried [Phys. Rev. **174**, 1725 (1968); New Phys. (Korean Phys. Soc.) Suppl. **7**, 23 (1968)] which suggests applying the EF methods to chiral Lagrangians, though derivative couplings were not considered.

plings so essential in nonlinear chiral Lagrangians) can be stated as follows:

(i) Assign to each scalar field $\varphi(x)$ (with the propagator $\langle T\{\varphi(x)\varphi(0)\}\rangle=\Delta(x)\approx x^{-2}$ as $x^2\to 0$) the "singularity" behavior $\varphi(x)\approx 1/\sqrt{(x^2)}\approx 1/x$ as $x\to 0$ or equivalently $\varphi\sim M$ with $M\to\infty$.

(ii) Likewise assign the behaviors

$$\partial_\mu\varphi(x)\underset{x\to 0}{\approx}1/x^2 \quad\text{or}\quad \partial_\mu\varphi\underset{M\to\infty}{\sim}M^2;$$

$$\psi(x)\underset{x\to 0}{\approx}1/x^{3/2} \quad\text{or}\quad \psi\underset{M\to\infty}{\sim}M^{3/2},\quad \psi=\text{spin-}\tfrac{1}{2}\text{ field};$$

$$U_\mu(x)\underset{x\to 0}{\approx}1/x^2 \quad\text{or}\quad U_\mu\underset{m\to\infty}{\sim}M^2,\quad U=\text{spin-1 field}.$$

A theory is expected to be renormalizable, with only a few types of integrals that are ultraviolet infinite, if $\mathfrak{L}_{\text{int}}\underset{M\to\infty}{\sim}M^4$. *This criterion applies equally to integrals in conventional polynomial Lagrangians like* $\mathfrak{L}_{\text{int}}=g\varphi^4$ *or* $g\bar{\psi}\psi\varphi$, *as well as to EF sums in theories with Lagrangians like* $g\varphi^2(\partial\varphi)^2/(1+\varphi^2)$. We shall call such theories *normal*. These like $\mathfrak{L}_{\text{int}}=g\varphi^3$ or $g(\partial\varphi)^2/(1+\varphi^2)$ which behave like $M^3$ or $M^2$ or lower ($\mathfrak{L}_{\text{int}}\sim M^n$; $n<4$) will be called *supernormal*. All theories which behave worse than $\varphi^4$, i.e., for which $\mathfrak{L}_{\text{int}}\sim M^n$, $n>4$, will be called *abnormal*. For supernormal theories there is the attractive possibility that when $n<2$ all integrals, including those for self-mass and self-charge, may be finite.

The plan of the paper is as follows: In Sec. II we give an outline of the EF method which has two ingredients: (i) Hori's exponential representation[10] of Wick's normal-ordering theorem and (ii) the EF integral representation[7,8] of Hori's exponential operator, which essentially performs a "Borel" sum of the divergent perturbation series. The power-counting rules for estimating *over-all* ultraviolet infinities of EF sums is given in Sec. III. We consider derivative couplings in Sec. IV and formulate the rules for writing EF sums in such a manner that the ultraviolet power-counting estimate can also be stated here. Section V contains the application of these results to the nonlinear (chiral type) Lagrangians in an $SU(2)\otimes SU(2)$ symmetric theory. Since equivalence theorems, which state that on-mass-shell $S$-matrix elements are unaltered by contact transformations in field space, play such a critical role,[11] we devote Sec. VI to a nonrigorous discussion of the circumstances in which such transformations are permissible. Not discussed in this paper is the problem of absorbing these infinities into counter-term Lagrangians.

[10] S. Hori, Progr. Theoret. Phys. (Kyoto) **7**, 589 (1952).

[11] The type of problem which one meets is that $\mathfrak{L}_{\text{int}}(\varphi)$ may look abnormal when expressed in terms of one set of fields but normal in another formulation. Consider, for example, a theory with $\mathfrak{L}_{\text{int}}=2\varphi(1+\varphi)(\partial\varphi)^2+\varphi^3(1+\varphi)^3$, which looks hideously abnormal but can be transformed into the familiar supernormal $\varphi^3$ theory with substitution $\varphi\to\varphi(1+\varphi)$.

## II. THEORIES WITH NONDERIVATIVE COUPLINGS

We summarize below the steps needed to arrive at the EF representation[7,8] of the $S$ matrix, assuming that the interaction Lagrangian contains no field derivatives. (In Sec. IV we extend the techniques to cover situations where derivatives are encountered.) An illustrative example is presented to demonstrate the power of the EF method.

*Step 1.* Begin with the standard perturbation expansion of the $S$ matrix,

$$S=\sum_N \frac{i^N}{N!}S^{(N)},$$

where

$$S^{(N)}=g^N\int d^4z_1\cdots d^4z_N T[L\{\varphi(z_1)\}\cdots L\{\varphi(z_N)\}] \quad (1)$$

and we are supposing in this section that

$$\mathfrak{L}_{\text{int}}=gL\{\varphi(x)\}, \quad (2)$$

where $\varphi$ denotes a real scalar field.

The further expansion of the $S$ matrix into normal Wick products can be compactly expressed through *Hori's functional operator*[10] as follows:

$$S^{(N)}=g^N\int d^4z_1\cdots d^4z_N \times\exp\left(\frac{1}{2}\int d^4x_1d^4x_2\Delta(x_1-x_2)\frac{\delta^2}{\delta\varphi(x_1)\delta\varphi(x_2)}\right) \times[L\{\varphi(z_1)\}\cdots L\{\varphi(z_N)\}], \quad (3)$$

where $\Delta(x_1-x_2)$ denotes the bare causal propagator for the scalar field $\varphi$. This formula can be simplified to read

$$S^{(N)}=g^N\int d^4z_1\cdots d^4z_N \exp\left(\tfrac{1}{2}\sum_{i,j=1}^N \Delta_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\right) \times[L\{\varphi_1\}\cdots L\{\varphi_N\}]_{\varphi_k=\varphi^{\text{ext}}(z_k),\Delta_{kl}=\Delta(z_k-z_l)}. \quad (4)$$

Here $\varphi_k=\varphi^{\text{ext}}(z_k)$ is the wave function of any external particle which may be acting at the point $z_k$. One may rewrite (4) in a form where these external wave functions are exhibited separately by writing

$$S^{(N)}=\int d^4x_1\cdots d^4x_n:\varphi(x_1)\cdots\varphi(x_n):S^{(N)}(x_1,\cdots,x_n), \quad (5)$$

where the $n$-point function in the $N$th order equals

$$S^{(N)}(x_1,\cdots,x_n)=\langle 0|\frac{\delta^n}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}S^{(N)}|0\rangle$$
$$=g^N\int d^4z_1\cdots d^4z_N\sum_{m_i}\delta^{m_1}(x-z_1)\cdots\delta^{(m_N)}(x-z_N)S_{m_1\cdots m_N}(\Delta), \quad (6)$$

