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Radiative Corrections to an Electron-Positron Scattering Experiment*

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We present the radiative corrections for a precision experiment on electron-positron scattering performed by Browman, Grossetête, and Yount. It is necessary to extend the earlier work by Redhead and Polovin by evaluating the unobserved real bremsstrahlung without infrared approximations. To accomplish this efficiently and accurately, the very lengthy algebra is handled by a computer. Higher-order corrections are studied with care, and it is believed that our results are adequate for present and foreseeable experiments of this type.

I. INTRODUCTION

A VERY precise measurement of electron-positron scattering in the neighborhood of 180° c.m. scattering angles has been made by Browman, Grossetête, and Yount (BGY).[1] Because of the high precision of their experiment and its particular experimental arrangement, their results provide a very sensitive test of radiative corrections at high energies. No previous calculation of radiative corrections is directly applicable to this particular experiment, but the experimental data have been compared[1] with an extrapolation of some approximate calculations made by Meister and Yennie[2] for a general class of scattering experiments. It is the purpose of the present paper to improve those earlier calculations and extend them to this new experiment.

The BGY experiment is essentially 180° scattering in the c.m. system. A high-energy positron beam enters a target and electrons of nearly the same energy are detected. Because these electrons emerge in a very narrow cone, no attempt is made to resolve their direction. Only their energy distribution is measured. The top 2% of the energy spectrum is examined in detail and energies down to 90% of the incident energy are measured for over-all normalization. If there were no radiative effects, the cross section would be quite flat in this region. The effect of the radiative corrections is to reduce the cross section. The physical reason for this is quite clear: at the upper end of the spectrum there is no energy available for real photon emission. From the Bloch-Nordsieck argument, the cross section for scattering without emitting photons is zero. For a slightly smaller energy for the electron, there is some phase space for emitting photons and the cross section rises. When the energy is far below the incident energy, the phase space for real photons is large and the cross section is roughly the same as the uncorrected one.

For purposes of orientation, it is helpful to study this phase-space variation by looking at the mass of the undetected system. Let p_1 and p_3 be the initial and final positron's four-momenta, p_2 and p_4 the corresponding electron's four-momenta, and K the total four-momentum of the photons. Then

$$\begin{aligned} M^2 &= (p_3+K)^2 = (p_1+p_2-p_4)^2 \\ &= 3m^2+2m(E_1-E_4)-2E_1E_4+2p_1p_4\cos\theta_4 \\ &\cong m^2[1-(p_1-p_4)^2/p_1p_4] \\ &\qquad +2m(p_1-p_4)-2p_1p_4(1-\cos\theta_4)\,, \quad (1.1) \end{aligned}$$

where θ_4 is the angle of the emitted electron relative to the incident positron. Elastic scattering is determined by $M^2=m^2$, giving

$$\cos\theta_4^{\rm el}\cong 1-m(p_1-p_4)/p_1p_4 \qquad (1.2)$$

or

$$\theta_4^{\rm el}\cong[2m(p_1-p_4)/p_1p_4]^{1/2}.$$

For fixed p_4 and angles less than $\theta_4^{\rm el}$, the mass of the final positron-photon system is greater than m, and it as-

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[1] A. Browman, B. Grossetête, and D. Yount, Phys. Rev. **151**, 1094 (1966).

[2] N. Meister and D. R. Yennie, Phys. Rev. **130**, 1210 (1963).

sumes its maximum for $\theta_4=0$

$$(M^2)_{\max}\cong m^2+2m(p_1-p_4). \qquad (1.3)$$

In the frame in which the total momentum of the positron and the photons is zero (missing-mass frame),

$$\hat{\mathbf{p}}_3+\hat{\mathbf{K}}=0, \qquad (1.4)$$

the energy is given by

$$(m^2+\hat{\mathbf{K}}^2)^{1/2}+\hat{K}_0=M. \qquad (1.5)$$

Now suppose $M-m\ll m$. Then $\hat{K}_0\ll m$ and $|\hat{\mathbf{K}}|\leq\hat{K}_0\ll m$ and

$$\hat{K}_0\cong M-m. \qquad (1.6)$$

We call this the nonrelativistic region since the kinetic energy of the positron is negligible and all the energy available goes to the photons. This situation attains for all θ_4 if $(p_1-p_4)\ll m$. In the nonrelativistic region there is no kinematical constraint on the photons except that their total energy must add up to a constant. They are also very soft and the usual infrared arguments hold, permitting us to sum the contributions from all numbers of photons. In the opposite, extreme relativistic situation, $M\gg m$, Eq. (1.5) gives us an additional kinematic constraint between the photons depending on their direction of emission. However, for a single photon, Eq. (1.5) is still relatively simple since $|\hat{\mathbf{K}}|=\hat{K}_0$.

The earliest calculations of radiative corrections to electron-positron scattering were made by Redhead[3] and Polovin[4] who obtained the same result. They calculated the correction of order α under the assumption that the real photons were very soft ($k\leq k_{\max}\ll m$) in the laboratory frame. This is just the nonrelativistic region. The Meister-Yennie (MY) calculation[2] extends into the relativistic region, but is restricted by $\Delta\theta_4=\theta_4^{\mathrm{el}}-\theta_4^{\min}\ll\theta_4^{\mathrm{el}}$; it also contains the approximation of neglecting all nonlogarithmic terms. These calculations must now be extended to the entire θ_4 range without neglect of the nonlogarithmic terms. Many approximations used by MY are no longer valid, and it is necessary to make a complete new calculation of the hard real-photon contributions. In order to retain some contact with the older work, we shall calculate separately the most important contribution using the infrared factorization.

Since the radiative corrections are large for this experiment, multiple photon corrections also become important. In the past it has generally been accepted that these terms can be obtained with some accuracy by the "exponentiation" of the lowest-order term. Intuitively, this exponentiation is based on the fact that soft photons can be emitted and absorbed independently by the charged particles, and a kind of Poisson distribution in the real and virtual photons results. As indicated earlier, these arguments continue to be valid in the nonrelativistic region ($M-m\ll m$). For larger values of M, exponentiation should not be strictly valid, but may still yield a reasonable estimate. Fortunately the radiative corrections are largest for small $M_{\max}$, where exponentiation is valid. To investigate the validity of the exponentiation, the following procedure will be employed: The leading logarithmic terms of α^2 will be calculated and compared with the corresponding terms obtained by expanding the exponential of the lowest-order correction. The difference will give us an estimate of the errors associated with exponentiation and enable us to improve the usual exponential approximation.

Section II contains the derivation of the radiative correction to order α, and Sec. III discusses the higher-order corrections. The result is summarized in Sec. IV.

II. LOWEST-ORDER RADIATIVE CORRECTIONS

The derivation given here parallels in many respects the treatment given in MY.[2] However, since we wish to eliminate the approximations made there, the main points will be repeated briefly. The general procedure for the calculation is the following. The integration over the unobserved angle of the electron is converted into an integration over the mass of the undetected system using (1.1). For each value of M^2, the integration over the momentum of the unobserved positron and photon is carried out in the special Lorentz frame in which the spatial part of $(p_1+p_2-p_4)$ vanishes:

$$\hat{I}\text{ frame: }\hat{\mathbf{p}}_1+\hat{\mathbf{p}}_2-\hat{\mathbf{p}}_4=\hat{\mathbf{K}}+\hat{\mathbf{p}}_3=0. \qquad (2.1)$$

The usual infrared divergence makes its appearance at the lower limit of the M^2 integration ($M^2\sim m^2$). It is not difficult to circumvent this divergence by using a photon mass, ultimately cancelling the divergence against the virtual photon corrections. Fortunately, it is not necessary to do this in detail since Redhead[3] and Polovin[4] have already calculated the radiative corrections arising from one virtual photon plus one very soft real photon whose energy is restricted by

$$k=M-m\leq M_1-m=k_{\max}\ll m \quad \text{(Redhead and Polovin}^{3,4}\text{)}. \qquad (2.2)$$

The principal work is then concentrated on obtaining a sufficiently accurate evaluation of the contribution from M_1 to $M_{\max}$

$$M_1\leq M\leq M_{\max} \quad \text{(present work)}. \qquad (2.3)$$

A. Kinematical Preliminaries

The cross section for scattering with emission of one real photon is

$$\frac{d\hat{\sigma}_1}{dp_4}=\int\cdots\int\frac{d^3p_3}{E_3}\frac{d^3k}{k}\frac{p_4{}^2d\Omega_4}{E_4}\times\delta(p_1+p_2-p_3-p_4-k)\hat{\rho}_1(p_1p_4;k). \qquad (2.4\text{a})$$

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[3] M. L. G. Redhead, Proc. Roy. Soc. (London) **A220**, 219 (1953).

[4] R. V. Polovin, Zh. Eksperim. i Teor. Fiz. **31**, 449 (1956) [English transl.: Soviet Phys.—JETP **4**, 385 (1957)].

