Electromagnetic Decays of Pseudoscalar Mesons*

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The electromagnetic 2γ decays of the neutral pseudoscalar mesons are discussed in the context of broken $SU(3)\times SU(3)$ and anomalous partial conservation of axial-vector current (PCAC). Particular attention is paid to the mixed ηX channel. It is predicted that the X width is enhanced beyond 80 keV. The observed value for the η width is used to compute $(F_K/F_\pi)^2\approx 1.3$.

I. INTRODUCTION

THE breaking of chiral $SU(3)\times SU(3)$ in the absence of electrodynamics was recently studied[1–3] under the hypothesis that the symmetry-breaking Lagrangian transforms like a member of the $(3,\bar{3})+(\bar{3},3)$ representation of $SU(3)\times SU(3)$. When electrodynamics is present, it has been shown[4–6] that neutral axial-vector currents develop anomalous properties. Our intent is to combine these two results in order to calculate the relative electrodynamic (two-photon) decay rates of the three neutral pseudoscalar mesons η, $X(960)$, and π^0. Just as in the prior discussion of π^0 decay, our analysis uses the low-energy theorems following from the presence of anomalous electromagnetic terms in the divergence equations.[5]

In Sec. II the divergence equations that obtain in the absence of electrodynamics for the three neutral axial currents are reviewed. Here, the broken symmetry group $SU(3)\times SU(3)\times U(1)$ is exploited.

The electromagnetic anomalies are included in Sec. III and the three two-photon decay widths Γ_π, Γ_η, and Γ_X are computed. An inequality is immediately found which is compatible with the observed values of Γ_π and Γ_η but requires the yet unmeasured Γ_X to be of order $(\mu_X/\mu_\pi)^5\Gamma_\pi\approx 130$ keV or larger. With the hypothesis that the electromagnetic current is constructed as in the normal quark model, the inequality becomes a sum rule

$$\mu_\eta^{-5}\Gamma_\eta+\mu_X^{-5}\Gamma_X\approx(25/9)\mu_\pi^{-5}\Gamma_\pi. \qquad (1)$$

With the insertion of experimental values for Γ_η and Γ_π, this gives $\Gamma_X\approx 350$ keV. An additional hypothesis of the Gell-Mann–Okubo variety is discussed in the Appendix. It enables us to deduce the ratio of the kaon and pion leptonic decay amplitudes F_K/F_π in terms of Γ_η and Γ_π. We obtain $(F_K/F_\pi)^2\approx 1.3$, independent of the charge structure of the basic fermions and in agreement with other analyses.

In the final Sec. IV we discuss our results and our use of the controversial $SU(3)\times SU(3)\times U(1)$ group.

II. THREE NEUTRAL AXIAL-VECTOR CURRENTS AND THEIR DIVERGENCE EQUATIONS

The starting point for a calculation of the decay rate $\pi^0\to 2\gamma$ is the effective divergence equation

$$\partial^\mu A_\mu^{(3)}=F_\pi\mu_\pi^2\Phi_\pi+(e^2/16\pi^2)a^{(3)}F^{\mu\nu}\tilde{F}_{\mu\nu}, \qquad (2)$$

where F_π is the pion leptonic decay amplitude, Φ_π is a smooth and renormalized pion field, $F_{\mu\nu}$ ($\tilde{F}_{\mu\nu}$) is the electromagnetic (dual) tensor, and $a^{(3)}$ is a dimensionless model-dependent parameter related to the electric charges of the fundamental fermions. As we are concerned with the decays of η and X, we must consider the analogous divergence equations for two $I=0$ neutral axial-vector currents: the $SU(3)$ octet $A_\mu^{(8)}$ and the $SU(3)$ singlet $A_\mu^{(0)}$. It is more convenient, however, to write

$$A_\mu^{(8)}=(\sqrt{\tfrac{1}{3}})A_\mu^{(1)}-(\sqrt{\tfrac{2}{3}})A_\mu^{(2)},$$
$$A_\mu^{(0)}=(\sqrt{\tfrac{2}{3}})A_\mu^{(1)}+(\sqrt{\tfrac{1}{3}})A_\mu^{(2)}. \qquad (3)$$