($\sum m_i=n$) with

$$\delta^m(x-z)=\delta(x_{i_1}-z)\delta(x_{i_2}-z)\cdots\delta(x_{i_m}-z)\,, \quad (7)$$

and

$$S_{m_1\cdots m_N}(\Delta)\equiv\exp\left(\tfrac{1}{2}\sum_{ij}\Delta_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\right)\left(\frac{\partial}{\partial\varphi_1}\right)^{m_1}\cdots\left(\frac{\partial}{\partial\varphi_N}\right)^{m_N}[L(\varphi_1)\cdots L(\varphi_N)]_{\varphi=0}. \quad (8)$$

The vacuum graphs are given in their entirety by

$$S=\sum_N\frac{(ig)^N}{N!}\int d^4z_1\cdots d^4z_N S_{00\cdots0}(\Delta)\,, \quad (9)$$

$$S_{00\cdots0}(\Delta)=\exp\left(\tfrac{1}{2}\sum_{ij}^{N}\Delta_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\right)L(\varphi_1)\cdots L(\varphi_N)\,, \quad (10)$$

while the two-point (self-energy) graphs are completely described by

$$S(x_1,x_2)=\sum_{N\geqslant 2}\frac{(ig)^N}{N!}\int d^4z_1\cdots d^4z_N \times\{\delta(x_1-z_1)\delta(x_2-z_2)S_{200\cdots}(\Delta)+\delta(x_1-z_2)\delta(x_2-z_2)S_{020\cdots}(\Delta)+[\delta(x_1-z_1)\delta(x_2-z_2)+\delta(x_1-z_2)\delta(x_2-z_1)]S_{110\cdots}(\Delta)\}\,, \quad (11)$$

with

$$S_{200\cdots}(\Delta)=\exp\left[\tfrac{1}{2}\sum\Delta\frac{\partial^2}{\partial\varphi\partial\varphi}\right]\frac{\partial^2}{\partial\varphi_1{}^2}L(\varphi_1)\cdots L(\varphi_N)\Big|_{\varphi=0}$$

$$S_{110\cdots}(\Delta)=\exp\left[\tfrac{1}{2}\sum\Delta\frac{\partial^2}{\partial\varphi\partial\varphi}\right]\times\frac{\partial^2}{\partial\varphi_1\partial\varphi_2}L(\varphi_1)\cdots L(\varphi_N)\Big|_{\varphi=0,\text{etc.}}\,, \quad (12)$$

which expressions are represented graphically in Figs. 1(a) and 1(b), respectively.

*Step 2.* Give a simple integral representation of Hori's exponential operator by making use of the EF lemma[7,8]

$$\exp\left(\Delta\frac{\partial^2}{\partial\varphi\partial\varphi'}\right)F(\varphi,\varphi')=\frac{1}{\pi}\int d^2u\exp\left(-|u|^2+uc\frac{\partial}{\partial\varphi}+u^*c'\frac{\partial}{\partial\varphi}\right)F(\varphi,\varphi')=\frac{1}{\pi}\int d^2u\exp(-|u|^2)F(\varphi+uc,\,\varphi'+u^*c')\,, \quad (13)$$

with the parameters $c$ and $c'$ constrained to satisfy $cc'=\Delta$, but otherwise arbitrary. (They can be chosen to suit one's purpose. Thus $c=c'=\sqrt{\Delta}$ would correspond to the most symmetric choice, one we often make; $c=\Delta$, $c'=1$ to the most asymmetric choice. In any event, the final result cannot explicitly involve any square roots of $\Delta$ and must depend only on the product $cc'=\Delta$.) Since the final expression on the right-hand side of (13) involves as integrand the function $F$ shifted from its value at $\varphi$, $\varphi'$ to $\varphi+uc$, $\varphi'+u^*c'$, we shall call this the *exponential-shift* lemma.

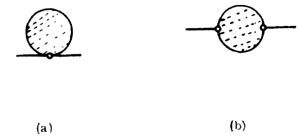



FIG. 1. (a) Self-energy diagrams $S_{200\cdots}$. (b) Self-energy diagrams $S_{110\cdots}$.

Applying the lemma to the $N$th order $S$ matrix by introducing complex variables $u_{ij}$, $c_{ij}$ between every two pairs of points $ij$, one has the representation

$$\exp\left(\tfrac{1}{2}\sum_{ij}\Delta_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\right)L(\varphi_1)\cdots L(\varphi_N)=\prod_{i\geqslant j}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\times\exp(-\tfrac{1}{2}\sum_{ij}|u_{ij}|^2)L(\varphi_1+\sum_k c_{1k}u_{1k})\cdots L(\varphi_N+\sum_k c_{Nk}u_{Nk})\,, \quad (14)$$

with

$$c_{ij}c_{ji}=\Delta_{ij}\ (\text{no summation over } ij)$$

and

$$u_{ij}=u_{ji}{}^*. \quad (15)$$

As an application of the lemma, consider all vacuum graphs of order $g^N$. These are given by

$$S_{00\cdots0}(\Delta)=\prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp(-\sum|u_{ij}|^2)\times L(\sum_k c_{1k}u_{1k})\cdots L(\sum_k c_{Nk}u_{Nk}). \quad (16)$$

Likewise, the self-energy graphs of order $g^N$ are given in terms of

$$S_{20\cdots0}(\Delta)=\prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp(-\sum|u_{ij}|^2)\times L''(\sum_k c_{1k}u_{1k})\cdots L(\sum_k c_{Nk}u_{Nk})\,,$$

$$S_{110\cdots}(\Delta)=\prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp(-\sum|u_{ij}|^2)\times L'(\sum_k c_{1k}u_{1k})L'(\sum_k c_{2k}u_{2k})\cdots\,, \quad (17)$$

and so on. Hence $L'\equiv\partial L/\partial\varphi$, $L''=\partial^2L/\partial\varphi^2$, etc.

To see how this works in practice, take the model for which $gL(\varphi)=g\varphi^4/(1+\lambda^2\varphi^2)$. The power of the technique, which explicitly displays sums of perturbation series to each order in $g$, is already apparent since *all*

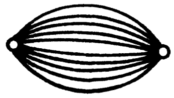


FIG. 2. Typical vacuum transition $S_{00}$.