(The caret on σ_1 and ρ_1 is used in this section to emphasize that corrections due to virtual photons are not included.) It is convenient to replace the integration over the solid angle Ω_4 by one over M^2 using (1.1). Each value of M^2 then defines a special frame (taking the azimuthal angle of $\mathbf{p}_4$ to be zero, for example) and (2.4a) reduces to

$$\frac{d\hat{\sigma}_1}{dp_4}=\frac{\pi}{p_1}\int_{m^2}^{M_{\max}^2}dM^2\int\frac{\hat{k}d\hat{\Omega}}{\hat{k}+\hat{E}_3}\hat{\rho}_1(p_1\cdots p_4;k). \quad (2.4b)$$

The various energies in the special frame are easily expressed in terms of the laboratory energies and M^2. Thus we have

$$M=\hat{k}+\hat{E}_3$$

or

$$\hat{k}=(M^2-m^2)/2M, \quad (2.5a)$$

$$\hat{E}_3=(M^2+m^2)/2M, \quad (2.5b)$$

and

$$\hat{E}_1=p_1\cdot(p_1+p_2-p_4)/M$$
$$=[mE_4+\tfrac{1}{2}(M^2-m^2)]/M, \quad (2.5c)$$

$$\hat{E}_2=p_2\cdot(p_1+p_2-p_4)/M$$
$$=m(E_1+m-E_4)/M\equiv mE_3/M, \quad (2.5d)$$

$$\hat{E}_4=[mE_1-\tfrac{1}{2}(M^2-m^2)]/M. \quad (2.5e)$$

Our integral (2.4) is conveniently rewritten

$$\frac{d\hat{\sigma}_1}{dp_4}=\frac{2\pi}{p_1}\int_{m^2}^{M_{\max}^2}\frac{dM^2}{M^2-m^2}\int\hat{k}^2d\hat{\Omega}_3\hat{\rho}_1. \quad (2.6)$$

B. Hard Real-Photon Contributions—Infrared Estimate

We want to evaluate (2.6) under the assumption that the cross section for scattering with photon emission is the elastic scattering cross section times an emission factor. This means

$$\frac{2\pi}{p_1}\hat{\rho}_1=\frac{d\hat{\sigma}_0}{dp_4}\tilde{S}, \quad (2.7a)$$

where

$$\tilde{S}=-\frac{\alpha}{4\pi^2}\left(\frac{p_1^\mu}{k\cdot p_1}-\frac{p_3^\mu}{k\cdot p_3}-\frac{p_2^\mu}{k\cdot p_2}+\frac{p_4^\mu}{k\cdot p_4}\right)^2$$
$$=-\frac{\alpha}{4\pi^2}\left(\frac{p_1^\mu}{k\cdot p_1}-\frac{f^\mu}{f\cdot k}-\frac{p_2^\mu}{k\cdot p_2}+\frac{p_4^\mu}{k\cdot p_4}\right)^2, \quad (2.7b)$$

with

$$f=p_3+k=p_1+p_2-p_4. \quad (2.7c)$$

The basic integrals are

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i^2}{(k\cdot p_i)^2}=\hat{k}^2\int d\hat{\Omega}_3\frac{f^2}{(k\cdot f)^2}=4\pi, \quad (2.8a)$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot p_j}{k\cdot p_ik\cdot p_j}=4\pi\rho\left(\frac{p_i\cdot p_j}{m^2}\right), \quad (2.8b)$$

and

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot f}{k\cdot p_ik\cdot f}=4\pi\rho\left(\frac{\hat{E}_i}{m}\right), \quad (2.8c)$$

where

$$\rho(x)=\frac{x}{(x^2-1)^{1/2}}\ln[x+(x^2-1)^{1/2}] \quad (2.8d)$$
$$\cong\ln 2x \quad \text{for} \quad x\gg 1$$
$$\cong 1 \quad \text{for} \quad x\sim 1.$$

To further simplify the integral over M^2 in (2.6) we evaluate the results of the angular integral with the infrared condition $\hat{k}=0$ $(M=m)$. This means

$$\hat{E}_1\to E_4,\quad \hat{E}_4\to E_1,\quad \hat{E}_3\to E_3,\quad p_1\cdot p_4\to mE_3.$$

With this approximation, (2.6) reduces to

$$\left(\frac{d\hat{\sigma}_1}{dp_4}\right)_{>M_1}=\frac{d\hat{\sigma}_0}{dp_4}\frac{4\alpha}{\pi}\left[\rho\left(\frac{E_1}{m}\right)+\rho\left(\frac{E_4}{m}\right)-\rho\left(\frac{E_3}{m}\right)-1\right]\times\ln\frac{M_{\max}^2-m^2}{M_1^2-m^2}. \quad (2.9)$$

For the purpose of the BGY experiment, E_1 and E_4 are always much greater than m; E_3 is also at least several times m so that the approximate form in (2.8d) for large argument can be used. Equation (2.9) then becomes

$$\left(\frac{d\hat{\sigma}_1}{dp_4}\right)_{>M_1}=\frac{d\hat{\sigma}_0}{dp_4}\frac{4\alpha}{\pi}\left(\ln\frac{2E_1E_4}{mE_3}-1\right)\ln\frac{M_{\max}^2-m^2}{M_1^2-m^2}. \quad (2.10)$$

C. Matching to Redhead's Result

Redhead's calculation appears to have been donefor slightly different conditions than those specified by $M<M_1$; namely, he restricts the photon energy in the laboratory by condition (2.2), whereas we need (2.2) in the rest frame of the unobserved positron. However, since the radiative corrections are completely symmetric under the interchange $p_1\leftrightarrow p_4$, $p_2\leftrightarrow p_3$, it turns out that the photon phase space given by the rest frame of p_2 gives the same result as that defined by p_3. We may accordingly take over Redhead's result with the simple substitution

$$k_{\max}=M_1-m\cong(M_1^2-m^2)/2m.$$

There are a few complications about using Redhead's result which should be mentioned. He gives two high-energy expressions for e^+-e^- scattering. The first of these, his Eq. (5.4), is valid in the region

$$E_1,\ E_4,\ E_1-E_4\gg m.$$

The second, his Eq. (5.6), is valid for

$$E_1=E_4\gg m.$$

The first equation does not extrapolate precisely to the value given by the second equation. The differences however are very small. One such difference occurs in the coefficient of $\ln k_{\max}$ and may be understood by noting that this coefficient contains the term $\rho(E_3/m)$, whose argument can approach 1. If the approximate form of ρ is used, its limiting value is ln2 rather than 1, as it should be. This error is to be compared with the large terms in the coefficient of $\ln k_{\max}$ such as $\ln(2E_1/m)$, and we see that the fractional error is extremely small. In any case, we can see that if E_4 is only 1 MeV less than its maximum value (giving $x=3$), the error in this rather small term is only 1/18 of itself, producing negligible error in the cross section. Although we have not investigated the terms independent of $\ln k_{\max}$ in detail, the error of the extrapolated function at the upper limit is only of order 1.5% of the cross section. Assuming that the error behaves in the same general manner as the coefficient of $\ln k_{\max}$, it should be completely unimportant for $(E_1-E_4)\gtrsim 1$ MeV. Since the cross section is forced to zero at $E_4=E_1$ by the infrared divergence, and since experimental energy resolutions are not likely to approach 1 MeV at higher energies, we conclude that Redhead's equation (5.4) should be valid for all presently foreseeable experimental conditions.

The second complication in Redhead's result is that not all the terms are simply proportional to the uncorrected cross section. It seems convenient to extract the terms proportional to the uncorrected cross section (including the ones associated with the infrared divergence) and arrange the remaining terms in ascending powers of

$$\beta=(E_1-E_4)/E_1. \qquad (2.11)$$

For the BGY experiment these extra terms are completely negligible; they are recorded here only to illustrate this point. If β is not small (i.e., not less than 0.1) and the experimental accuracy is good ($\sim$1%), it would be necessary to completely redo the present calculations. Redhead's result may then be expressed in the form of a fractional correction

$$\delta_{\mathrm{Red.}}=\frac{\alpha}{\pi}\Bigg[\left(\ln\frac{2E_1E_4}{mE_3}-1\right)\ln\frac{k_{\max}{}^4}{mE_1E_3E_4}-\frac{1}{2}\left(\ln\frac{E_1E_4}{mE_3}\right)^2 +\frac{11}{6}\ln\frac{4E_1E_4}{m^2}+\tfrac{1}{2}\ln\frac{E_1E_4}{E_3{}^2}+\frac{3}{4}\left(\ln\frac{E_1}{E_4}\right)^2 -\tfrac{1}{2}\ln\frac{E_1}{E_3}\ln\frac{E_4}{E_3}-\frac{37}{9}+\frac{3\pi^2}{4}-\ln 2-\tfrac{1}{2}\ln^2 2-\tfrac{1}{2}\pi^2\beta -\left(\frac{25}{6}+\tfrac{1}{2}\ln^2\frac{E_1}{E_3}+\ln\frac{E_1}{E_3}+\tfrac{1}{2}\pi^2\right)\beta^2\Bigg]. \qquad (2.12)$$

Adding Redhead's result to (2.10), we find for the sum of the infrared and noninfrared virtual one-photon corrections

$$\delta_{\mathrm{ir}}+\delta_V{}^{(1)}=\frac{\alpha}{\pi}\Bigg[\left(\ln\frac{2E_1E_4}{mE_3}-1\right)\ln\frac{(E_3-m)^4}{mE_1E_3E_4}-\frac{1}{2}\left(\ln\frac{E_1E_4}{mE_3}\right)^2 +\frac{11}{6}\ln\frac{4E_1E_4}{m^2}+\tfrac{1}{2}\ln\frac{E_1E_4}{E_3{}^2}-\tfrac{1}{2}\ln^2\frac{E_1}{E_3}-\frac{37}{9}+\frac{3\pi^2}{4} -\ln 2-\tfrac{1}{2}\ln^2 2\Bigg], \qquad (2.13\text{a})$$

where terms which are negligible (<0.1%) for $\beta<0.1$ have been dropped. The infrared part of (2.13a) is defined by

$$\delta_{\mathrm{ir}}=\frac{\alpha}{\pi}\Bigg[\left(\ln\frac{2E_1E_4}{mE_3}-1\right)\ln\frac{(E_3-m)^4}{mE_1E_3E_4} -\frac{1}{2}\left(\ln\frac{E_1E_4}{mE_3}\right)^2\Bigg]. \qquad (2.13\text{b})$$

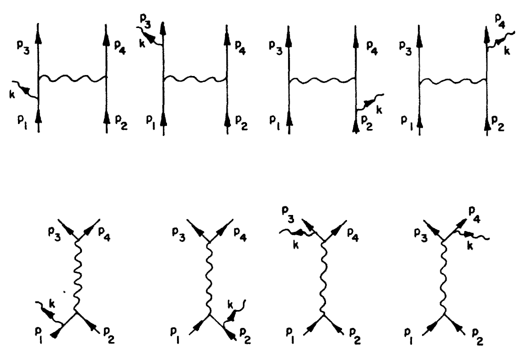

FIG. 1. Lowest-order Feynman diagrams for electron-positron bremsstrahlung.