(In a quark model, $A_\mu^{(1)}$ would be the axial-vector doublet-quark number and $A_\mu^{(2)}$ the axial-vector singlet-quark number.) We must construct the effective divergence equations

$$\partial^\mu A_\mu^{(1)}=G_\eta\mu_\eta^2\Phi_\eta+G_X\mu_X^2\Phi_X+(e^2/16\pi^2)a^{(1)}F^{\mu\nu}\tilde{F}_{\mu\nu}, \qquad (4)$$

$$\partial^\mu A_\mu^{(2)}=H_\eta\mu_\eta^2\Phi_\eta+H_X\mu_X^2\Phi_X+(e^2/16\pi^2)a^{(2)}F^{\mu\nu}\tilde{F}_{\mu\nu}, \qquad (5)$$

where the leptonic decay amplitudes G_η, H_η, G_X, and H_X are not directly experimentally measurable, but are at least partially determined by symmetry arguments. The dimensionless parameters $a^{(1)}$ and $a^{(2)}$ are again related to fermion charges.

We begin by assuming that the Lagrangian without electrodynamics consists of two parts,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{SB}, \qquad (6)$$

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where $\mathcal{L}_0$ is invariant under $SU(3)\times SU(3)\times U(1)$ and $\mathcal{L}_{SB}$ breaks this symmetry, leaving only isospin and hypercharge exactly conserved.[7] $\mathcal{L}_{SB}$ is assumed to transform like a member of the $(3,\bar{3})+(\bar{3},3)$ representation of $SU(3)\times SU(3)$, and under the additional chiral $U(1)$ factor, corresponding states of opposite parity are mixed. No term is present which breaks only $U(1)$ but leaves $SU(3)\times SU(3)$ intact.[8] Thus, we may write, in general,

$$\mathcal{L}_{SB}=\sum \epsilon_i\varphi_i. \tag{7}$$

Here the φ_i are a basis of 18 local fields with the required transformation properties. There are only two nonvanishing ϵ_i corresponding to the $SU(3)$ singlet and $I=0$ octet scalar fields, but again it is convenient [as in Eq. (3)] to use the linear combinations ϵ_1 and ϵ_2.

These are just the symmetry properties of the quark model with an $SU(3)$-singlet neutral vector meson, the symmetry being broken only by quark masses: ϵ_1/ϵ_2 is just the ratio of bare masses of the doublet and singlet quarks, $\epsilon_1\to 0$ is the limit of exact chiral $SU(2)$, and $\epsilon_1\to\epsilon_2$ is the limit of exact $SU(3)$.

The 17 currents which would be conserved if ϵ were zero we call $J_\mu^{(a)}$, and their charges [whose commutation relations are those of the generators of $SU(3)\times SU(3)\times U(1)$] we call $Q^{(a)}$. The transformation behavior of the φ is given by

$$[Q^{(a)},\varphi_i]=iT_{ij}^{(a)}\varphi_j, \tag{8}$$

and the divergence equations are simply

$$\partial^\mu J_\mu^{(a)}=\epsilon_i T_{ij}^{(a)}\varphi_j. \tag{9}$$

We define the one-point and two-point functions

$$\langle\varphi_i\rangle_0=\lambda_i, \tag{10}$$

$$\frac{1}{i}\int d^4x\, e^{ipx}\langle T(\varphi_i(X),\varphi_j(0))\rangle_0=\Delta_{ij}(p^2), \tag{11}$$

in terms of which we immediately deduce the relation

$$-\Delta^{-1}{}_{ij}(0)T_{jk}^{(a)}\lambda_k=T_{ik}^{(a)}\epsilon_k. \tag{12}$$

In order to obtain useful physical results, we make the following smoothness assumption: At sufficiently small p^2 (say, $p^2\leq 1$ GeV^2), we assume that $\Delta^{-1}(p^2)$ is well approximated by a linear function of p^2,

$$\Delta^{-1}(p^2)=Z^{-\frac{1}{2}}{}_T(p^2-\mu^2)Z^{-\frac{1}{2}}, \tag{13}$$

where $Z^{-\frac{1}{2}}$ is an arbitrary matrix, $Z^{-\frac{1}{2}}{}_T$ is its transpose, and μ^2 is the diagonolized mass (squared) matrix. We define the renormalized fields Φ by

