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Nonuniqueness of the Spin-$\frac{1}{2}$ Equation*

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We show that the Dirac equation is not the only first-order differential equation that is form-invariant under Lorentz transformations, irreducible, and derivable from a Lagrangian, and whose solutions correspond to mass m and spin $\frac{1}{2}$. It is, however, the only such equation, except for spin 0 and 1, for which β_0 is diagonalizable.

I. INTRODUCTION

SINCE Dirac's discovery[1] in 1928 of the equation that now bears his name, many particles with spin $\frac{1}{2}$ have been found experimentally. It would be aesthetically pleasing to have only one equation, namely Dirac's, capable of describing these particles. If one makes sufficiently stringent assumptions, this is certainly true. Thus, Naimark[2] has shown that the Dirac equation is the only finite-dimensional equation of the form

$$(\beta_\mu p^\mu - m)\psi = 0 \tag{1}$$

with a diagonalizable β_0 matrix for which the charge density is definite. As we will show later on, the requirement of definiteness of the charge density may be dropped, so that to obtain uniqueness of the Dirac equation for spin $\frac{1}{2}$ requires only that β_0 be diagonalizable. Unfortunately, there seems to be no compelling reason for making this assumption. In fact, if one requires that β_0 be diagonalizable, one not only makes the Dirac equation unique for spin $\frac{1}{2}$, but one also excludes all equations of the form (1) for all other spins, except spin 0 and spin 1. So it seems that this requirement is too strong. We have not succeeded in finding a weaker requirement that would make the Dirac equation unique.

In Sec. II, we give an outline of the general theory required, as well as a proof of the uniqueness of the Dirac equation in case β_0 is diagonalizable. In Sec. III, we work out in detail the next simplest case of a spin-$\frac{1}{2}$ equation and give an explicit representation of the matrices β_μ and the Hermitianizing matrix η. In Sec. IV, we list speculations regarding the use of these equations.

II. GENERAL THEORY

Once we drop the requirement that β_0 be diagonalizable, the equation resulting may be studied in the framework of higher-spin equations. As we shall see, this results in a hierarchy of spin-$\frac{1}{2}$ equations. The general theory of such higher-spin equations was discussed in a previous paper.[3] The main features of that paper will be repeated here without proof.

We are concerned with finding equations of the form

$$(\beta_\mu p^\mu - m)\psi = 0 \tag{1}$$

that are form-invariant under Lorentz transformations, irreducible, and derivable from a Lagrangian, and whose solutions transform according to mass-m spin-$\frac{1}{2}$ representations under the inhomogeneous Lorentz group. We shall now discuss each of these requirements in turn.

Form invariance of (1) demands that if ψ transforms under Lorentz transformations according to

$$\psi(x) \rightarrow \psi'(x') = D(\Lambda)\psi(\Lambda^{-1}x'),$$

then the β's satisfy

$$D(\Lambda)^{-1}\beta^\mu D(\Lambda) = \Lambda^\mu{}_\nu \beta^\nu. \tag{2}$$

In terms of generators this reads

$$[\beta^\mu, M^{\rho\sigma}] = g^{\mu\sigma}\beta^\rho - g^{\mu\rho}\beta^\sigma, \tag{3}$$

so that

$$\beta^k = [\beta^0, M^{0k}], \quad k = 1, 2, 3. \tag{4}$$

Bhabha[4,5] found the most general solution of Eq. (3). This brings us to the question of irreducibility. To discuss this requires more details of Bhabha's solutions.

The solutions obtained by Bhabha are expressible in terms of certain matrices u^α and v^β and a number of arbitrary constants. The u^α and v^β are related to the decomposition of the representation $\mathfrak{D}^k \otimes \mathfrak{D}^{1/2}$ of $SU(2)$ into $\mathfrak{D}^{k\pm 1/2}$ and are thus closely related to Clebsch-Gordan coefficients. The solutions are nonzero only if the representation $D(\Lambda)$ in Eq. (2) is such that, for every irreducible representation $D^{(j,k)}$ in $D(\Lambda)$, there also occurs at least one irreducible $D^{(j',k')}$ such that $|j-j'| = |k-k'| = \frac{1}{2}$. We say that two such representations are linked. If the irreducible representations occurring in $D(\Lambda)$ can be split into independently linked sets of representations with no cross linkage between the sets, then the resultant β's are reducible. Otherwise they are irreducible.

The condition that Eq. (1) be derivable from a Lagrangian is equivalent to requiring that a Hermitianizing matrix η exist such that

$$\eta\beta^{\mu\dagger}\eta^{-1} = \beta^\mu. \tag{5}$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928).
[2] M. A. Naimark, *Linear Representations of the Lorentz Group* (The Macmillan Co., New York, 1964), pp. 411–415.
[3] A. Z. Capri, Phys. Rev. **178**, 2427 (1969).
[4] H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).
[5] H. J. Bhabha, Rev. Mod. Phys. **21**, 451 (1949).