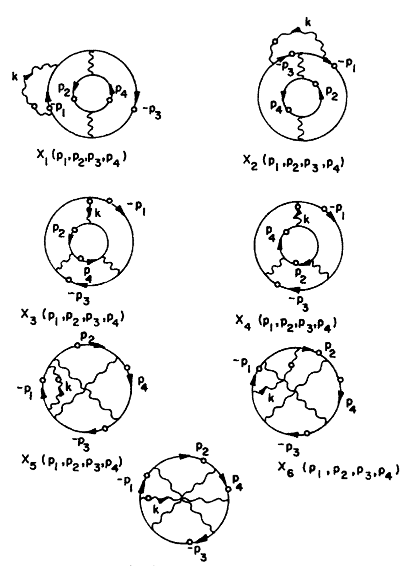

FIG. 2. Graphical representation of the products of matrix elements for some of the Feynman diagrams in Fig. 1. A circle on a line indicates that no propagator factor of $(p^2-m^2)^{-1}$ appears.

D. Hard Real-Photon and Noninfrared Contributions

Having isolated the infrared part of the cross section for single-photon emission, it is now possible to express the remaining noninfrared real contributions to the cross section in the form of a convergent integral

$$\left(\frac{d\hat{\sigma}_1}{dp_4}\right)_R=\frac{2\pi}{p_1}\int_{m^2}^{M_{\max}^2}\frac{dM^2}{M^2-m^2}\int \hat{k}^2 d\hat{\Omega}_3(\hat{\rho}_1-\hat{\rho}_1^{\text{ir}}), \quad (2.14)$$

where $\hat{\rho}_1^{\text{ir}}$ is the infrared limit of $\hat{\rho}_1$ extracted in the previous subsection, as given by (2.7) and the approximations preceding (2.9). We write

$$\hat{\rho}_1=\lambda\tfrac{1}{4}\sum_{\text{spins}}|F|^2,\quad \hat{\rho}_1^{\text{ir}}=\lambda\tfrac{1}{4}\sum_{\text{spins}}|F_{\text{ir}}|^2, \quad (2.15)$$

where F is the usual Feynman amplitude and λ contains the incident flux and other factors.

The form of $|F|^2$ for this process may be readily computed by standard Feynman techniques. To this order, there are eight Feynman graphs contributing to F as shown in Fig. 1. The traces which arise from the spin sums in Eq. (2.15) may be represented pictorially in terms of closed loop diagrams as shown in Fig. 2. Because of the identical fermion symmetries in the cross section, however, only seven such diagrams need be considered, and all other traces may be expressed in terms of these by momentum interchange.

The seven basic traces depicted in Fig. 2 may be written in the form

$$\begin{aligned}
X_1(p_1,p_2,p_3,p_4)=&-\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\nu(\gamma\cdot p_2+m)\gamma_\mu]\,\mathrm{Tr}[(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\\
&\times\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\tau(\gamma\cdot k-\gamma\cdot p_1+m)\gamma_\nu]/[64m^4(p_1\cdot k)^2(p_2-p_4)^4],\\
X_2(p_1,p_2,p_3,p_4)=&\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\nu(\gamma\cdot p_2+m)\gamma_\mu]\,\mathrm{Tr}[(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\\
&\times\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\nu(-\gamma\cdot k-\gamma\cdot p_3+m)\gamma_\tau]/[64m^4p_1\cdot kp_3\cdot k(p_2-p_4)^4],\\
X_3(p_1,p_2,p_3,p_4)=&-\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\nu(\gamma\cdot p_2-\gamma\cdot k+m)\gamma_\tau(\gamma\cdot p_2+m)\gamma_\mu]\,\mathrm{Tr}[(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\\
&\times\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\nu]/[64m^4p_1\cdot kp_2\cdot k(p_1-p_3)^2(p_2-p_4)^2],\\
X_4(p_1,p_2,p_3,p_4)=&-\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\tau(\gamma\cdot k+\gamma\cdot p_4+m)\gamma_\nu(\gamma\cdot p_2+m)\gamma_\mu]\,\mathrm{Tr}[(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\\
&\times\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\nu]/[64m^4p_1\cdot kp_4\cdot k(p_1-p_3)^2(p_2-p_4)^2],\\
X_5(p_1,p_2,p_3,p_4)=&-\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\nu(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\tau(\gamma\cdot k-\gamma\cdot p_1+m)\\
&\times\gamma_\nu(\gamma\cdot p_2+m)\gamma_\mu]/[64m^4(p_1\cdot k)^2(p_2-p_4)^2(p_3+p_4)^2],\\
X_6(p_1,p_2,p_3,p_4)=&-\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\nu(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\gamma_\tau(-\gamma\cdot p_1+m)\gamma_\nu(\gamma\cdot p_2-\gamma\cdot k+m)\\
&\times\gamma_\tau(\gamma\cdot p_2+m)\gamma_\mu]/[64m^4p_1\cdot kp_2\cdot k(p_2-p_4)^2(p_3+p_4)^2],\\
X_7(p_1,p_2,p_3,p_4)=&\mathrm{Tr}[(\gamma\cdot p_4+m)\gamma_\tau(\gamma\cdot k+\gamma\cdot p_4+m)\gamma_\nu(-\gamma\cdot p_3+m)\gamma_\mu(\gamma\cdot k-\gamma\cdot p_1+m)\gamma_\tau(-\gamma\cdot p_1+m)\\
&\times\gamma_\nu(\gamma\cdot p_2+m)\gamma_\mu]/[64m^4p_1\cdot kp_4\cdot k(p_2-p_4)^2(p_3+p_4)^2],
\end{aligned}\quad (2.16)$$

where conservation of momentum requires that

$$k=p_1+p_2-p_3-p_4.$$

The cross section as written in terms of these six traces contains 36 terms. However, it is possible to combine the general traces further to produce a more compact form of the cross section. For example, it was pointed out by Swanson[5] that if the original eight diagrams are paired according to identical photon propagators, then only three combinations of X_1 through X_7 appear in the cross section, namely,

$$\begin{aligned}
A_1(p_1,p_2,p_3,p_4)&=X_1(p_1,p_2,p_3,p_4)+X_1(p_3,p_4,p_1,p_2)+2X_2(p_1,p_2,p_3,p_4),\\
A_2(p_1,p_2,p_3,p_4)&=X_3(p_1,p_2,p_3,p_4)+X_4(p_1,p_2,p_3,p_4)+X_3(p_3,p_4,p_1,p_2)+X_4(p_3,p_4,p_1,p_2),\\
A_3(p_1,p_2,p_3,p_4)&=X_5(p_1,p_2,p_3,p_4)+X_6(p_1,p_2,p_3,p_4)+X_6(p_1,p_3,p_2,p_4)+X_7(p_1,p_2,p_3,p_4).
\end{aligned}\quad (2.17)$$

In terms of A_1, A_2, and A_3, the cross section is then

$$\begin{aligned}
\sum|F|^2=&A_1(p_1,p_2,p_3,p_4)+A_1(-p_2,-p_1,-p_4,-p_3)+A_1(p_1,-p_3,-p_2,p_4)+A_1(p_3,-p_1,-p_4,p_2)\\
&+2A_2(p_1,p_2,p_3,p_4)+2A_2(p_1,-p_3,-p_2,p_4)-2A_3(p_1,p_2,p_3,p_4)\\
&-2A_3(p_3,-p_1,-p_4,p_2)-2A_3(-p_2,p_4,p_1,-p_3)-2A_3(-p_4,-p_3,-p_2,-p_1).
\end{aligned}\quad (2.18)$$

The expressions A_1, A_2, and A_3 are essentially Swanson's A, B, and C, bearing in mind that we have named our momenta differently.

The reduction of A_1 through A_3 to scalar products was performed by computer,[6] and the results are as follows:

[5] S. M. Swanson, Phys. Rev. **154**, 1601 (1967).
[6] A. C. Hearn, Comm. A. C. M. **9**, 573 (1966).