$$\Phi=Z^{-\frac{1}{2}}\varphi, \tag{14}$$

in terms of which the divergence equations become

$$\partial^\mu J_\mu^{(a)}=\epsilon T^{(a)}Z^{\frac{1}{2}}\Phi. \tag{15}$$

The generalized leptonic decay amplitudes $\mathfrak{F}^{(a)}$ are then defined by

$$\mathfrak{F}^{(a)}{}_T\mu^2Z^{-\frac{1}{2}}=\epsilon T^{(a)}, \tag{16}$$

so that

$$\partial^\mu J_\mu^{(a)}=\mathfrak{F}^{(a)}{}_T\mu^2\Phi. \tag{17}$$

From Eqs. (12), (13), and (16), we obtain

$$Z^{\frac{1}{2}}\mathfrak{F}^{(a)}=T^{(a)}\lambda. \tag{18}$$

By multiplying these two equations, a useful relation results from which Z has been eliminated:

$$\mathfrak{F}^{(a)}{}_T\mu^2\mathfrak{F}^{(b)}=\epsilon T^{(a)}T^{(b)}\lambda. \tag{19}$$

Explicitly, (19) implies

$$F_\pi^2\mu_\pi^2=4\epsilon_1\lambda_1, \tag{20a}$$

$$F_K^2\mu_K^2=(\epsilon_1+\epsilon_2)(\lambda_1+\lambda_2), \tag{20b}$$

$$F_\kappa^2\mu_\kappa^2=(\epsilon_1-\epsilon_2)(\lambda_1-\lambda_2), \tag{20c}$$

$$G_\eta^2\mu_\eta^2+G_X^2\mu_X^2=4\epsilon_1\lambda_1, \tag{20d}$$

$$H_\eta^2\mu_\eta^2+H_X^2\mu_X^2=4\epsilon_2\lambda_2, \tag{20e}$$

$$G_\eta H_\eta\mu_\eta^2+G_XH_X\mu_X^2=0. \tag{20f}$$

These equations yield a determination of F_η, G_η, H_X, H_η in terms of the other parameters and a single undetermined parameter θ:

$$G_\eta=F_\pi(\mu_\pi/\mu_\eta)\sin\theta, \tag{21a}$$

$$G_X=F_\pi(\mu_\pi/\mu_X)\cos\theta, \tag{21b}$$

$$H_\eta=\Lambda^{-1}F_\pi(\mu_\pi/\mu_\eta)\cos\theta, \tag{21c}$$

$$H_X=-\Lambda^{-1}F_\pi(\mu_\pi/\mu_X)\sin\theta, \tag{21d}$$

where Λ is a small parameter given by

$$\Lambda^2=F_\pi^2\mu_\pi^2(2F_K^2\mu_K^2+2F_\kappa^2\mu_\kappa^2-F_\pi^2\mu_\pi^2)^{-1}. \tag{22}$$

In Sec. III we use these results in Eqs. (4) and (5) to compute the values of Γ_π, Γ_η, and Γ_X.

Finally, we quote[3] the consequences of the additional hypothesis that the entire inverse propagator (13), restricted to the pseudoscalar octet, transforms like a mixture of a unitary singlet and a unitary octet (i.e., both the matrices $Z^{-\frac{1}{2}}{}_TZ^{-\frac{1}{2}}$ and $Z^{-\frac{1}{2}}{}_T\mu^2Z^{-\frac{1}{2}}$ satisfy the Gell-Mann-Okubo hypothesis). One consequence is that $\lambda_1=\lambda_2$, and hence that $F_\kappa^2\mu_\kappa^2=0$, allowing an evaluation of Λ^2 in terms of measurable quantities. The other result is that some of the generalized leptonic decay amplitudes satisfy the Gell-Mann–Okubo formula

$$[(\sqrt{\tfrac{1}{3}})G_\eta-(\sqrt{\tfrac{2}{3}})H_\eta]^2+[(\sqrt{\tfrac{1}{3}})G_X-(\sqrt{\tfrac{2}{3}})H_X]^2 =\tfrac{4}{3}F_K^2-\tfrac{1}{3}F_\pi^2. \tag{23}$$

[7] There, of course, exists also the additional symmetry group $U(1)$ associated with baryon number. Since it is irrelevant to our discussion, we make no reference to it.