*orders* in $\lambda^2$ are automatically taken into account by the EF expressions. Thus, to second order in $g$ and all orders in $\lambda^2$, the vacuum contribution equals

$$g^2S_{00}(x_1,x_2)=g^2\int \frac{d^2u}{\pi}e^{-|u|^2}\frac{c^4u^4}{1+\lambda^2c^2u^2}\,\frac{c'^4u^{*4}}{1+\lambda^2c'^2u^{*2}},$$

where $cc'=\Delta(x_1-x_2)$. Likewise, the two relevant self-energy terms to second order in $g$ but all orders in $\lambda^2$ are

$$g^2S_{20}=g^2\int \frac{d^2u}{\pi}e^{-|u|^2}\frac{d^2}{c^2du^2}\left(\frac{c^4u^4}{1+\lambda^2c^2u^2}\right)\frac{c'^4u^{*4}}{1+\lambda^2c'^2u^{*2}}$$

and

$$g^2S_{11}=g^2\int \frac{d^2u}{\pi}e^{-|u|^2}\frac{d}{cdu}\left(\frac{c^4u^4}{1+\lambda^2c^2u^2}\right)\frac{d}{c'du^*}\left(\frac{c'^4u^{*4}}{1+\lambda^2c'^2u^{*2}}\right).$$

The simplification of these integrals rests on the pair of relations[12]

$$\frac{1}{\pi}\int d^2u\, u^{*m}u^nf(|u|^2)=\delta_{nm}\int_0^\infty d\xi\,\xi^nf(\xi)\,,$$

$$\frac{1}{\pi}\int d^2u\frac{f(|u|^2)}{(1+\alpha u^2)(1+\beta u^{*2})}=\int_0^\infty d\xi\frac{f(\xi)}{1-\alpha\beta\xi^2},$$

and derivatives thereof. Thus we find, as expected, that the integrals only involve the product $cc'=\Delta$ and not the parameters $c$ and $c'$ separately. Explicitly (see Figs. 2 and 3),

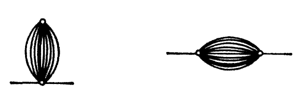
$$g^2S_{00}=g^2\int_0^\infty d\xi\frac{\Delta^4\xi^4e^{-\xi}}{1-\lambda^4\Delta^2\xi^2},\tag{18}$$

$$g^2S_{20}=-g^2\int_0^\infty d\xi\frac{\lambda^2\Delta^4\xi^6e^{-\xi}}{1-\lambda^4\Delta^2\xi^2},$$

$$g^2S_{11}=g^2\int_0^\infty d\xi\Delta^3\xi^4e^{-\xi}\left(\frac{2}{1-\lambda^4\Delta^2\xi^2}+\frac{2}{(1-\lambda^4\Delta^2\xi^2)^2}\right),\tag{19}$$

In particular, when we set $\lambda=0$, we recover the $\varphi^4$ perturbation-theory results, viz.,

$$S_{00}=4!\Delta^4,\quad S_{20}=S_{02}=0,\quad S_{11}=4(4!)\Delta^3.$$



(a) (b)

FIG. 3. (a) Self-energy diagram $S_{20}$. (b) Self-energy diagram $S_{11}$.

[12] It is clear from these identities that the $E$-$F$ summation method is equivalent to a Borel summation of divergent series.

We shall return to the ultraviolet properties of these integrals after we have discussed the question of infinities.

## III. ULTRAVIOLET INFINITIES OF EF SUMS

Physically, we are only concerned with $S$-matrix elements in momentum space,[13] i.e., the Fourier transforms

$$\tilde{S}(p)=\prod\left(\int d^4x\,e^{ipx}\right)S(\Delta(x_{ij}))\,.\tag{20}$$

On account of the causal character of the propagators $\Delta(x)$, the task of defining the $x$-space contours of integration in integrals like (20) is not trivial. As is well known,[14] the light-cone singularity of $\Delta(x)$ is given by the following expression:

$$\begin{aligned}4\pi i\Delta(x;\mu)&=\delta(x^2)-\frac{\mu\theta(x^2)}{2\sqrt{(x^2)}}J_1(\mu\sqrt{(x^2)})+i\mu\left[\frac{\theta(x^2)}{2\sqrt{(x^2)}}N_1(\mu\sqrt{(x^2)})+\frac{\theta(-x^2)}{\pi\sqrt{(-x^2)}}K_1(\mu\sqrt{(-x^2)})\right]\\&=\delta(x^2)-\frac{i}{\pi x^2}-\tfrac{1}{4}\mu^2\left[\theta(x^2)-\frac{2i}{\pi}\ln(\tfrac{1}{2}\mu\sqrt{|x^2|})\right]+O((\sqrt{|x^2|})\ln|x^2|).\end{aligned}\tag{21}$$

The crucial part of Efimov's work is a method of carrying out the $x$-space integrals, with the demonstration that one may define them so as to preserve the unitarity of the $S$ matrix in the perturbation sense, i.e., in the expansion of $S(\Delta)$ in powers of $\Delta$. Efimov's procedure consists in concentrating firstly on the Euclidean or Symanzik region of the external momenta.[15] For this region of $p$ space, it must be assumed that $x$-space contours of integration have been rotated from the Minkowskian into the Euclidean region of $x$. (For the theories under consideration, the Minkowskian integrals may not be well defined.) For other regions of $p$-space, Efimov makes suitably defined continuations from the

[13] The integrands (18) and (19) exhibit poles on the real axis, $\xi>0$, and pose the problem of defining the correct contour of integration in the $\xi$ plane, such that power series expansion of EF integrals coincides with the perturbation expansion and satisfies causality and unitarity requirements. We believe that this problem is bound up with the problem of defining the Fourier integral (20) away from the Symanzik region in $p$ space. Efimov (Ref. 7) in his calculation of self-masses takes the ***principal-value*** integral in $\xi$ space. This has been further discussed by B. W. Lee and B. Zumino, CERN Report No. TH1053, 1969 (unpublished).

[14] See, e.g., N. N. Bogolubov and D. V. Shirkov, ***Introduction to the Theory of Quantized Fields*** (Wiley-Interscience, Inc., New York, 1959).

[15] The Symanzik region is defined by the condition that the linear combinations $\sum_j\alpha_jp_j$ be spacelike for any choice of real parameters $\alpha_j$; i.e., the Gram determinants of $-(p_i\cdot p_j)$ are positive for all sets of $i$ and $j$.

Symanzik region. In this paper we are only concerned with the ultraviolet infinities associated with integrals (20), so for our purpose it is sufficient to remain in the Symanzik region—or, to make matters simpler, on its edge, where all external momenta $p_\mu$ are zero. Thus we examine the infinities associated with the Euclidean $x$-space integrals ($x_{ij}^2<0$)

$$\tilde{S}(0)=\int \Pi d^4x\, S(\Delta)\,,$$

*where the* $\Delta$ *assume real values.* A naive power count of the over-all[16] infinities can be made by considering the appropriate proper diagrams and retaining the most singular parts of all the propagators $\Delta$. Applying the lower cutoff $x^2=M^{-2}$ ($M^2\to\infty$) to all $x$-space integrations, it is evident that we can associate a factor $M$ to each $\sqrt{\Delta}$ that occurs; and since what in fact determines the infinities is the powers of $L(\varphi\approx\sqrt{\Delta})$, we may easily estimate the over-all infinity to be expected by setting $\varphi=M$ in $L(\varphi)$ and letting $M\to\infty$.