$$A_1(p_1,p_2,p_3,p_4)$$

$$=\left\{p_1\cdot kp_3\cdot k\left[2(m^2-p_2\cdot p_4)(p_1\cdot p_3-2m^2)+\frac{(p_2\cdot p_4-2m^2)(p_1\cdot k^2+p_3\cdot k^2)}{(m^2-p_2\cdot p_4)}-2(p_1\cdot p_2p_2\cdot p_4+p_2\cdot p_3p_3\cdot p_4)\right]\right.$$

$$\left.-m^2\left[2w_2(m^2w_2+2p_1\cdot kp_3\cdot k)+w_3{}^2(m^2-p_2\cdot p_4)+2(p_1\cdot kp_1\cdot p_2-p_3\cdot kp_2\cdot p_3)\frac{(p_1\cdot kp_1\cdot p_4-p_3\cdot kp_3\cdot p_4)}{(m^2-p_2\cdot p_4)}\right]\right\}\Big/$$

$$[2m^4p_1\cdot k^2p_3\cdot k^2(m^2-p_2\cdot p_4)],$$

$$A_2(p_1,p_2,p_3,p_4)$$

$$=\{\tfrac{1}{4}(w_5w_7-w_4w_6)(\tfrac{1}{2}w_1{}^2+\tfrac{1}{2}w_3{}^2+w_4{}^2+w_5{}^2)-m^2[w_4{}^2w_6+w_5{}^2w_7+w_2(w_4+w_5)(p_2\cdot kp_4\cdot k-p_1\cdot kp_3\cdot k)$$

$$-w_1w_3w_4w_5+w_1w_3(p_1\cdot kp_3\cdot k+p_2\cdot kp_4\cdot k)]\}/[8m^4p_1\cdot kp_2\cdot kp_3\cdot kp_4\cdot k(m^2-p_2\cdot p_4)(m^2-p_1\cdot p_3)], \quad (2.19)$$

$$A_3(p_1,p_2,p_3,p_4)$$

$$=\left\{(4m^4-w_8{}^2)\left(w_9+2p_1\cdot k^2+w_8p_1\cdot k+4p_1\cdot kp_4\cdot k-2m^2p_4\cdot k+\frac{2m^2p_2\cdot kp_3\cdot k}{p_1\cdot k}\right)+4m^2p_4\cdot k(p_2\cdot k-p_3\cdot k)\right.$$

$$\times(2m^2+w_8-p_1\cdot k)-2m^2p_2\cdot kp_3\cdot k(6p_1\cdot k+2p_4\cdot k-3w_8)+2(p_1\cdot k+p_4\cdot k)[(m^2-w_8)w_9-2p_1\cdot kp_4\cdot kw_8]$$

$$\left.-2(p_1\cdot k^2+p_4\cdot k_2)(w_9+2p_1\cdot kp_4\cdot k+p_1\cdot kw_8)+6m^2p_1\cdot k^2w_8\right\}\Big/$$

$$\times[32m^4p_1\cdot kp_2\cdot kp_3\cdot k(m^2-p_2\cdot p_4)(m^2+p_3\cdot p_4)],$$

where

$$\begin{aligned}
w_1&=p_2\cdot k+p_4\cdot k,\\
w_2&=p_1\cdot k-p_3\cdot k=p_4\cdot k-p_2\cdot k,\\
w_3&=p_1\cdot k+p_3\cdot k,\\
w_4&=p_1\cdot p_4+p_2\cdot p_3,\\
w_5&=p_1\cdot p_2+p_3\cdot p_4,\\
w_6&=p_1\cdot kp_4\cdot k+p_2\cdot kp_3\cdot k,\\
w_7&=p_1\cdot kp_2\cdot k+p_3\cdot kp_4\cdot k,\\
w_8&=2(m^2-p_2\cdot p_4)+2(m^2+p_3\cdot p_4),\\
w_9&=2p_2\cdot k(m^2-p_2\cdot p_4)-2p_3\cdot k(m^2+p_3\cdot p_4).
\end{aligned} \quad (2.20)$$

These expressions are more compact than those given by Swanson, but may easily be shown to be equivalent.

Equations (2.16)–(2.18) indicate the wide varieties of symmetries which exist in the complete cross section, and various parts of it. In particular, it is easy to see that the total cross section is invariant under the interchange

$$p_1\leftrightarrow -p_4$$

or

$$p_2\leftrightarrow -p_3$$

expressing symmetry under the interchange of an initial particle (antiparticle) and final antiparticle (particle).

It is now a relatively straightforward matter to perform analytically the angular integration of $|F|^2$ in the special frame $\hat{\mathbf{p}}_3+\hat{\mathbf{K}}=0$. By using the kinematical relations among scalar products, it is possible to express all angular dependent terms in terms of the standard integrals given in Appendix A. The integration can then be performed, and the corresponding angular integral over $|F_{\rm ir}|^2$ subtracted to give a convergent integrand for the M^2 integral in Eq. (2.14). The computer was again used to help with this reduction both as a store for the large expressions encountered in transforming the expressions to a form suitable for integration, and for checking the accuracy of any hand algebra used.

We note in passing that in integrating the cross section in our special frame, much of the symmetry which is expressed in Eqs. (2.17) and (2.18) is lost. However, it is useful to recognize that the over-all cross section is still invariant under the interchange $p_1\leftrightarrow -p_4$ in this frame.

The answer at this point contains of the order of 500 terms, and so a complete analytic integration over M^2 was not performed. Since our discussion of the infrared estimate was only accurate to order β, only those terms up to linear in β were computed exactly. The remaining terms were found numerically to be completely negligible for the range of variables considered.

Keeping then only terms linear in β, the required correction to the cross section may be written

$$\delta_R{}^{(1)}=\int_{m^2}^{M_{\max}{}^2}dM^2I, \quad (2.21)$$

where

$$I=\frac{\alpha}{\pi}\left\{\frac{4\rho(p_1\cdot p_4/m^2)}{p_1\cdot p_4}-\frac{3p_2\cdot f[\rho(p_1\cdot p_4/m^2)/p_1\cdot p_4-\rho(p_2\cdot f/m^2)/p_2\cdot f]}{p_3\cdot k}+\frac{7\rho(p_4\cdot f/Mm)}{2p_4\cdot f}\right.$$
$$+\frac{(p_1\cdot p_2-\frac{5}{2}p_2\cdot f)[\rho(p_4\cdot f/Mm)/p_4\cdot f-\rho(p_1\cdot p_2/m^2)/p_1\cdot p_2]}{p_3\cdot k}-\frac{\rho(p_2\cdot f/Mm)}{p_2\cdot f}-\frac{[\rho(p_2\cdot f/Mm)-\rho(p_2\cdot f/m^2)]}{p_3\cdot k}$$
$$+\frac{2p_3\cdot k}{M^2p_2\cdot f}-\frac{2}{p_2\cdot f}-\frac{7\rho(p_1\cdot f/Mm)}{2p_1\cdot f}+\frac{(p_1\cdot p_2+\frac{3}{2}p_2\cdot f)[\rho(p_1\cdot f/Mm)/p_1\cdot f-\rho(p_2\cdot p_4/m^2)/p_2\cdot p_4]}{p_3\cdot k}$$
$$\left.+\frac{2(p_3\cdot k/M^2-1)\rho[(m^2+p_3\cdot k)/Mm]}{m^2+p_3\cdot k}+\frac{7}{2}\left[\frac{\rho(p_2\cdot p_4/m^2)}{p_2\cdot p_4}-\frac{\rho(p_1\cdot p_2/m^2)}{p_1\cdot p_2}\right]+\frac{4p_3\cdot k}{M^4}\right\}. \quad (2.22)$$

For the range of variables considered, the approximation

$$\rho(x)\cong\ln 2x$$

is valid for all arguments which remain large over the whole integration region. With this approximation, the integrand may then be written to order β in the form

$$I=\frac{\alpha}{\pi}\left\{\frac{4\rho(p_1\cdot p_4/m^2)}{p_1\cdot p_4}-\frac{3p_2\cdot f[\rho(p_1\cdot p_4/m^2)/p_1\cdot p_4-\rho(p_2\cdot f/m^2)/p_2\cdot f]}{p_3\cdot k}+\frac{2p_3\cdot k}{M^2p_2\cdot f}-\frac{2}{p_2\cdot f}\right.$$
$$\left.-\frac{\ln(M^2/m^2)}{p_3\cdot k}-\frac{\rho(p_2\cdot f/Mm)}{p_2\cdot f}-\frac{[\rho(p_2\cdot f/Mm)-\rho(p_2\cdot f/m^2)]}{p_3\cdot k}+\frac{2(p_3\cdot k/M^2-1)\rho[(m^2+p_3\cdot k)/Mm]}{m^2+p_3\cdot k}+\frac{4p_3\cdot k}{M^4}\right\}, \quad (2.23)$$

which can be evaluated to order β using the integrals in Appendix B. The final result is then

$$\delta_R^{(1)}=(\alpha/2\pi)[\ln(2E_3/m)+\tfrac{1}{2}\pi^2-5]. \quad (2.24)$$

It is interesting to compare this result with the calculation of MY, whose value for the same quantity may be written in the form

$$\delta_R^{(1)}=(\alpha/2\pi)[-\ln^2(2E_3/m)+\cdots]. \quad (2.25)$$

The kinematic range of variables considered in the two calculations is different. However, it is possible to make a comparison of the leading $\ln^2$ terms by looking at the energy-dependent contribution from the lower part of the M^2 integral in the present calculation. This contribution has the form

$$\delta_{\text{lower}}=-(\alpha/2\pi)\ln^2(2E_3/m) \quad (2.26)$$

which agrees with the $\ln^2$ term in (2.25).

The absence of a $\ln^2$ term in our result is quite unexpected, and gives a much smaller contribution to the over-all radiative corrections than extrapolation of previous calculations would suggest. This absence is due to an exact cancellation of the $\ln^2$ contributions from the upper and lower parts of the M^2 integral, which is hard to explain by any simple physical argument, since the mechanisms responsible for the radiation at each end of the spectrum are so different.

The absence of any term linear in β in the result is also worth noting, and is probably due to the symmetries existing in the cross section under the interchange $E_1\leftrightarrow -E_4$.