[8] This controversial assertion is further discussed in Sec. IV. Note that in the limit of exact chiral $SU(2)$ an additional one-parameter chiral symmetry appears corresponding to the matrix

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}.$$

Evidently, (23) gives a determination of $(F_K/F_\pi)^2$ in terms of the parameter θ.

These consequences of the assumption about $\Delta^{-1}(p^2)$ are derived in the Appendix.

III. ELECTROMAGNETIC ANOMALIES IN DIVERGENCE EQUATIONS

In the presence of electromagnetic interactions, coupled minimally and gauge invariantly to the electromagnetic current, the divergence equations (17) are modified by the addition of the effective electromagnetic terms displayed in (2), (4), and (5). The constants $a^{(i)}$ are given by (a) the electric charge structure of the underlying fermion fields or quarks, (b) renormalization effects, if any, on (a). We shall assume, for lack of any definite knowledge, that these renormalization effects are the same for all the $a^{(i)}$'s, so that the ratios $a^{(i)}/a^{(j)}$ are independent of renormalizations. The charge structure then determines the ratios completely:

$$a^{(i)}/a^{(j)}=\mathrm{Tr}QM^{(i)}Q/\mathrm{Tr}QM^{(j)}Q. \qquad (24)$$

Here Q is the charge matrix for the electronmagnetic current, and $M^{(i)}$ the matrix for the axial-vector current $A_\mu^{(i)}$. (In a fermion model, the electromagnetic current is $\bar\psi\gamma^\mu Q\psi$, while the axial-vector current becomes $i\bar\psi\gamma^5\gamma^\mu M^{(i)}\psi$.)

In the remainder we shall confine the discussion to a $SU(3)$ triplet of fermions with charges q, $q-1$, and $q-1$. In this model the matrices Q and $M^{(i)}$ are given by

$$Q=\begin{bmatrix} q & 0 & 0 \\ 0 & q-1 & 0 \\ 0 & 0 & q-1\end{bmatrix},$$

$$M^{(3)}=\begin{bmatrix} \frac12 & 0 & 0 \\ 0 & -\frac12 & 0 \\ 0 & 0 & 0\end{bmatrix},$$

$$M^{(1)}=\begin{bmatrix} \frac12 & 0 & 0 \\ 0 & \frac12 & 0 \\ 0 & 0 & 0\end{bmatrix}, \qquad (25)$$

$$M^{(2)}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt2\end{bmatrix},$$

When $q=\frac23$, this is just the quark model. With (25), (24) becomes

$$\alpha\equiv\frac{a^{(1)}}{a^{(3)}}=\frac{q^2+(q-1)^2}{q^2-(q-1)^2},$$
$$\beta\equiv\frac{a^{(2)}}{a^{(3)}}=\frac{\sqrt2(q-1)^2}{q^2-(q-1)^2}. \qquad (26)$$

Note that for all q,[9]

$$\alpha^2\geq1 \quad\text{and}\quad 0\leq\beta/\alpha\leq\sqrt2. \qquad (27)$$

When q has the value $\frac23$, as in the quark model, $\alpha=5/3$ and $\beta=\frac13\sqrt2$. Alternatively, with integrally charged fermions, $q=1$, $\alpha=1$, and $\beta=0$.

The decay width Γ_i of a pseudoscalar meson (π^0, η, or X) into 2γ is given by

$$\Gamma_i=\mu_i^3g_i^2C, \qquad (28)$$

where C is a common dimensionless factor, and g_i is the effective coupling constant determined by the vacuum-2γ matrix element of Φ_i. This field is gotten from (2), (4), and (5). Making the low-energy approximation, we know that the divergence of the axial-vector current will not contribute.[10] Hence, in calculating Φ_i, and thus the decay coupling constants, (2), (4), and (5) can be replaced by

$$F_\pi\mu_\pi^2\Phi_\pi=-(e^2/16\pi^2)F^{\mu\nu}\tilde F_{\mu\nu}a^{(3)},$$
$$\begin{bmatrix} G_\eta\mu_\eta^2 & G_X\mu_X^2 \\ H_\eta\mu_\eta^2 & H_X\mu_X^2\end{bmatrix}\begin{bmatrix}\Phi_\eta\\ \Phi_X\end{bmatrix}=-\frac{e^2}{16\pi^2}F^{\mu\nu}\tilde F_{\mu\nu}\begin{bmatrix}a^{(1)}\\ a^{(2)}\end{bmatrix}. \qquad (29)$$