Using the results of (3), it will follow that, for the theories here described, there is always one and only one such matrix η. The form of η is again explicitly given in (3).

The condition for mass m (Klein-Gordon condition) requires that the minimal equation for β_0 be of the form

$$(\beta_0^2-1)\beta_0^n=0. \quad (6)$$

The condition is also sufficient. Umezawa and Visconti[6] have shown that $n=2s-1$, where s is the maximum spin contained in the field ψ. From (6) it follows that β_0 has eigenvalues 0, ±1. The physical solutions of (1) correspond to the ±1 eigenvalues. The projection operator onto the space of physical solutions of (1) in the rest frame is β_0^{2s-1}. Therefore, one can write the condition for spin $\frac{1}{2}$ as

$$J^2\beta_0^{2s-1}=\tfrac{1}{2}(\tfrac{1}{2}+1)\beta_0^{2s-1}, \quad (7)$$

where J^2 is the square of the generators of rotations.

For the ensuing calculations, it is convenient to work in a basis in which J^2 is diagonal. The components of β_0 are then given by the following equations:

$$\langle(k,l)_s|\beta_0|(k-\tfrac{1}{2},l+\tfrac{1}{2})_t\rangle = C_{st}(k+j-l)^{1/2}(j+l+1-k)^{1/2}\delta_{jj'}, \quad (8a)$$

$$\langle(k,l)_s|\beta_0|(k-\tfrac{1}{2},l-\tfrac{1}{2})_t\rangle = C_{st}(-1)^{k+l+j}(k+l+j)^{1/2} \times(k+l+j+1)^{1/2}\delta_{jj'}, \quad (8b)$$

$$\langle(k,k+\tfrac{1}{2})_s|\beta_0|(k+\tfrac{1}{2},k)_t\rangle = C_{st}(-1)^{\{j\}+1}(j+\tfrac{1}{2})\delta_{jj'}, \quad (8c)$$

where $l\neq k$ and $l'\neq k'$,

$$\{j\}=\text{integral part of } j,$$

$$|k-l|\leqslant j'\leqslant|k+l|,$$

and the C_{st} are arbitrary coefficients. For $l=k$ or $l'=k'$, we have

$$\langle(k-\tfrac{1}{2},k+\tfrac{1}{2})_s|\beta_0|(k,k)_t\rangle = C_{st}(-1)^{2k+j-1}(j)^{1/2}(j+1)^{1/2}\delta_{jj'}. \quad (8d)$$

All other required components can be obtained from

$$\langle k,l|\beta_0|k',l'\rangle=(-1)^{2k+2}\langle k',l'|\beta_0|k,l\rangle \quad (8e)$$

and

$$\langle k,l|\beta_0|k',l'\rangle=-\langle l,k|\beta_0|l',k'\rangle. \quad (8f)$$

The indices (k,l) here refer to the indices in $D^{(k,l)}$ labeling the irreducible representations.

A. Diagonalizable β_0

We shall now show that the requirement that β_0 be diagonalizable, uniquely picks out the Dirac equation. Equation (6) implies that the matrix β_0 may be put in the following Jordan form:

$$\beta_0=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&N\end{bmatrix}, \quad (9)$$

where 1 is a unit matrix and N is a nilpotent matrix of order $n=2s-1$. If β_0 is diagonalizable, then $n\leq1$ so that $2s-1\leq1$ or $s\leq1$. Thus, the maximum allowed spin is 1. This limits us to the representations $D^{(0,0)}$, $D^{(\frac{1}{2},0)}$, $D^{(\frac{1}{2},\frac{1}{2})}$, $D^{(1,0)}$, and their conjugates. For these representations, however, the only possible equations of the form (1) with a Hermitianizing matrix η are the Dirac and Duffin-Kemmer equations. Thus we have proved our claim.

In general, of course, if we drop this requirement, all other irreducible representations are permitted.

B. General Case

In this case, we have chosen as the representation D in Eq. (2), any one of the following:

$$D(k)=\bigoplus_{j=0}^{k}\left[D^{(j+1/2,\,j/2)}\oplus D^{(j/2,\,j+1/2)}\right], \qquad k=0,1,2,\cdots. \quad (10)$$

The procedure to be followed is as follows: Using the relations (8), an explicit representation $\beta_0(k)$ of β_0 corresponding to the representation $D(k)$ is written down. The matrix β_0^2 then appears in block diagonal form, with the various blocks labeled by the values of J^2.