Combining the result of this calculation with the part of the radiative corrections given in Eq. (2.13), we find for the complete radiative corrections to order β for this process

$$\delta_{\text{ir}}+\delta_{\text{nir}}^{(1)}=\frac{\alpha}{\pi}\left[\left(\ln\frac{2E_1E_4}{mE_3}-1\right)\ln\frac{(E_3-m)^4}{mE_1E_3E_4}\right.$$
$$-\frac{1}{2}\left(\ln\frac{E_1E_4}{mE_3}\right)^2+\frac{11}{6}\ln\frac{4E_1E_4}{m^2}+\tfrac{1}{2}\ln\frac{E_1E_4}{E_3{}^2}$$
$$\left.-\tfrac{1}{2}\ln^2 2-\tfrac{1}{2}\ln^2\frac{E_1}{E_3}+\tfrac{1}{2}\ln\frac{E_3}{2m}+\pi^2-\frac{119}{18}\right]. \quad (2.27)$$

III. HIGHER-ORDER RADIATIVE CORRECTIONS

We noted in the previous section that when $M_{\max}{}^2$ is small (comparable to or only a few times m^2), the radiative corrections become quite large. In this situation one cannot expect the lowest-order corrections to be adequate. The higher-order corrections may be significant and it is necessary to include them as well as we can. Fortunately, it is possible to to this reasonably well. We note that the main reason the correction is large is that the phase space for real photon emission is so strongly limited for small M^2 that the positive contribution from real photons is much smaller than the negative contribution from virtual photons. The usual tendency for infrared contributions to cancel is therefore inhibited and we are left with a large negative contribution. In order to improve on the lowest-order contribution, it is mainly necessary to include the infrared contributions from higher order. In simpler situations, it is known how to do this exactly and the result is that the lowest-order infrared contributions are exponentiated

$$(1+\delta_{\rm ir}+\delta_{\rm nir}{}^{(1)}) \to e^{\delta_{\rm ir}}(1+\delta_{\rm nir}{}^{(1)}). \tag{3.1}$$

We will review this analysis and see how to apply it to the present situation.

A. Review of Exponentiation of Infrared Part of Radiative Corrections

A treatment of the exponentiation of the radiative corrections is contained in Ref. 7, Secs. II C and II E. In the problem studied there was electron scattering with energy loss ϵ which was carried away by real photons. This energy loss was so small that kinematical restrictions other than energy conservation could be ignored. In particular, the recoil energy of the unobserved nucleus depends on the momentum of the real photons, but that dependence was neglected. Suppose the probability for energy loss ϵ without radiative corrections is

$$\left(\frac{d^2\sigma}{d\epsilon d\Omega}\right)_0=\frac{\alpha A}{\epsilon}\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}, \tag{3.2}$$

where αA arises from the integration over all angles of the probability of emission of a single real photon. The result of including all contributions from real and virtual infrared photons such that the total energy carried away is ϵ is obtained by multiplying (3.2) by the factor

$$F(\alpha A)\exp[\delta_{\rm ir}(\epsilon)], \tag{3.3}$$

where

$$F(\alpha A)=1-\tfrac{1}{12}\pi^2(\alpha A)^2+\cdots$$

and

$$\delta_{\rm ir}=2\alpha[\hat{B}+\tilde{B}(\epsilon)]\cong\alpha A\ln(\epsilon/E). \tag{3.4}$$

[The present discussion differs from Yennie, Frautschi, and Suura (YFS)[7] in that we use $\hat{B}$ rather than B. We are omitting the singly logarithmic terms arising from J_{ij}, which really have an ultraviolet origin.] There are also noninfrared radiative corrections from virtual photons which multiply (3.2) by $(1+\delta_{\rm nir})$, giving the result

$$\frac{d^2\sigma}{d\epsilon d\Omega}=\frac{\alpha A}{\epsilon}\left(\frac{\epsilon}{E}\right)^{\alpha A}F(\alpha A)(1+\delta_{\rm nir})\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}. \tag{3.5}$$

If, instead of looking at the differential cross section in energy, we make an energy-resolution-type measurement ($\epsilon\leq\Delta E$), we integrate (3.5) to obtain

$$\frac{d\sigma}{d\Omega}(\Delta E)=\left(\frac{\Delta E}{E}\right)^{\alpha A}F(\alpha A)(1+\delta_{\rm nir})\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}$$

$$=e^{\delta_{\rm ir}(\Delta E)}(1+\delta_{\rm nir})F(\alpha A)\left(\frac{d\sigma}{d\Omega}\right)_{\rm el} \tag{3.6a}$$

$$\cong(1+\delta_{\rm ir}+\delta_{\rm nir})\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}. \tag{3.6b}$$

The final expression is what would be calculated directly to order α using an infrared cutoff. In practice, this gives us a convenient way of identifying $\delta_{\rm nir}$: Calculate δ using a cutoff and subtract $\delta_{\rm ir}$ which generally includes all the doubly logarithmic parts of δ as well as all terms in $\ln\Delta E$. The exponential (3.6a) is then obtained without doing any additional work beyond that already contained in the calculation of δ.

B. Adaptation of Exponentiation Argument to BGY Experiment

Consider electron-positron scattering with emission of n photons of definite momentum. The differential cross section for the process may be written

$$d\sigma_n=\tilde{\rho}_n(p_1,p_2,p_3,p_4;\,k_1\cdots k_n)$$
$$\times\exp(2\alpha\hat{B})\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\prod_{i=1}^{n}\frac{d^3k_i}{\omega_i}$$
$$\times\delta(p_1+p_2-p_4-p_3-\sum_{i=1}^{n}k_i). \tag{3.7}$$

The infrared virtual contributions are contained in $\exp[2\alpha\hat{B}(p_1,p_2,p_3,p_4)]$, while the noninfrared virtual contributions are contained as a power series in α in the function $\tilde{\rho}_n$. The complication of (3.7) over the corresponding expression for elastic scattering from a potential is that we include momentum as well as energy conservation. It is necessary to integrate (3.7) over all variables subject to the constraint ($E_4=$ const) and then sum over all n. As in the one-photon case, the integral over the angles of p_4 can be converted into one over M^2 using (1.1). This gives

$$\frac{d\sigma_n}{dp_4}=\frac{\pi}{n!p_1}\int_{m^2}^{M_{\rm max}{}^2}dM^2\int\frac{d^3p_3}{E_3}\prod_{i=1}^{n}\frac{d^3k_i}{\omega_i}$$
$$\times\delta(p_1+p_2-p_3-p_4-\sum k_i)\tilde{\rho}_n\exp(2\alpha\hat{B}). \tag{3.8}$$

If we compare (3.8) with the corresponding expression in potential scattering, we see that the present kinematical restriction on the photons is much more complicated. In the potential-scattering case, one has simply the condition that the energy lost by the electron is carried off by the photons. The momentum carried off by the photons is compensated by the momentum transfer to the potential, and the dependence of the matrix element on that momentum transfer is usually neglected. In the present situation, the combined

[7] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 379 (1961).

phase space of the photons is much more complicated because the energy of the unobserved positron is a function of all the photon momenta. Further, the factor $\tilde{\rho}_n$ depends on p_3, and hence has an indirect dependence as well as a direct dependence on all the k's; this variation of $\tilde{\rho}_n$ with p_3 cannot be ignored. For small M ($\sim m$) these complications disappear and one recovers the simplification of the potential scattering case. (One finds $\sum \omega_i = M-m$ and $p_3 \leq M-m$, so the variation with p_3 becomes unimportant.) When M is large ($\gg m$), we have to invent some convenient separation of (3.8) into infrared and noninfrared parts. The precise method of doing this is not unique, but the complete answer is well defined.

Standard infrared arguments tell us that if we let any one of the photon momenta tend to zero, $\tilde{\rho}_n$ reduces to an infrared factor for that photon times $\tilde{\rho}_{n-1}$ for the remaining photons:

$$\lim_{k_n\to 0} \tilde{\rho}_n(k_1\cdots k_n) = \tilde{S}(k_n)\tilde{\rho}_{n-1}(k_1\cdots k_{n-1}). \quad (3.9)$$

Using this together with symmetry, we introduce new functions $\tilde{\beta}_n$ which have no infrared singularities:

$$\begin{aligned} \tilde{\rho}_0 &= \tilde{\beta}_0, \\ \tilde{\rho}_1(k_1) &= \tilde{S}(k_1)\tilde{\beta}_0 + \tilde{\beta}_1(k_1), \\ \tilde{\rho}_2(k_1,k_2) &= \tilde{S}(k_1)\tilde{S}(k_2)\tilde{\beta}_0 + \tilde{S}(k_1)\tilde{\beta}_1(k_2) \\ &\quad + \tilde{S}(k_2)\tilde{\beta}_1(k_1) + \tilde{\beta}_2(k_1,k_2). \end{aligned} \quad (3.10)$$

These equations give us a recursive definition of the $\tilde{\beta}_i$'s.[8]

Our first aim is to rearrange the series using (3.10) and to carry out the sum over all numbers of infrared photons. To do this, we use the identity

$$\delta(f-p_3-\sum k_i) = \int \frac{d^4x}{(2\pi)^4} e^{if\cdot x} e^{-ip_3\cdot x} \prod_i e^{-ik_i\cdot x}. \quad (3.11)$$

It will be convenient to treat the $\tilde{\beta}_0$ terms slightly differently from the rest. Before using (3.11), we use the identity (due to the δ function)

$$1 = (\hat{\omega}_1 + \cdots \hat{\omega}_n)/(M-\hat{E}_3). \quad (3.12)$$

Then we find

$$\begin{aligned} \frac{d\sigma}{dp_4} &= \sum_n \frac{d\sigma_n}{dp_4} \\ &= \frac{\pi}{p_1}\int_{m^2}^{M_{\max}^2} dM^2 \int \frac{d^4x}{(2\pi)^4} e^{if\cdot x} \int \frac{d^3p_3}{E_3} e^{-ip_3\cdot x} \\ &\quad \times \exp(2\alpha\hat{B}) \exp\left[\int \frac{d^3k}{\omega}\tilde{S}(k)e^{-ik\cdot x}\right] \\ &\quad \times \left\{ \int d^3k_1\, e^{-ik_1\cdot x}\left[\frac{\tilde{S}(k_1)\tilde{\beta}_0}{M-\hat{E}_3} + \frac{\tilde{\beta}_1(k_1)}{k_1}\right] \right. \\ &\quad \left. + \sum_{n=2} \frac{1}{n!} \int \prod_i \left[\frac{d^3k_i}{k_i} e^{-ik_i\cdot x}\right] \tilde{\beta}_n(k_1\cdots k_n) \right\}. \end{aligned} \quad (3.13)$$