The solution of these equations is now expressed with the aid of (21) and (22);

$$\Phi_\pi=-(e^2/16\pi^2)F^{\mu\nu}\tilde F_{\mu\nu}(a^{(3)}/F_\pi\mu_\pi^2),$$
$$\begin{bmatrix}\Phi_\eta\\ \Phi_X\end{bmatrix}=-\frac{e^2}{16\pi^2}F^{\mu\nu}\tilde F_{\mu\nu}\frac{1}{F_\pi\mu_\pi^2} \qquad (30)$$
$$\times\begin{bmatrix}(\mu_\pi/\mu_\eta)\sin\theta & \Lambda(\mu_\pi/\mu_\eta)\cos\theta \\ (\mu_\pi/\mu_X)\cos\theta & -\Lambda(\mu_\pi/\mu_X)\sin\theta\end{bmatrix}\begin{bmatrix}a^{(1)}\\ a^{(2)}\end{bmatrix}.$$

Evidently the effective coupling constants, apart from universal factors, are given by

$$g_\pi=F_\pi^{-1},$$
$$g_\eta=F_\pi^{-1}(\mu_\eta/\mu_\pi)(\alpha\sin\theta+\Lambda\beta\cos\theta), \qquad (31)$$
$$g_X=F_\pi^{-1}(\mu_X/\mu_\pi)(\alpha\cos\theta-\Lambda\beta\sin\theta).$$

From (31) now follows our principal result:

$$\frac{\Gamma_\eta}{\mu_\eta^5}+\frac{\Gamma_X}{\mu_X^5}=(\alpha^2+\Lambda^2\beta^2)\frac{\Gamma_\pi}{\mu_\pi^5}. \qquad (32)$$

This immediately implies that

$$\frac{\Gamma_\eta}{\mu_\eta^5}+\frac{\Gamma_X}{\mu_X^5}\geq\frac{\Gamma_\pi}{\mu_\pi^5}, \qquad (33)$$

since $\alpha^2\geq1$; see (27).

When Λ is approximated by the value $F_K\approx F_\pi$, we find from its definition (22) that $\Lambda^2\leq1/24$ and according to (27) $\Lambda^2\beta^2/\alpha^2\leq1/12$. Therefore, (32) may be approxi-

[9] These inequalities, in fact, remain valid in models with several fermion triplets with arbitrary charges, providing that for each, (25) is true. Our principal results remain valid for such models.

[10] M. Veltman, Proc. Roy. Soc. (London) **A301**, 107 (1967); D. Sutherland, Nucl. Phys. **B2**, 433 (1967).

mated:

$$\frac{\Gamma_\eta}{\mu_\eta^5}+\frac{\Gamma_X}{\mu_X^5}\approx\alpha^2\frac{\Gamma_\pi}{\mu_\pi^5}, \tag{34}$$

$\alpha^2=25/9$ (quarks),

$\alpha^2=1$ (integrally charged triplet).

Finally, experimental values are inserted for Γ_π and Γ_η [11]:

$$\Gamma_\pi=7.3\pm1.5 \text{ eV}, \quad \Gamma_\eta=1.00\pm0.25 \text{ keV}.$$

We find for (33)

$$\Gamma_X\geq 80 \text{ keV}, \tag{35}$$

and for (34)

$$\begin{aligned}\Gamma_X&\approx 350\pm80 \text{ keV} \quad \text{(quarks)},\\ \Gamma_X&\approx 120\pm30 \text{ keV} \quad \text{(integrally charged triplet)}.\end{aligned} \tag{36}$$

These results will be discussed in Sec. IV.