Consider, therefore, an $n$-point function and follow the Dyson power-counting procedure.[17] Suppose that $L(\varphi=M)$ behaves as $M^\nu$ for large $M$. The integrand of $S_{m_1\cdots m_N}(\Delta)$ in (14) contains the term (putting $c_{ij}=c_{ji}=\sqrt{(\Delta_{ij})}$ for simplicity)

$$\Delta^{-\Sigma m_i/2}[L((\sqrt{\Delta})u)]^N\sim M^{-n+N\nu}\,,$$

where $n$ denotes the number of external lines and $N$ the order of the graph (number of "vertices"). The singularity produced at $x^2=0$ ($M\to\infty$) is compensated by $4(N-1)$ integrations, four integrations being omitted because the integrand is independent of the over-all c.m. coordinates. There,

$$\int (d^4x)^{N-1}S(\Delta)\sim M^{-4(N-1)}M^{-n+N\nu}. \tag{22}$$

If the integral is to be regular in the limit $M\to\infty$, then

$$N(4-\nu)+n>4.$$

This is the same criterion which one encounters in renormalization theory of polynomial Lagrangians. In this count we have included tadpole contributions [$i=j$ terms in Eq. (14)]. If these were left out, the count would be jeopardized in a subtle manner to be discussed elsewhere.

We return to the example above to see that this naive infinity count is sensible. The self-energy contributions (19) to second order in $g$ (but all orders in $\lambda$) read, in momentum space,

$$\tilde{S}(p)=2g^2\int S_{20}(\Delta(x))d^4x+2g^2\int S_{11}(\Delta(x))e^{ipx}d^4x. \tag{23}$$

[16] There may, of course, be hidden infinities from subintegrations which, though they are not discussed here, form an integral part of the renormalization program. We hope to study these in future work.

[17] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

Taking $p^2\leqslant 0$ (Euclidean region), the integrals reduce to

$$i\tilde{S}(p)=4\pi^2g^2\int_{1/M}^{\infty}dr\, r^3S_{20}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right)+\frac{8\pi^2g^2}{\sqrt{(-p^2)}}\times\int_{1/M}^{\infty}dr\, r^2J_1(r\sqrt{(-p^2)})S_{11}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right), \tag{24}$$

where we cut off the integrations at $r=M^{-1}$ in order to estimate the infinity as $x^2\to 0$. Since the ultraviolet behavior of the integral is independent of the value of $p^2$, we set this equal to 0:

$$i\tilde{S}(0)=4\pi^2g^2\int_{1/M}^{\infty}dr\, r^3\left[S_{20}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right)+S_{11}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right)\right].$$

As $r\to 0$,

$$\frac{\mu K_1(\mu r)}{4\pi^2 r}\to\frac{\mu^2}{8\pi^2}\ln(\tfrac{1}{2}\mu r)+\frac{1}{4\pi^2r^2},$$

so that the lethal infinities at the lower limit are obtained, using (19), as

$$\begin{aligned}\lim_{M\to\infty}4\pi^2g^2\int_{1/M}dr\, r^3\left[S_{20}\left(\frac{1}{4\pi^2r^2}\right)+S_{11}\left(\frac{1}{4\pi^2r^2}\right)\right]&=\lim_{M\to\infty}4\pi^2g^2\int_{1/M}dr\, r^3\left(\frac{-12}{\lambda^2 16\pi^4r^4}-\frac{8}{\lambda^4}\right)\\&\sim\frac{3g^2}{\pi^2\lambda^2}\ln M;\end{aligned} \tag{25}$$

i.e., we meet a logarithmic infinity at most. A naive power count (up to these logarithms) would have agreed with this result since when we set $\Delta=M^2\to\infty$,

$$\int d^4x[S_{20}(\Delta)+S_{11}(\Delta)]\sim M^{-4}[S_{20}(M^2)+S_{11}(M^2)]\to -12/\lambda^2. \tag{26}$$

An interesting feature of the result is the pole $1/\lambda^2$ of the "leading infinity" in the $\lambda^2$ plane. This is not entirely surprising in view of the fact that for $\lambda=0$, we must necessarily recover the conventional quadratic perturbation infinity.

## IV. THEORIES WITH DERIVATIVE COUPLINGS

In this section we extend the summation technique to cases where $\mathfrak{L}_{\rm int}$ contains derivatives of the $\varphi$ field,

$$\mathfrak{L}_{\rm int}=gL(\varphi,\partial_\mu\varphi). \tag{27}$$

It is common knowledge that for such situations the Hamiltonian contains surface-dependent terms and formula (1) for the $S$ matrix holds only if suitable modifications are made to the definition of time-ordering products of $\partial_\mu\varphi$, More specifically, using a theorem[18] first proved by Matthews for Lagrangians involving one time derivative, and later extended by Dyson to Lagrangians with two time derivatives, the $S$ matrix is covariantly defined if we invert the order of differentiation and the time-ordering operation in vacuum expectation values of the following variety:

$$\langle T^*\varphi_\mu(x)\varphi(x')\rangle=\Delta_\mu(x-x')\equiv\partial_\mu\Delta(x-x')\,,$$
$$\langle T^*\varphi_\mu(x)\varphi_\nu(x')\rangle=\Delta_{\mu\nu}(x-x')\equiv-\partial_\mu\partial_\nu\Delta(x-x')\,,$$

where $\varphi_\mu\equiv\partial_\mu\varphi$, provided one leaves out all terms which involve $\delta^4(0)$ whenever it occurs. Given, then, the modified time-ordering operation $T^*$, we have

$$S^{(N)}=g^N\int d^4z_1\cdots d^4z_N T^*[L\{\varphi(z_1),\varphi_\mu(z_1)\}\cdots L\{\varphi(z_N),\varphi_\mu(z_N)\}]. \quad (28)$$