Unlike the potential-scattering case, the integrations over x cannot be carried out explicitly. To proceed, we write

$$\begin{aligned} \int \frac{d^3k}{\omega}\tilde{S}(k)e^{-ik\cdot x} &= \tilde{B}(\epsilon) + \int \frac{d^3k}{k} \\ &\quad \times \tilde{S}(k)[e^{-ik\cdot x} - \theta(\epsilon - \hat{k})] \\ &= \tilde{B}(\epsilon) + D(x,\epsilon). \end{aligned} \quad (3.14)$$

The constant ϵ may be chosen in any convenient manner. The choice suggested by potential theory is that ϵ be the energy carried by a single photon when no others are present, namely,

$$\epsilon = (M^2-m^2)/2M. \quad (3.15)$$

Then we find

$$2\alpha[\hat{B}+\tilde{B}(\epsilon)] = \tfrac{1}{4}\alpha A_s \ln\frac{(M^2-m^2)^4}{16M^4\hat{E}_1\hat{E}_2\hat{E}_3\hat{E}_4} - \frac{\alpha}{2\pi}\left(\ln\frac{\hat{E}_1\hat{E}_4}{\hat{E}_2\hat{E}_3}\right)^2, \quad (3.16a)$$

where

$$\alpha A_s = \frac{2\alpha}{\pi}\left(\ln\frac{4E_1E_4p_1\cdot p_3p_4\cdot p_3}{m^3E_3p_1\cdot p_4} - 2\right). \quad (3.16b)$$

Because these functions are complicated, we introduce reference functions

$$2\alpha[\hat{B}_0+\tilde{B}_0(\epsilon)] = \tfrac{1}{4}\alpha A_0 \ln\frac{(M^2-m^2)^4}{16m^5E_1E_3E_4} - \frac{\alpha}{2\pi}\left(\ln\frac{E_1E_4}{mE_3}\right)^2, \quad (3.17a)$$

where

$$\alpha A_0 = \frac{4\alpha}{\pi}\left(\ln\frac{2E_1E_4}{mE_3} - 1\right). \quad (3.17b)$$

We notice that δ_{ir}, (2.13b), is the value of (3.17a) at the upper limit of M^2.

Finally, we define

$$\alpha H = 2\alpha[\hat{B}+\tilde{B}(\epsilon)] - 2\alpha[\hat{B}_0+\tilde{B}_0(\epsilon)]; \quad (3.18)$$

then we can rewrite (3.13) as follows:

$$\begin{aligned} \frac{d\sigma}{dp_4} &= e^{\delta_{\mathrm{ir}}}\frac{\pi}{p_1}\int_{m^2}^{M_{\max}^2} dM^2 \left(\frac{M^2-m^2}{M_{\max}^2-m^2}\right)^{\alpha A_0} \\ &\quad \times \int \frac{d^4x}{(2\pi)^4} e^{if\cdot x} \int \frac{d^3p_3}{E_3} e^{-ip_3\cdot x} \\ &\quad \times e^{\alpha H} e^D \left\{ \int d^3k_1\, e^{-ik_1\cdot x}\left[\frac{\tilde{S}(k_1)\tilde{\beta}_0}{M-\hat{E}_3} + \frac{\tilde{\beta}_1(k_1)}{k_1}\right] \right. \\ &\quad \left. + \sum_{n=2} \frac{1}{n!} \int \prod_i \left(\frac{d^3k_i}{k_i} e^{-ik_i\cdot x}\right) \tilde{\beta}_n(k_1\cdots k_n) \right\} \end{aligned} \quad (3.19a)$$

$$= e^{\delta_{\mathrm{ir}}}(1+\delta_{\mathrm{nir}}^{(1)}+\delta_{\mathrm{nir}}^{(2)}+\cdots)\frac{d\hat{\sigma}_0}{dp_4}, \quad (3.19b)$$

[8] Note that in these defining equations we relax the condition that p_3 is related to the k_i's through energy-momentum conservation. Any ambiguity in the resulting definition of the $\tilde{\beta}_i$'s will of course not appear in the final answer. In particular, $\tilde{\beta}_0$ is specified to give elastic scattering at the experimental energy E_4.

where (3.19b) indicates the expansion of the integral in (3.19a) in powers of α.

If nothing has gone awry, (3.19b) should agree to first order in α with the results of Sec. II. Let us verify this quickly. We note that the integrand of (3.19a) contains at least one explicit power of α. However, any term which has an infrared singularity $[\sim(M^2-m^2)^{-1}]$ will have one power of α absorbed in the M^2 integration. To first order in α, we may therefore neglect H, D and the terms with $n>2$ in (3.19a). The x integration may immediately be carried out to give a δ function. Carrying out the momentum integrals in the I frame, we find a term

$$2\alpha A_0\tilde{\beta}_0/(M^2-m^2) \tag{3.20}$$

plus nonsingular terms. To order α,

$$(2\pi/p_1)\tilde{\beta}_0=(d\hat{\sigma}_0/dp_4)(1+\delta_V{}^{(1)}).$$

To lowest order in α, the nonsingular terms are just the ones evaluated in Sec. II D. Carrying out the M^2 integral, we recover (3.19b) to order α.

C. Identification and Estimates of Contributions to $\delta_{\text{nir}}{}^{(2)}$

It is probably feasible to evaluate the leading logarithmic contributions to $\delta_{\text{nir}}{}^{(2)}$. We shall not attempt this here except in the simplest cases. However, we hope to obtain reasonable estimates of the errors made in omitting $\delta_{\text{nir}}{}^{(2)}$ from an analysis of the data. The separate contributions will now be itemized and discussed.

1. Higher-Order Corrections to $\tilde{\beta}_0$

So far we have included the terms of order unity and of order α (relative to the uncorrected cross section). The term of order $\alpha(\delta_V{}^{(1)})$ arises from the noninfrared part of the one-virtual-photon contribution to the radiative corrections, and it is given explicitly in (2.13). Its dominant term is

$$(\alpha/\pi)[(11/6)\ln 4E_1E_4/m^2]. \tag{3.21}$$

The sources of such large logarithms are rather readily identified, and the coefficient 11/6 is found to break down into the following pieces: $\frac{1}{3}$ from vacuum polarization, $\frac{1}{2}$ from the large virtual-momentum region, and 1 from cross terms between the spin and convection parts of the current in the intermediate-momentum region. The last of these is more sensitive to the momenta of the external lines than to the short-range details of the interaction. We might expect it to have a counterpart in higher order.

The terms of order α^2 in $\tilde{\beta}_0$ of course equal the square of the order α term in the amplitude plus twice the order α^2 term in the amplitude. The first of these is $\frac{1}{4}(\delta_V{}^{(1)})^2$ and the second requires the evaluation of all diagrams in which there are two virtual photons (beyond lowest order). To evaluate the leading logarithms in the amplitude to order α^2 would require a massive effort. First one would have to eliminate the infrared parts of the amplitude; this would involve subtraction of both first- and second-order infrared contributions. Diagrams involving two vacuum polarization bubbles or one vacuum polarization bubble together with one vertex or two separate vertices could of course be trivially evaluated as they would have multiplicative four-dimenaional momentum integrals. These trivially calculated terms are either the square of lowest-order terms or $\frac{1}{2}$ the cross terms contained in the square of the lowest-order amplitude. We can only guess at the nature of the result from the diagrams having eight-dimensional momentum integrals. Intuition tells us that they are probably quadratic in the logarithm of (3.21). It seems reasonable that the correction to the elastic amplitude has the form

$$1+\tfrac{1}{2}\delta_V{}^{(1)}+\eta(\tfrac{1}{2}\delta_V{}^{(1)})^2+\cdots$$

with η of order 1 (actually η can vary for the different terms in the square of $\frac{1}{2}\delta_V{}^{(1)}$; we have seen that it is 1 or $\frac{1}{2}$ for the trivially calculable terms). The elastic cross section then has the form

$$\tilde{\beta}_0\propto 1+\delta_V{}^{(1)}+\tfrac{1}{4}(1+2\eta)(\delta_V{}^{(1)})^2+\cdots. \tag{3.22a}$$

If we want to include the known α^2 contributions, the best choice of η is $\frac{1}{2}$ (this mistreats the square of the vacuum polarization term, but that is very small anyway). If we assume that the uncalculated terms introduce an uncertainty of one in η (this seems fairly generous), then we find

$$\tilde{\beta}_0\propto 1+\delta_V{}^{(1)}+(\tfrac{1}{2}\pm\tfrac{1}{2})(\delta_V{}^{(1)})^2+\cdots. \tag{3.22b}$$

How big is this α^2 correction and its uncertainty in a typical case? If we take $E_1=500$ MeV and let E_3/E_1 vary from 0.01 to 0.1, we find that $\delta_V{}^{(1)}$ varies from approximately 6% to approximately $7\frac{1}{2}$%. The α^2 term and its uncertainty vary from 0.2 to 0.3%.

2. Virtual Photon Corrections to $\delta_R{}^{(1)}$

Since $\delta_R{}^{(1)}$ is almost negligible for typical situations, we expect the corrections to it to be completely unimportant. These corrections arise from the order α term in $\tilde{\beta}_0$ in the part of the integral which is left after (3.20) has been removed plus the order α^2 term in $\tilde{\beta}_1$. The first contribution is of relative order $\delta_V{}^{(1)}$ and the second one is also estimated to be of this magnitude. The complete result, of order $\delta_V{}^{(1)}\,\delta_R{}^{(1)}$, is completely negligible.