It is to be emphasized that the above does not make use of the $SU(3)$ relations derived in the Appendix. To make progress beyond (35) and (36), we now incorporate these $SU(3)$ assumptions and consequences into our framework. Thus we have from (23)

$$\Lambda^2[4(F_K^2/F_\pi^2)-1]=(\mu_\pi^2/\mu_\eta^2)(\Lambda\sin\theta-\sqrt{2}\cos\theta)^2 + (\mu_\pi^2/\mu_X^2)(\Lambda\cos\theta+\sqrt{2}\sin\theta)^2. \tag{37}$$

In the above, Λ^2 is given by (22), except that $F_\kappa^2\mu_\kappa^2=0$ by virtue of the results in the Appendix. We define also

$$\begin{aligned}R_\eta&\equiv\alpha\sin\theta+\Lambda\beta\cos\theta,\\ R_X&\equiv\alpha\cos\theta-\Lambda\beta\sin\theta.\end{aligned} \tag{38a}$$

From (28) and (31) it follows that

$$R_\eta^2=\Gamma_\eta\mu_\pi^5/\Gamma_\pi\mu_\eta^5, \quad R_X^2=\Gamma_X\mu_\pi^5/\Gamma_\pi\mu_X^5. \tag{38b}$$

This implies [see also (32)]

$$R_\eta^2+R_X^2=\alpha^2+\beta^2\Lambda^2. \tag{39}$$

With the definition of two angles,

$$\tan\varphi=R_\eta/R_X, \tag{40}$$

$$\tan\omega=\Lambda(2^{-1/2}-\beta/\alpha)(1+2^{-1/2}\Lambda^2\beta/\alpha)^{-1}, \tag{41}$$

we may recast Eq. (37) as

$$\Lambda^2(2+\Lambda^2)^{-1}(4F_K^2/F_\pi^2-1)=\mu_\pi^2/\mu_\eta^2 - (\mu_\pi^2/\mu_\eta^2-\mu_\pi^2/\mu_X^2)\sin^2(\varphi+\omega). \tag{42}$$

Equations (39)–(42) give *one* relation between F_K/F_π and R_η (or R_X), in terms of known masses and the model-dependent parameters α and β. In this relation, R_η depends sensitively on the precise value of F_K/F_π, which is not well determined experimentally. Thus, we use our formula, rather, to calculate F_K/F_π in terms of the known experimental value of R_η.

Since $0\leq\beta/\alpha\leq\sqrt{2}$ [see (27)], it follows that ω is a small angle. Experimentally, $R_\eta\ll1$, so that φ is also a small angle. Thus, independent of the precise value of Γ_η, and independent of the fermion charge structure, we may drop the last term in (42) and incur a negligible error. We then obtain

$$F_K^2/F_\pi^2\approx1.3. \tag{43}$$

IV. DISCUSSION

It is necessary to clarify some rather controversial points in our analysis. We have assumed that the extra chiral $U(1)$ factor—chiral quark number—is broken only by terms which transform like a member of the $(3,\bar{3})+(\bar{3},3)$ representation of $SU(3)\times SU(3)$.[12] In particular, as in the usual quark models, this $U(1)$ factor becomes an exact symmetry when $\epsilon\to0$. More troubling is that *four* chiral generators become exact symmetries in the chiral $SU(2)$ limit $\epsilon_1\to0$. This is clearly reflected in (20), from which we may deduce

$$F_\pi^2\mu_\pi^2=G_\eta^2\mu_\eta^2+G_X^2\mu_X^2=4\epsilon_1\lambda_1. \tag{44}$$

When $\epsilon_1=0$, we must have either $G_\eta=G_X=0$ (which would make the corresponding chiral generator a good conventional symmetry), or at least one of the mesons (presumably η) must become massless. Presumably, the latter is the more attractive alternative.

Thus, we envisage a world in which both π and η become Goldstone bosons in the chiral limit $\epsilon_1\to0$. Of course, soft-η calculations must evidently be less reliable than soft-pion calculations because of the larger extrapolation involved, although 0.3 GeV^2 is still rather small compared to baryon masses or Regge-spacing parameters. Our hypothesis seems tenable since we encounter no difficulty in enforcing the smoothness assumption nor in accommodating the observed pseudoscalar-meson mass spectrum. Moreover, our results are consistent with weak $SU(3)$ breaking and a Gell-Mann–Okubo structure of the pseudoscalar propagator. Our scheme for $SU(3)\times SU(3)\times U(1)$ breaking can only be tested by its experimental predictions, which thus far seem to be satisfied.