The Wick reduction can be carried through by extending Hori's exponential operator to include differentiation with respect to the derived fields $\varphi_\mu$ as follows:

$$S^{(N)}=g^N\int d^4z_1\cdots d^4z_N \exp\left(\frac{\partial}{\partial\boldsymbol{\varphi}}\boldsymbol{\Delta}\frac{\partial}{\partial\boldsymbol{\varphi}}\right)\times[L\{\varphi_1,\varphi_{\mu1}\}\cdots L\{\varphi_N,\varphi_{\mu N}\}]_{\varphi k=\varphi^{\mathrm{ext}}(z_k),\,\varphi_{\mu k}=\partial_\mu\varphi^{\mathrm{ext}}(z_k)}\,, \quad (29)$$

where

$$\frac{\partial}{\partial\boldsymbol{\varphi}}\boldsymbol{\Delta}\frac{\partial}{\partial\boldsymbol{\varphi}}\equiv\tfrac{1}{2}\sum_{ij}\frac{\partial}{\partial\boldsymbol{\varphi}_i}\boldsymbol{\Delta}_{ij}\frac{\partial}{\partial\boldsymbol{\varphi}_j}$$
$$\equiv\tfrac{1}{2}\sum_{ij}\Delta(z_i-z_j)\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}+\Delta_\mu(z_i-z_j)\frac{\partial^2}{\partial\varphi_{\mu i}\partial\varphi_j}+\Delta_\nu(z_i-z_j)\frac{\partial^2}{\partial\varphi_i\partial\varphi_{\nu j}}+\Delta_{\mu\nu}(z_i-z_j)\frac{\partial^2}{\partial\varphi_{\mu i}\partial\varphi_{\nu j}}. \quad (30)$$

In order to give a simple integral representation of this generalized operator we must be prepared to introduce auxiliary vector variables. (This representation will be needed in its full generality only for interactions which are not polynomials in the vector variables.) To see how this is achieved, it is enough to consider a pair of points, since the extension to the whole series of points is easily performed by the method outlined in Sec. II. Since

$$\Delta_\mu(x)=2x_\mu\frac{d}{dx^2}\Delta(x^2)\equiv2x_\mu\Delta'(x^2)\,,$$
$$\Delta_{\mu\nu}(x)=-4x_\mu x_\nu\Delta''(x^2)-2g_{\mu\nu}\Delta'(x^2)\,, \quad (31)$$

we need to introduce at most one auxiliary vector and four auxiliary scalar complex integrations. Showing this in detail,

$$\exp\left(\frac{\partial}{\partial\boldsymbol{\varphi}}\Delta(x)\frac{\partial}{\partial\boldsymbol{\varphi}'}\right)F(\boldsymbol{\varphi};\boldsymbol{\varphi}')$$
$$=\exp\left(\Delta\frac{\partial^2}{\partial\varphi\partial\varphi'}+2x_\mu\Delta'\frac{\partial^2}{\partial\varphi_\mu\partial\varphi'}-2x_\nu\Delta'\frac{\partial^2}{\partial\varphi\partial\varphi_\nu'}-4x_\mu x_\nu\Delta''\frac{\partial^2}{\partial\varphi_\mu\partial\varphi_\nu}-2\Delta'\frac{\partial^2}{\partial\varphi_\mu\partial\varphi_\mu}\right)F(\varphi,\varphi_\lambda;\varphi',\varphi_\lambda')$$
$$=\frac{1}{\pi^8}\int d^8c_{(\mu)}d^2cd^2b_1d^2b_2d^2a\,\exp\left(-|a|^2+\alpha a\frac{\partial}{\partial\varphi}+\alpha'a^*\frac{\partial}{\partial\varphi'}\right)\exp\left(-|b_2|^2+\beta_2b_22x_\mu\frac{\partial}{\partial\varphi_\mu}+\beta_2'b_2^*\frac{\partial}{\partial\varphi'}\right)$$
$$\times\exp\left(-|b_1|^2+\beta_1b_1\frac{\partial}{\partial\varphi}+\beta_1'b_1^*2x_\nu\frac{\partial}{\partial\varphi_\nu'}\right)\exp\left(-|c|^2+\gamma_1c2x_\mu\frac{\partial}{\partial\varphi_\mu}+\gamma_1'c^*2x_\nu\frac{\partial}{\partial\varphi_\nu'}\right)$$
$$\times\exp\left(-|c_\lambda c_\lambda|^2+\gamma_2c_\mu\frac{\partial}{\partial\varphi_\mu}+\gamma_2'c_\nu^*\frac{\partial}{\partial\varphi_\nu'}\right)F(\varphi,\varphi_\lambda;\varphi',\varphi_\lambda')$$
$$=\frac{1}{\pi^8}\int d^8c_{(\mu)}d^2cd^2b_1d^2b_2a\,\exp[-(|a|^2+|b_1|^2+|b_2|^2+|c|^2+|c_\lambda c_\lambda|)]F(\varphi+\alpha a+\beta_1b_1,$$
$$\varphi_\lambda+2x_\lambda\beta_2b_2+2x_\lambda\gamma_1c+\gamma_2c_\lambda;\ \varphi'+\alpha'a^*+\beta_2'b_2^*,\ \varphi_\lambda'+2x_\lambda\beta_1'b_1^*+2x_\lambda\gamma_1'c^*+\gamma_2'c_\lambda^*)\,, \quad (32)$$

where

$$\alpha\alpha'=\Delta\,,\quad \gamma_1\gamma_1'=-\Delta''\,,$$
$$\beta_2\beta_2'=-\beta_1\beta_1'=-\tfrac{1}{2}\gamma_2\gamma_2'=\Delta'. \quad (33)$$

[18] P. T. Matthews, Phys. Rev. **75**, 1270 (1949); F. J. Dyson, *ibid.* **83**, 608 (1951).

The result cannot depend on the individual $\alpha,\beta,\cdots$, but only on the products $\alpha\alpha'=\Delta$, etc. For the remainder of this discussion we choose to make the quasisymmetric split

$$\alpha=\alpha'=\beta_1=\beta_2'=\sqrt{\Delta}\,,\quad \gamma_1=\gamma_1'=\sqrt{(-\Delta'')}\,,$$
$$\gamma_2=\gamma_2'=\sqrt{(-2\Delta')}\,,\ \text{and}\ \beta_2=-\beta_1'=\Delta'/\sqrt{\Delta}. \quad (34)$$