3. Contributions from $\tilde{\beta}_2$

Even if these are several times their expected order of $(\delta_R{}^{(1)})^2$, they are still completely unimportant. However, we recall that there was a fortuitous cancellation in $\delta_R{}^{(1)}$. Perhaps this cancellation will fail in higher order and terms like

$$\left(\frac{\alpha}{2\pi}\right)^2\left(\ln\frac{2E_3}{m}\right)^4$$

may occur. Even these terms seem entirely negligible for foreseeable experiments.

4. Corrections from Variation of Factor $[(M^2-m^2)/(M_{\max}^2-m^2)]^{\alpha A_0}$

This factor

$$[(M^2-m^2)/(M_{\max}^2-m^2)]^{\alpha A_0} \tag{3.23}$$

was essential for the convergence of the infrared contribution (3.20) at the lower limit. However, it also modifies the noninfrared contributions in a way which has not yet been taken into account. We must look at the M^2 dependence of the contribution from $\tilde{\beta}_0$ and $\tilde{\beta}_1$ after the infrared term (3.20) has been removed. We recall from Sec. II D that although the net contribution $\delta_R^{(1)}$ from these terms was small, there was a somewhat fortuitous cancellation of larger contributions from the upper and lower regions of integration.

For typical experimental parameters, the exponent αA_0 is approximately 0.1. Consequently, the factor (3.23) deviates significantly from unity only near the lower limit. We shall now consider the contribution obtained by inserting this factor into (2.12). In any given situation, the resulting integral could be evaluated numerically; however, for our present purposes, we shall only evaluate the leading contribution. Selecting only those terms in (2.23) which contribute doubly logarithmic terms in (2.21), we find

$$\int_{m^2}^{M^2} dM^2 I\alpha A_0 \ln\left(\frac{M^2-m^2}{M_{\max}^2-m^2}\right) \times\frac{\alpha}{\pi}\alpha A_0\int_1^{x_m} dx\left[\frac{1}{x_m+2-x}\ln(x_m+2-x) - \frac{1}{x-1}\ln x\right]\ln\left(\frac{x-1}{x_m-1}\right), \tag{3.24}$$

where $x=M^2/m^2$ and $x_m=M_{\max}^2/m^2$. The two terms in the brackets give large and compensating corrections to (2.21); but because of the extra factor in (3.24), only the second term is important here. The contribution of (3.24) is then estimated to be

$$\frac{\alpha}{6\pi}\alpha A_0\left(\ln\frac{2E_3}{m}\right)^3. \tag{3.25}$$

Typical values of this expression are (using $\alpha A_0=0.1$)

for $E_3=5$ MeV: 0.1%,

for $E_3=50$ MeV: 0.6%.

Except for high-precision experiments, this term seems safely negligible. If it should become important, the complete expression should be evaluated numerically since the terms with one less logarithm might not be completely negligible under those circumstances.

5. Contributions from Expanding e^D

Both the first and second powers of D lead to contributions of order α^2. The first power eliminates the infrared singularity in the M^2 integration and the entire range of M^2 must be studied. On the other hand, the second power of D leaves an infrared singularity and one power of α is absorbed in the M^2 integration. The contribution from $\frac{1}{2}D^2$ is easy to evaluate since we need to study only the region $M^2\sim m^2$. But in this region the analysis is equivalent to potential scattering and we have the result given in (3.3)

$$\tfrac{1}{2}D^2 \text{ contribution: } -\tfrac{1}{12}\pi^2(\alpha A_0)^2. \tag{3.26}$$

This term is of order 1% in typical circumstances.

The contribution from the term linear in D is rather more difficult to evaluate. The choice of ϵ given in (3.15) assures that the M^2 integration is not infrared divergent. The nature of D is perhaps indicated by carrying out the x integration in (3.13). Calling k_2 the momentum variable in D, we find

$$\delta(f-p_3-k_1-k_2)-\theta(\epsilon-\hat{k}_2)\delta(f-p_3-k_1).$$

Thus D represents a correction to the kinematics of the emitted photons. It is not too hard to show that the leading terms are of order

$$\frac{\alpha}{\pi}\alpha A_0\left(\ln\frac{2E_3}{m}\right)^2. \tag{3.27}$$

These are probably smaller than (3.25) (depending on the coefficient), and they can therefore be neglected whenever (3.25) can. If they become important, it will be necessary to carry out a sixfold integration of a fairly lengthy and complicated expression. It might be feasible to extract the leading logarithmic terms, but it would probably be better to develop numerical methods for handling such problems.

6. Contributions from Expanding $e^{\alpha H}$

Since $H\to 0$ as $M^2\to m^2$, it is necessary only to consider the first power of H. To an adequate approximation, we may use

$$M^4\hat{E}_1\hat{E}_2\hat{E}_3\hat{E}_4=m^3E_1E_3E_4\tfrac{1}{2}(M^2+m^2),$$

and

$$\hat{E}_1\hat{E}_4/\hat{E}_2\hat{E}_3=(E_1E_4/mE_3)2m^2/(m^2+M^2),$$

which gives

$$\alpha H=\frac{\alpha}{2\pi}\ln\left(\frac{p_1\cdot p_3p_4\cdot p_3E_3^2}{p_2\cdot p_3p_1\cdot p_4E_1E_4}\right)\ln\frac{(M^2-m^2)^4}{16m^5E_1E_3E_4} -\frac{\alpha}{2\pi}\left[\ln\left(\frac{4p_1\cdot p_3p_4\cdot p_3E_3^2}{p_2\cdot p_3p_1\cdot p_4E_1E_4}\right)-2\right]\ln\frac{M^2+m^2}{2m^2} -\frac{\alpha}{2\pi}\ln^2\left(\frac{M^2+m^2}{2m^2}\right). \tag{3.28}$$

Now we must consider the k_1 integration in (3.19). When M^2 is not large (say $M^2<\frac{1}{2}M_{\max}{}^2$), an important contribution to the $\hat{k}_1$ integration arises from $\hat{\mathbf{k}}_1$ nearly parallel to $\hat{\mathbf{p}}_1$ or $\hat{\mathbf{p}}_4$. In this region

$$p_1\cdot p_3\cong mE_4,\quad p_4\cdot p_3\cong mE_1,$$
$$p_2\cdot p_3\cong mE_3,\quad p_1\cdot p_4\cong mE_3-\tfrac{1}{2}(M^2-m^2),$$

so that

$$\alpha H\cong -\frac{\alpha}{2\pi}\ln\left(1-\frac{M^2-m^2}{2mE_3}\right)\ln\frac{(M^2-m^2)^4}{16m^5E_1E_3E_4} -\frac{\alpha}{2\pi}\left[\ln4-2-\ln\left(1-\frac{M^2-m^2}{2mE_3}\right)\right]\ln\left(\frac{M^2+m^2}{2m^2}\right) -\frac{\alpha}{2\pi}\ln^2\left(\frac{M^2+m^2}{2m^2}\right).$$

The correction is obtained by multiplying this by $\alpha A_0/(M^2-m^2)$ (from the k_1 integration) and then integrating with respect to M^2. The leading term in the result is

$$-\frac{\alpha}{6\pi}\alpha A_0\left(\ln\frac{2E_3}{m}\right)^3. \tag{3.29}$$

This just cancels (probably fortuitously) the contribution (3.25).

A contribution of similar importance probably arises from the region of large M^2, where $\tilde{\beta}_1$ is important. In this region, $\hat{\mathbf{p}}_3$ tends to be parallel to $\hat{\mathbf{p}}_1$ so that αH varies rapidly with the angle of $\hat{\mathbf{k}}_1$. We shall not attempt to evaluate this contribution here, but we may guess that it contains terms of the form

$$\left(\frac{\alpha}{2\pi}\right)^2\left(\ln\frac{E_1E_4}{E_3{}^2}\right)^2\left(\ln\frac{2E_3}{m}\right)^2. \tag{3.30}$$

If the coefficient is not large, these terms are not likely to be important.

IV. SUMMARY

Our best expression for the radiative corrections is

$$\frac{d\sigma}{dp_4}=e^{\delta_{\rm ir}}\left\{1+\frac{\alpha}{\pi}\left[\frac{11}{6}\ln\frac{4E_1E_4}{m^2}+\tfrac{1}{2}\ln\frac{E_1E_4}{E_3{}^2} -\frac{1}{2}\left(\ln\frac{E_1}{E_3}\right)^2-\tfrac{1}{2}\ln^2 2+\tfrac{1}{2}\ln\frac{2E_3}{m}+\pi^2-\frac{119}{18}\right] -\tfrac{4}{3}\alpha^2\left(\ln\frac{2E_1E_4}{mE_3}-1\right)^2\right\}\frac{d\sigma_0}{dp_4}, \tag{4.1}$$

where $\delta_{\rm ir}$ is given by (2.13b). The approximations to order α should be entirely adequate so long as $E_3<0.1E_1$. The approximations to order α^2 should also be adequate at the 1% level of accuracy. For more accurate experiments, the various contributions in Sec. III C should be reconsidered. A rough gauge of the importance of these terms is provided by (3.25). Because of possible cancellations, this may even yield an overestimate of the neglected terms.

When our present result is compared with the radiative correction employed by BGY[1] (which was in turn based on an extrapolation from the results of MY[2]), it is found that there are significant changes at the 1–2% level. While these changes might affect the interpretation of the BGY experiment slightly, they would not alter the general conclusion that there is excellent agreement between theory and experiment at the 1% level.