A second unconventional aspect of our analysis is the inclusion of anomalous electromagnetic terms in the divergences of the neutral axial-vector currents. These terms are forced by the infinities of a local fermion field theory, and it is not obvious that they must be present in nature. The evidence for the anomaly in the isospin axial-vector current, associated with the pion, is that ordinary PCAC seems to predict a strong suppression in Γ_π, which contradicts experiment.[10] We consider our prediction of the enhancement for Γ_X an important test for these anomalies in the ηX channels. It is encouraging that our calculation of $(F_K/F_\pi)^2$, which also relies on these anomalies, is consistent with experiment.

We have discussed the speculative ideas upon which our work is based. What are our results?

From the assumption that the inverse propagator

[11] N. Barash-Schmidt *et al.*, Rev. Mod. Phys. 41, 109 (1969).

[12] Certainly one can introduce terms which break $U(1)$ but preserve $SU(3)\times SU(3)$. The simplest possibility would be a trilinear expression in φ. However, such terms are unappealing from the point of view of a quark model, where they would correspond to six-fermion interactions.

$\Delta^{-1}(p^2)$ at low momenta transforms like a mixture of $SU(3)$ singlet and octet [which is what is expected if $SU(3)$ breaking may be treated perturbatively], we have deduced that $\lambda_1 \approx \lambda_2$ and that $F_\kappa^2 \approx 0$. Thus, the notion of weak $SU(3)$ breaking is shown to be consistent. In agreement with experiment, we find that there should not exist a low-lying (≤ 1 GeV) scalar kaon. Another experimental indication that F_κ^2 must be small may be found in the analysis of ρ, ω, and φ decays into lepton pairs.[3]

Although we do not predict the value of Γ_η, our analysis is consistent with the observed value. We unambiguously predict a large value for Γ_X (i.e., 350 keV in the quark model, greater than 80 keV in any model). Experimentally,[11] there is only an upper limit of 4 MeV on the total X decay width, while the branching ratio for $X \to 2\gamma$ is measured to be $(6.2 \pm 3.1)\%$. We conclude from experiment that $\Gamma_X \leq 370$ keV, consistent with our prediction.

Finally, we obtain the model-independent evaluation of the ratio of leptonic decay constants $(F_K/F_\pi)^2 \approx 1.3$. Let us compare this result with other analyses. Experimentally, from kaon decay data, one obtains[13]

$$(F_K/F_\pi)[f^+(0)]^{-1}(\tan\theta_A/\tan\theta_V) = 1.23 \pm 0.05 .$$

If equality of vector and axial-vector Cabibbo angles is assumed, and use is made of the $SU(3) \times SU(3)$ result[1]

$$f^+(0) = \tfrac{1}{2}(F_K^2 + F_\pi^2 - F_\kappa^2)F_K F_\pi ,$$

with $F_\kappa^2 \approx 0$ according to our analysis, we obtain

$$(F_K/F_\pi)^2 \approx 1.6 \pm 0.2.$$

On the other hand, $(F_K/F_\pi)^2$ can also be determined from the spectral function sum rules, one-particle dominance, and the known masses of spin-1 masses,[1,3]

$$(2F_\pi^2 - F_\kappa^2)M^2(K^*) = (2F_\pi^2 - F_K^2)M^2(K_A) .$$

Use of $M(K^*) = 890$ MeV, $M(K_A) = 1330$ MeV, and $F_\kappa^2 \approx 0$ yields

$$(F_K/F_\pi)^2 \approx 1.1 .$$

The agreement between the three evaluations of $(F_K/F_\pi)^2$ is far from perfect, but it is impressive when one considers that there is no *a priori* reason for $(F_K/F_\pi)^2$ to be near unity, and that the three determinations are based upon entirely different kinds of data: electromagnetic decay widths, weak decay widths, and the masses of vector and axial-vector mesons. Indeed, the third determination depends upon entirely different hypotheses than do the others.

APPENDIX

In broken $SU(3) \times SU(3)$, it is our point of view that chiral $SU(2) \times SU(2)$ is not strongly broken in the Lagrangian (i.e., $\epsilon_1 \ll \epsilon_2$), but that we are near the point of spontaneous symmetry breakdown, so that chiral perturbation theory is a bad approximation. On the other hand, $SU(3)$ is badly broken in the Lagrangian, but apparently $SU(3)$ breaking can be treated perturbatively. Thus, we anticipate $\lambda_1 \approx \lambda_2$. [Indeed, from (20c) we see immediately that $\lambda_1 - \lambda_2$ occurs only to second order in symmetry breaking.]