Now because in the limit $x^2 \to 0$,

$$\Delta'(x^2) \sim 1/x^4 \text{ and } \Delta''(x^2) \sim 1/x^6, \tag{35}$$

one can see that, consistently for all integrations over the shifted functional, we can ascribe the "singularity factors"

$$\varphi \sim M \quad \text{and} \quad \varphi_\mu \sim M^2 \tag{36}$$

owing to the terms $\Delta$ and $(x\Delta'/\sqrt{\Delta}+x\sqrt{\Delta''})$ occurring, respectively, in the shifted arguments. Perhaps the clearest way to appreciate this conclusion is to realize that most of the auxiliary integrations are redundant and that for the simple case treated above, only one auxiliary vector and one auxiliary scalar variable suffice to make the exponential shift defined in Sec. II. Thus, write

$$\exp\left(\frac{\partial}{\partial \boldsymbol{\varphi}}\boldsymbol{\Delta}\frac{\partial}{\partial \boldsymbol{\varphi}'}\right)=\exp\left[\left(c_{\mu\lambda}\frac{\partial}{\partial \varphi_\mu}+c_\lambda\frac{\partial}{\partial \varphi}\right)\times\left(c_{\lambda\nu}'\frac{\partial}{\partial \varphi_\nu'}+c_\lambda'\frac{\partial}{\partial \varphi'}\right)+cc'\frac{\partial^2}{\partial\varphi\partial\varphi'}\right], \tag{37}$$

with

$$cc'+c_\lambda c_\lambda'=\Delta, \quad c_{\mu\lambda}c_\lambda=\Delta_\mu, \quad c_{\mu\lambda}c_{\lambda\nu}'=\Delta_{\mu\nu}. \tag{38}$$

$$\exp\left(\frac{\partial}{\partial \boldsymbol{\varphi}}\boldsymbol{\Delta}\frac{\partial}{\partial \boldsymbol{\varphi}'}\right)F(\boldsymbol{\varphi};\boldsymbol{\varphi}')=\frac{1}{\pi^5}\int d^8u_{(\mu)}d^2u \times\exp[-|u|^2+|u_\lambda u_\lambda|)]F(\varphi+cu+c_\mu u_\mu,\, \varphi_\lambda+c_{\lambda\mu}u_\mu;\, \varphi'+c'u^*+c_\mu'u_\mu^*,\, \varphi_\lambda'+c_{\lambda\nu}u_\nu^*). \tag{39}$$

Again the result can only depend on the products $cc'=\Delta$; if we make the symmetrical choice $c=c'$ for simplicity, then (see the Appendix) in the (Euclidean) limit $x\to 0$,

$$c\sim 1/x, \quad c_\mu \sim 1/x, \quad c_{\mu\nu}\sim 1/x^2.$$

The association (36) of the ultraviolet factors $\varphi\sim M$ and $\varphi_\mu \sim M^2$ then becomes more obvious. The following identities prove useful for the vector integrations:

$$\frac{1}{\pi^4}\int d^8u_{(\mu)}u_\kappa u_\lambda^* e^{-u_\mu u_\mu^*}=\tfrac{1}{4}g_{\kappa\lambda}, \tag{40}$$

$$\frac{1}{\pi^4}\int d^8u_{(\mu)}u_\kappa u_\lambda f(u_\nu u_\nu^*)=0, \text{ etc.} \tag{41}$$

Here we are concerned only with a superficial count of the over-all infinities to be expected in a given $S$-matrix element.

The procedure is the same as before and will not be repeated, since the result is that derivatives make no essential difference to the infinity count beyond what is expected for conventional polynomial Lagrangian theories.

## V. NONLINEAR REALIZATIONS OF $SU(2)\otimes SU(2)$

The simplest practical applications of our conclusions about derivative couplings are to be found in the nonlinear realizations of chiral groups. We shall study the case of $SU(2)\otimes SU(2)$ symmetry for definiteness, because the features which emerge will apply to more complicated cases as well.

Describe the mesons of the $(\frac{1}{2},\frac{1}{2})$ representation by the field matrix

$$\mathcal{S}=\sigma+i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\Lambda(\boldsymbol{\varphi}^2), \tag{42}$$

where the nonlinearity is introduced by imposing the constraint

$$\mathcal{S}\mathcal{S}^\dagger=1 \text{ or } \sigma^2+\boldsymbol{\varphi}^2\Lambda^2(\boldsymbol{\varphi}^2)=1. \tag{43}$$

The choice of the function $\Lambda(\boldsymbol{\varphi}^2)$ corresponds to different parametrizations of the nonlinear coordinates [$\sigma$ and $\boldsymbol{\varphi}$ are coordinates of the differential manifold (43)], and with each such choice of $\Lambda$ the corresponding interpolating field $\varphi$ is different.[19] (However, we shall use the same symbol in every case.)

The ***unique***[20] $SU(2)\otimes SU(2)$-invariant Lagrangian which contains only two derivatives of the fields is

$$\mathcal{L}=\tfrac{1}{4}\Lambda^{-2}(0)\operatorname{Tr}[(\partial_\mu\mathcal{S})(\partial_\mu\mathcal{S}^\dagger)]=\tfrac{1}{4}\Lambda^{-2}(0)\operatorname{Tr}[\mathcal{J}_\mu\mathcal{J}_\mu], \tag{44}$$

where we write

$$\mathcal{J}_\mu\equiv -i\mathcal{S}^\dagger\partial_\mu\mathcal{S}=\mathcal{J}_\mu^\dagger. \tag{45}$$

If we substitute for $\mathcal{S}$ the expression (42) and eliminate $\sigma$ by means of the constraint equation (43), we then find

$$\begin{aligned}\mathcal{J}_\mu&=\boldsymbol{\tau}\cdot[\Lambda(\sigma\partial_\mu\boldsymbol{\varphi}-\boldsymbol{\varphi}\partial_\mu\sigma+\Lambda\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})+2\Lambda'\sigma\boldsymbol{\varphi}(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})]\\&=\Lambda\boldsymbol{\tau}\cdot\left((1-\Lambda^2\varphi^2)^{1/2}\partial_\mu\boldsymbol{\varphi}+\frac{\boldsymbol{\varphi}(\boldsymbol{\varphi}\partial_\mu\boldsymbol{\varphi})}{(1-\Lambda^2\varphi^2)^{1/2}}\times(\Lambda^2+2\Lambda^{-1}\Lambda')+\Lambda\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}\right)\end{aligned} \tag{46}$$

and

$$\begin{aligned}\mathcal{L}_{\text{int}}&=\tfrac{1}{4}\Lambda^{-2}(0)\operatorname{Tr}[\mathcal{J}_\mu\mathcal{J}_\mu]-\tfrac{1}{2}(\partial_\mu\boldsymbol{\varphi})\cdot(\partial_\mu\boldsymbol{\varphi})\\&=\tfrac{1}{2}[\Lambda^{-2}(0)\Lambda^2-1](\partial_\mu\boldsymbol{\varphi})\cdot(\partial_\mu\boldsymbol{\varphi})\\&\quad+\frac{(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})}{2\Lambda^2(0)(1-\Lambda^2\varphi^2)}(\Lambda^4+4\Lambda\Lambda'+4\Lambda'^2\varphi^2),\end{aligned} \tag{47}$$

where $\Lambda'=d\Lambda/d\varphi^2$. The ensuing equations of motion can be conveniently remembered in the Sugawara form[21]

$$\partial_\mu\mathcal{J}_\mu=0, \quad \partial_\mu\mathcal{J}_\nu-\partial_\nu\mathcal{J}_\mu+i[\mathcal{J}_\mu,\mathcal{J}_\nu]=0. \tag{48}$$

[19] For criteria when the $S$ matrices are equivalent, see Sec. VI and Refs. 23 and 24.