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APPENDIX A: TABLE OF ANGULAR INTEGRALS

The following integrals are required for the complete calculation of the lowest-order bremsstrahlung cross section:

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot p_j}{[k\cdot p_i k\cdot p_j]}=4\pi\rho\left(\frac{p_i\cdot p_j}{m^2}\right), \tag{A1}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i{}^2}{k\cdot p_i{}^2}=4\pi, \tag{A2}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot f}{k\cdot p_i}=4\pi p_3\cdot k\rho\left(\frac{p_i\cdot f}{Mm}\right), \tag{A3}$$

$$\hat{k}^2\int d\hat{\Omega}_3 p_i\cdot k=4\pi p_3\cdot k^3\frac{p_i\cdot f}{M^4}, \tag{A4}$$

$$\hat{k}^2\int d\hat{\Omega}_3 p_i\cdot k^2=4\pi p_3\cdot k^4\frac{(5p_i\cdot f^2-2m^2M^2)}{3M^6},\tag{A5}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot k}{p_j\cdot k}=4\pi p_3\cdot k^2\left(\frac{p_i\cdot f}{M^2p_j\cdot f}-(p_i\cdot p_jp_j\cdot f-m^2p_i\cdot f)\frac{\sigma(p_j\cdot f/mM)}{mM(m^2M^2-p_j\cdot f^2)}\right),\tag{A6}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot k}{p_j\cdot k^2}=4\pi p_3\cdot k\left(\frac{p_i\cdot p_j}{m^2p_j\cdot f}-(p_i\cdot fp_j\cdot f-M^2p_i\cdot p_j)\frac{\sigma(p_j\cdot f/mM)}{mM(m^2M^2-p_j\cdot f^2)}\right),\tag{A7}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot k^2}{p_j\cdot k}=4\pi p_3\cdot k^3\left(\frac{3p_i\cdot f^2-m^2M^2+(p_i\cdot fp_j\cdot f-M^2p_i\cdot p_j)^2/(m^2M^2-p_j\cdot f^2)}{2M^4p_j\cdot f}\right.$$
$$+\{(p_i\cdot p_jp_j\cdot f-m^2p_i\cdot f)^2+m^2[(p_i\cdot fp_j\cdot f-M^2p_i\cdot p_j)^2$$
$$\left.-(m^2M^2-p_i\cdot f^2)(m^2M^2-p_j\cdot f^2)]/2M^2\}\right)\frac{\sigma(p_j\cdot f/mM)}{mM(m^2M^2-p_j\cdot f^2)^2},\tag{A8}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_i\cdot k^2}{p_j\cdot k^2}=4\pi p_3\cdot k^2\left\{\frac{-p_i\cdot f(p_i\cdot fp_j\cdot f-M^2p_i\cdot p_j)}{M^2p_j\cdot f}-\frac{p_i\cdot p_j(p_i\cdot p_jp_j\cdot f-m^2p_i\cdot f)}{m^2p_j\cdot f}+\sigma(p_j\cdot f/mM)\right.$$
$$\left.\times\frac{[3(m^2p_i\cdot f-p_i\cdot p_jp_j\cdot f)(M^2p_i\cdot p_j-p_i\cdot fp_j\cdot f)+(m^2p_j\cdot f-p_i\cdot p_jp_i\cdot f)(m^2M^2-p_j\cdot f^2)]}{Mm(m^2M^2-p_j\cdot f^2)}\right\}\Big/$$
$$(m^2M^2-p_j\cdot f^2),\tag{A9}$$

$$\hat{k}^2\int\frac{d\hat{\Omega}_3}{[p_j\cdot k(p_i-p_3)^2]}=\frac{4\pi p_3\cdot k\hat{\rho}(p_i\cdot p_l/m^2)}{m^2(p_j-p_l)^2},\tag{A10}$$

$$\hat{k}^2\int\frac{d\hat{\Omega}_3}{[p_j\cdot k(p_i-p_3)^4]}=\frac{4\pi p_3\cdot k[B(p_i,p_j,p_l)\sigma(p_i\cdot p_l/m^2)+C(p_j-p_l)/p_i\cdot p_l]}{m^2(p_i-p_l)^4},\tag{A11}$$

$$\hat{k}^2\int\frac{d\hat{\Omega}_3}{[p_j\cdot k^2(p_i-p_3)^2]}=\frac{4\pi[p_j\cdot f/p_i\cdot p_l+A(p_i,p_j,p_l)\sigma(p_i\cdot p_l/m^2)]}{m^2(p_j-p_l)^2},\tag{A12}$$

$$\hat{k}^2\int\frac{d\hat{\Omega}_3}{[p_j\cdot k^2(p_i-p_3)^4]}=4\pi\left\{\frac{8p_j\cdot fC(p_j-p_l)/p_i\cdot p_l-4[C(p_j-p_l)^2+p_j\cdot f^2]/m^2}{(p_i+p_l)^2(p_i-p_l)^2}\right.$$
$$+\left[\frac{\sigma(p_i\cdot p_l/m^2)}{m^2}\right]\left[3A(p_i,p_j,p_l)B(p_i,p_j,p_l)\right.$$
$$\left.\left.-\frac{2p_3\cdot k\{1+2p_3\cdot k[1-2p_j\cdot f/(p_j-p_l)^2]/(p_i-p_l)^2\}}{(p_i+p_l)^2}\right]\Big/(p_j-p_l)^4\right\},\tag{A13}$$

$$\hat{k}^3\int\frac{d\hat{\Omega}_3}{(p_i-p_3)^2}=\frac{4\pi p_3\cdot k^2\hat{\rho}[C(p_j-p_l)/mM]}{mM(p_j-p_l)^2},\tag{A14}$$

$$\hat{k}^2\int\frac{d\hat{\Omega}_3}{(p_i-p_3)^4}=\frac{4\pi p_3\cdot k^2}{m^2(p_j-p_l)^4},\tag{A15}$$

$$\hat{k}^2\int d\hat{\Omega}_3\frac{p_j\cdot k}{(p_i-p_3)^2}=4\pi p_3\cdot k^2\left\{\frac{p_j\cdot fp_3\cdot k\hat{\rho}[C(p_j-p_l)/mM]}{(p_j-p_l)^2}\right.$$
$$\left.+\frac{(p_i\cdot fp_j\cdot f-M^2p_i\cdot p_j)C(p_j-p_l)\sigma[C(p_j-p_l)/mM]}{2(m^2M^2-p_i\cdot f^2)}\right\}\Big/mM^3,\tag{A16}$$

$$\hat{k}^2 \int \frac{d\hat{\Omega}_3}{[(p_3-p_i)^2(p_3+p_l)^2]} = 4\pi p_3\cdot k^2 \hat{\rho} \left\{1 - \frac{4p_3\cdot k^2(m^2+p_i\cdot p_l)}{m^2(p_i+p_j)^2(p_j-p_l)^2}\right\} \Big/ [m^2(p_i+p_j)^2(p_j-p_l)^2], \tag{A17}$$

where

$$A(p_i,p_j,p_l) = 2p_3\cdot k\left\{1 + \frac{4p_i\cdot p_l p_3\cdot k}{(p_j-p_l)^2(p_i-p_l)^2}\right\} \Big/ (p_i+p_l)^2, \tag{A18}$$

$$B(p_i,p_j,p_l) = 1 + 2p_3\cdot k\left\{1 - \frac{4m^2 p_3\cdot k}{(p_j-p_l)^2(p_i-p_l)^2}\right\} \Big/ (p_i+p_l)^2, \tag{A19}$$

$$C(p_j-p_l) = p_3\cdot k + m^2 - \frac{2p_3\cdot k^2}{(p_j-p_l)^2}, \tag{A20}$$

$$\hat{\rho}(x) = \rho(x)/x = \ln[x+(x^2-1)^{1/2}]/(x^2-1)^{1/2}, \tag{A21}$$

$$\sigma(x) = [\rho(x)-1]/x, \tag{A22}$$

and

$$f = p_1+p_2-p_4 = p_i+p_j-p_l, \tag{A23}$$

where p_i, p_j, p_l are some permutation of p_1, p_2, p_4 satisfying Eq. (A23).

APPENDIX B: SCALAR INTEGRALS

The following integrals are required for the calculation of the noninfrared part of the lowest-order bremsstrahlung cross section to first order in β. The results given are not exact, as approximations to order β have been made for all logarithms and Spence functions encountered. It is also assumed that $m \ll \beta$.

$$\int_{m^2}^{M_{\max}^2} p_3\cdot k \frac{dM^2}{M^2} \sim p_2\cdot f, \tag{B1}$$

$$\int_{m^2}^{M_{\max}^2} p_3\cdot k \frac{dM^2}{M^4} \sim \tfrac{1}{2}[\ln(2\kappa)-1], \tag{B2}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{\ln(M^2/m^2)}{(M^2-m^2)} \sim \tfrac{1}{2}\ln^2(2\kappa) + \tfrac{1}{6}\pi^2, \tag{B3}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{\rho(p_1\cdot p_4/m^2)}{p_1\cdot p_4} \sim \ln^2(2\kappa), \tag{B4}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{[\rho(p_1\cdot p_4/m^2)/p_1\cdot p_4 - \rho(p_2\cdot f/m^2)/p_2\cdot f]}{(M^2-m^2)} \sim \frac{[\tfrac{1}{2}\ln^2(2\kappa) - \tfrac{1}{6}\pi^2]}{p_2\cdot f}, \tag{B5}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{\rho(p_2\cdot f/Mm)}{p_2\cdot f} \sim \ln(2\kappa)+1, \tag{B6}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{[\rho(p_2\cdot f/Mm) - \rho(p_2\cdot f/m^2)]}{(M^2-m^2)} \sim -\tfrac{1}{4}[\ln^2(2\kappa)+\pi^2/3], \tag{B7}$$

$$\int_{m^2}^{M_{\max}^2} dM^2 \frac{[\rho(m^2+p_3\cdot k)/Mm](p_3\cdot k/M^2-1)}{(m^2+p_3\cdot k)} \sim -\tfrac{1}{4}\ln^2(2\kappa) - \tfrac{1}{6}\pi^2, \tag{B8}$$

where

$$\kappa = p_2\cdot f/m^2. \tag{B9}$$