Treating $SU(3)$ breaking perturbatively, we should expect that the Gell-Mann–Okubo hypothesis is satisfied, so that it is reasonable to require that the matrices occurring in (13), $Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}}$ and $Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}}$, transform like a mixture of the singlet and the neutral member of an octet.

From Eqs. (16) and (18) we may write

$$\lambda T^{(a)} Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}} = \epsilon T^{(a)} , \qquad (A1)$$

from which, explicitly, we may deduce

$$(Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}})_{\pi,\pi} = \epsilon_1/\lambda_1 , \qquad (A2)$$

$$(Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}})_{K,K} = (\epsilon_2 + \epsilon_1)/(\lambda_1 + \lambda_2) , \qquad (A3)$$

$$(Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}})_{8,8} = \tfrac{1}{3}\epsilon_1/\lambda_1 + \tfrac{2}{3}\epsilon_2/\lambda_2 . \qquad (A4)$$

The $1+8$ hypothesis for $Z^{-\frac{1}{2}}{}_T \mu^2 Z^{-\frac{1}{2}}$ then becomes

$$\frac{3}{4}\left(\frac{1}{3}\frac{\epsilon_1}{\lambda_1} + \frac{2}{3}\frac{\epsilon_2}{\lambda_2}\right) + \frac{1}{4}\frac{\epsilon_1}{\lambda_1} = \frac{\epsilon_1 + \epsilon_2}{\lambda_1 + \lambda_2} , \qquad (A5)$$

which is easily shown to be satisfied if and only if

$$\lambda_1 = \lambda_2 \qquad (A6)$$

or

$$\epsilon_1/\epsilon_2 = \lambda_1/\lambda_2 . \qquad (A7)$$

The second possibility, being completely incompatible with our outlook, may be ignored, and we conclude that $\lambda_1 = \lambda_2$ and that $F_\kappa^2 \mu_\kappa^2 = 0$. [This corresponds to $F_\kappa \to 0$ and $\mu_\kappa \to \infty$, but should not be regarded as a literal statement about the scalar kaon, since our analysis, especially (13), is only valid at low momenta.]

From (18), we may write

$$\mathfrak{F}^{(a)}{}_T \mathfrak{F}^{(b)} = \lambda T^{(a)} Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}} T^{(b)} \lambda . \qquad (A8)$$

When we make use of our result that λ is $SU(3)$-invariant, this gives us the explicit relations

$$(Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}})_{\pi\pi} = F_\pi^2 \lambda^{-2} , \qquad (A9a)$$

$$(Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}})_{KK} = F_K^2 \lambda^{-2} , \qquad (A9b)$$

$$(Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}})_{88} = (F_{8\eta}^2 + F_{8X}^2)\lambda^{-2} , \qquad (A9c)$$

where $F_{8\eta}$ and F_{8X} are the leptonic decay amplitudes appearing in the divergence equation for $A_\mu{}^8$,

$$\partial^\mu A_\mu{}^{(8)} = F_{8\eta}\mu_\eta^2 \Phi_\eta + F_{8X}\mu_X^2 \Phi_X . \qquad (A10)$$

From (3)–(5), we obtain

$$F_{8\eta} = (\surd\tfrac{1}{3})G_\eta - (\surd\tfrac{2}{3})H_\eta , \qquad (A11a)$$

$$F_{8X} = (\surd\tfrac{1}{3})G_X - (\surd\tfrac{2}{3})H_X . \qquad (A11b)$$

The requirement that $Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}}$ is a mixture of $SU(3)$ singlet and octet gives us one relation

$$(Z^{-\frac{1}{2}}{}_T Z^{-\frac{1}{2}})_{88} = \tfrac{4}{3}Z_K{}^{-1} - \tfrac{1}{3}Z_\pi{}^{-1} , \qquad (A12)$$

which, with (A9) and (A11), yields (23).

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[13] See C. Callan, in *Topical Conference on Weak Interactions*, CERN Report No. 69-7, 1969, p. 264 (unpublished).