[20] The coordinate independence of the Lagrangian on the differential manifold has been proved by S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); and C. Isham (Ref. 4).

[21] H. Sugawara and M. Yoshimura, Phys. Rev. **173**, 1419 (1968).

We may now inquire about the "ultraviolet behavior" of the interaction Lagrangian with a view to possible renormalizability.[22] Begin by supposing that for large $\varphi\sim M$

$$\Lambda(\varphi^2)\rightarrow\varphi^k\sim M^k,\quad \sigma\sim[1-M^{2+2k}]^{1/2},$$

$$\mathcal{J}_\mu\sim M^{k+2}\left[(1-M^{2+2k})^{1/2}+\frac{M^2(M^{2k}+M^{-2})}{(1-M^{2+2k})^{1/2}}+M^{k+1}\right], \tag{49}$$

and

$$\mathcal{L}_{\rm int}\sim(M^{2k}-1)M^4+\frac{M^6}{1-M^{2+2k}}(M^{4k}+M^{2k-2}). \tag{50}$$

Hence

for $k>0$, $\mathcal{J}_\mu\sim M^{2k+3}$ and $\mathcal{L}_{\rm int}\sim M^{2k+4}$;

for $-1<k<0$, $\mathcal{J}_\mu\sim M^{2k+3}$ and $\mathcal{L}_{\rm int}\sim M^4$;

and

for $k<-1$, $\mathcal{J}_\mu\sim M^{k+2}$ and $\mathcal{L}_{\rm int}\sim M^4$.

This shows that nonlinear realizations of chiral groups, for the *preferred meson fields*, yield normal ($k<0$) or seemingly abnormal ($k>0$) Lagrangians, *but not supernormal ones*. The reason for this is not far to seek. For $k<0$, $\mathcal{L}\sim M^{2k+4}$, so that subtracting off $\mathcal{L}_f=(\partial_\mu\varphi)^2\sim M^4$, we meet a normal situation.

The question now poses itself: Since we can pass from one set of coordinates $\varphi$ to another, $\varphi'$, by a point transformation

$$S=\sigma+i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\Lambda(\boldsymbol{\varphi}^2)=\sigma'+i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}'\Lambda'(\boldsymbol{\varphi}'^2), \tag{51}$$

what is the significance of the abnormal parametrizations ($k>0$)? In Sec. VI we argue that the invariance of the total Lagrangian ($\mathcal{J}_\mu\mathcal{J}_\mu$) should imply that the $S$-matrix elements on the mass shell do not differ from one parametrization to the next, so that the theory is normal irrespective of the possibility $k>0$. We list below some special choices of parametrization.

(i) Gasiorowicz-Geffen coordinates.

$$\Lambda(\varphi^2)=\lambda,\ \text{a constant (i.e., } k=0)$$
$$\mathcal{J}_\mu=\lambda\boldsymbol{\tau}\cdot[\sigma\partial_\mu\boldsymbol{\varphi}-\boldsymbol{\varphi}\partial_\mu\sigma+\lambda\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}],$$

with $\sigma=(1-\lambda^2\varphi^2)$. Also,

$$2\mathcal{L}_{\rm int}=\lambda^2(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})/(1-\lambda^2\varphi^2)\sim M^4. \tag{52}$$

(ii) Schwinger coordinates.

$$\Lambda(\varphi^2)=\lambda(1+\lambda^2\varphi^2)^{-1/2};\ \lambda\text{ constant (i.e., } k=-1).$$

Thus $\sigma=(1+\lambda^2\varphi^2)^{-1/2}$,

$$\mathcal{J}_\mu=\frac{\lambda\boldsymbol{\tau}}{1+\lambda^2\varphi^2}[\partial_\mu\boldsymbol{\varphi}+\lambda\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}],$$

and

$$2\mathcal{L}_{\rm int}=-\frac{\lambda^2}{1+\lambda^2\varphi^2}\left(\varphi^2(\partial_\mu\boldsymbol{\varphi})\cdot(\partial_\mu\boldsymbol{\varphi})+\frac{(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})}{1+\lambda^2\varphi^2}\right)\sim M^4. \tag{53}$$

(iii) Weinberg coordinates.

$$\Lambda(\varphi^2)=2\lambda(1+\lambda^2\varphi^2)^{-1};\ \lambda\text{ constant (i.e., } k=-2)$$
$$\sigma=(1-\lambda^2\varphi^2)(1+\lambda^2\varphi^2)^{-1},$$

giving

$$\mathcal{J}_\mu=\frac{2\lambda\boldsymbol{\tau}}{(1+\lambda^2\varphi^2)^2}[(1-\lambda^2\varphi^2)\partial_\mu\boldsymbol{\varphi}+2\lambda\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}+2\lambda^2\boldsymbol{\varphi}(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})]$$

and

$$2\mathcal{L}_{\rm int}=(\partial_\mu\boldsymbol{\varphi})\cdot(\partial_\mu\boldsymbol{\varphi})\left(\frac{1}{(1+\lambda^2\varphi^2)^2}-1\right)\sim M^4. \tag{54}$$

(iv) Harmonic coordinates. A set of coordinates which may prove useful in the vector problem is defined by the condition

$$\Lambda(1-\varphi^2\Lambda^2)^{1/2}=\lambda^2,$$

where $\lambda$ is a constant. In these coordinates, which we shall call harmonic, the current operator is given by

$$\mathcal{J}_\mu=\left[\partial_\mu\boldsymbol{\varphi}+\frac{2\lambda}{1+(1-4\lambda^2\varphi^2)^{1/2}}\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}+\frac{2\lambda\varphi(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})}{(1-4\lambda^2\varphi^2)^{1/2}}\right]\lambda\boldsymbol{\tau}.$$

In this form the linear term $\partial_\mu\varphi$ appears multiplied by a constant rather than by a function of $\varphi^2$.

## VI. FIELD TRANSFORMATIONS

In Sec. V we assumed the correctness of the basic equivalence theorem, which states that if a local point transformation of fields is made such that the physical spectrum associated with these fields is unaltered—and therefore also the Hilbert spaces of in and out states remains the same—then the on-mass-shell (physical) $S$-matrix elements, computed using either the original or the transformed Lagrangians, are identical. This theorem,[23] first stated by Chisholm, Kamefuchi, O'Raifeartaigh, and Salam, has been proved to varying degrees of restrictiveness on field transformations and rigor by the above-mentioned authors and in axiomatic field theory by Borchers. It has latterly been extended by Coleman, Wess, and Zumino[20] who claim to sharpen the result to apply even to diagrams with equal numbers of closed loops. The weak point, when one comes to applying the theorem in practical cases, is the lack of criteria whereby one may judge what transformations

[22] In applying the exponential-shift lemma for making an ultraviolet count, one has to introduce isotopic labels $U^i$ to the auxiliary variables of integration.

[23] J. S. R. Chisholm, Nucl. Phys. **26**, 469 (1961); H. J. Borchers, Nuovo Cimento **15**, 784 (1960); S. Kamefuchi, L. O'Raifeartaigh, and Abdus Salam, Nucl. Phys. **28**, 529 (1961).

leave unchanged the in and out limits of the interpolating fields. For practical purposes, the only procedure known to us is the adiabatic switching on and off of charges; this implies that a point transformation is allowed if:

(i) In the limit $g \to 0$ for a transformation like $\varphi(x) \to \varphi'(x) = a_1\varphi(x) + a_2\varphi^2(x) + \cdots$, the $a_i \to 0$, $i > 1$, and $a_1 \to \text{const} \neq 0$. ($a_1 \neq 1$ implies a wave-function renormalization).

(ii) In the language of axiomatic field theory, all transformations $\varphi \to \varphi'$ are allowed, provided $\varphi$ and $\varphi'$ are mutually local operators, $[\varphi'(x), \varphi(y)] = 0$, $(x-y)^2 < 0$ and provided $\langle 0|\varphi|p\rangle = z\langle 0|\varphi'|p\rangle$, $z \neq 0$, where $|p\rangle$ is the appropriate one-particle state.

(iii) The only known procedure for computing $S$-matrix elements for given Lagrangians is essentially the Dyson perturbation procedure which relies on identifying that part of the Lagrangian which depends *bilinearly* on field variables as $\mathcal{L}_f$. In this paper, when making point transformations we have separated out *all* bilinear terms; thus a term like $\mathcal{L} = (\partial_\mu\varphi)^2/(1+\varphi^2)$ will contribute $(\partial_\mu\varphi)^2$ to $\mathcal{L}_f$ and $[\varphi^2/(1+\varphi^2)](\partial_\mu\varphi)^2$ to $\mathcal{L}_{\text{int}}$.

(iv) A consequence of the split mentioned in (iii) is that in our power-counting theorem, $\mathcal{L} = (\partial_\mu\varphi)^2/(1+\varphi^2)$, does not behave supernormally like $M^2$ (assuming $\varphi \sim M$, $\partial\varphi \sim M^2$) but normally like $[\varphi^2/(1+\varphi^2)] \times (\partial\varphi)^2 \sim M^4$. This may mean that our estimates of singularity behavior are likely to be overestimates and that a future formulation of a new computational procedure may depress our estimates of likely infinities.

(v) Regarding our discussion of nonlinear realizations of chiral groups in Sec. V, it is important to realize that the interpolating fields for two different choices of coordinates can be related to each other; thus, writing

$$S = \sigma(\varphi^2) + i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\Lambda(\varphi^2) = \sigma'(\varphi'^2) + i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}'\Lambda'(\varphi'^2), \quad (51')$$

one can express $\boldsymbol{\varphi}$ fields in terms of $\boldsymbol{\varphi}'$ fields by comparing terms of the power series in the $\varphi$. We have assumed that the adiabatic limits of both $\varphi$ and $\varphi'$ are the same, so that the on-mass-shell $S$ matrices are equal and so is the singularity behavior of $S$-matrix elements. It is well known that this result does not apply to the $n$-point Green's functions.

## VII. CONCLUSIONS

We have shown in this paper that a simple power count of ultraviolet infinite integrals in Efimov-Fradkin sums of perturbation diagrams suggests that nonlinear meson theories may behave in the same way as polynomial Lagrangian theories so far as the infinity count is concerned.

A number of fundamental problems remain, basic to the whole approach, which are unresolved. There is the difficult problem of *uniqueness* of the sums, the renormalization program, and the problem of defining the contours in the auxiliary variable planes. It is important to realize that the proof of the absence of infinities in this paper has been given with all vectors $x_\mu$ Wick-rotated and Eulidean. It appears that for nonlinear theories a "Euclidean continuation postulate" must be an essential feature of the theories to render their matrix elements finite. This principle is not new. It has been suggested by Schwinger,[24] Symanzik,[25] Fradkin,[8] and others. The only thing one must guarantee is that the unitarity relation $T + T^\dagger + TT^\dagger = 0$ is preserved when the continuation in external momenta to the physical region is made.

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## APPENDIX

We give here proofs of the singular behaviors of the shifted arguments occurring in (39) for derivative-coupling theories. Our only concern is the (Euclidean) limit $x \to 0$, of Eqs. (38), where $\Delta(x) \sim 1/x^2$. To solve Eqs. (38), let

$$c_{\mu\nu} \equiv d_{\mu\nu}(x)c_1 + e_{\mu\nu}(x)c_0,$$

$$c_\mu = x_\mu c_2,$$

and

$$d_{\mu\nu}(x) \equiv g_{\mu\nu} - x^{-2}x_\mu x_\nu \equiv g_{\mu\nu} - e_{\mu\nu}(x),$$

and make the symmetrical choice $c = c'$ as in the text. Since

$$\Delta_{\mu\nu} = -2\Delta' d_{\mu\nu} - (2\Delta' + 4x^2\Delta'')e_{\mu\nu}$$

and

$$\Delta_\mu = 2x_\mu\Delta',$$

we obtain the equations

$$c_1^2 = -2\Delta', \quad c_0^2 = -2(\Delta' + 2x^2\Delta''),$$

$$c_0c_2 = 2\Delta', \quad c^2 + x^2c_2^2 = \Delta,$$

which are solved by

$$c_1 = [-2\Delta']^{1/2} \sim 1/x^2,$$

$$c_0 = [-2(\Delta' + 2x^2\Delta'')]^{1/2} \sim i(\sqrt{6})/x^2,$$

$$c_2 = 2\Delta'/c_0 \sim i\sqrt{2}/\sqrt{3}x^2,$$

and

$$c = [\Delta - x^2c_2^2]^{1/2} \sim 1/\sqrt{(3x^2)}.$$

This proves the statement that a correct estimate of the most singular behavior is given by

$$c_{\mu\nu} \sim 1/x^2, \quad c_\mu \sim 1/x, \quad \text{and} \quad c \sim 1/x.$$

[24] J. Schwinger, Proc. Natl. Acad. Sci. U. S. 44, 956 (1958); Phys. Rev. 115, 721 (1959).

[25] K. Symanzik, J. Math. Phys. 7, 510 (1966).