Thus the blocks are labeled by $k+\frac{1}{2}$, $k-\frac{1}{2}$, $k-\frac{3}{2}$, $\cdots$, $\frac{5}{2}$, $\frac{3}{2}$, $\frac{1}{2}$. We now fix the constants C_{st} in the blocks as follows: Starting with $k+\frac{1}{2}$, we make the successive blocks, except the spin-$\frac{1}{2}$ block, nilpotent. Finally, the spin-$\frac{1}{2}$ block is made to satisfy the minimal equation of the whole matrix. This ensures that both Eqs. (6) and (7) are satisfied and guarantees that we have a mass-m spin-$\frac{1}{2}$ representation of the inhomogeneous Lorentz group.

In applying the above procedure, one is doing in reverse what is usually done in higher-spin theories. There, one introduces lower spins and makes their corresponding submatrices nilpotent. Here we introduce higher spins and make their corresponding matrices nilpotent.

III. EXAMPLE

The case $k=0$ for Eq. (10) yields the Dirac equation. The next simplest case is $k=1$. In that case $D(1)$ is equivalent to $D^{(1,\frac{1}{2})}\oplus D^{(0,\frac{1}{2})}\oplus D^{(\frac{1}{2},0)}\oplus D^{(\frac{1}{2},1)}$. As stated above, we work in a basis with J^2 diagonal. Then J^2 has

[6] H. Umezawa and A. Visconti, Nucl. Phys. 1, 348 (1956).

the form

$$J^2=\left|\begin{array}{ccc|ccc} \frac{3}{2}(\frac{3}{2}+1) & 0 & 0 & & & \\ 0 & \frac{1}{2}(\frac{1}{2}+1) & 0 & & 0 & \\ 0 & 0 & \frac{1}{2}(\frac{1}{2}+1) & & & \\ \hline & & & \frac{1}{2}(\frac{1}{2}+1) & 0 & 0 \\ & 0 & & 0 & \frac{3}{2}(\frac{3}{2}+1) & 0 \\ & & & 0 & 0 & \frac{1}{2}(\frac{1}{2}+1) \end{array}\right|, \tag{11}$$

and in block form β_0 is given by

$$\beta_0=\left|\begin{array}{ccc|ccc} & & & 0 & 2C_2 & 0 \\ & 0 & & \sqrt{3}C_1 & 0 & -C_2 \\ & & & C_3 & 0 & \sqrt{3}C_4 \\ \hline 0 & \sqrt{3}C_4 & 0 & & & \\ 2C_2 & 0 & 0 & & 0 & \\ 0 & -C_2 & \sqrt{3}C_1 & & & \end{array}\right|. \tag{12}$$

Also

$$\beta_0^2=\left|\begin{array}{ccc|ccc} 4C_2^2 & 0 & 0 & & & \\ 0 & C_2^2+3C_1C_4 & \sqrt{3}C_4(C_3-C_2) & & 0 & \\ 0 & \sqrt{3}C_1(C_3-C_2) & C_3^2+3C_1C_4 & & & \\ \hline & & & C_3^2+3C_1C_4 & 0 & \sqrt{3}C_4(C_3-C_2) \\ & 0 & & 0 & 4C_2^2 & 0 \\ & & & \sqrt{3}C_1(C_3-C_2) & 0 & C_2^2+3C_1C_4 \end{array}\right|. \tag{13}$$

Therefore, we require $C_2=0$ and that $A^2=A$ as minimal equation, where

$$A=\begin{pmatrix} 3C_1C_4 & \sqrt{3}C_4C_3 \\ \sqrt{3}C_1C_3 & C_3^2+3C_1C_4 \end{pmatrix}. \tag{14}$$

This yields the following solutions:

(1) $C_4=0,\quad C_3^2=1,\quad C_1$ arbitrary; (3) $C_3=0,\quad C_1C_4=\frac{1}{3}$;

(2) $C_1=0,\quad C_3^2=1,\quad C_4$ arbitrary; (4) $C_3=0,\quad C_4=0,\quad C_1$ arbitrary.

Solutions 3 and 4 are excluded since they do not make $A^2=A$ a minimal equation. Solutions 1 and 2 lead to equivalent matrices so that we have only one solution. Also, in case 1, $C_1\neq 0$ and in case 2, $C_4\neq 0$, since then again $A^2=A$ is not a minimal equation. The $\pm$ sign associated with C_3 is also of no relevance, since it can be absorbed in C_1 or C_4 as the case may be.

We now present an explicit representation of the four β matrices just derived. These are not obtained using (4); instead, it was more convenient to use the actual definition in terms of spinors. The matrices are

$$\beta_0=\left|\begin{array}{c|cc|cc|c} & & & 0 & 0 & \\ & & & \sqrt{2}C & 0 & \\ & & & -C & 0 & \\ & & & 0 & C & \\ & & & 0 & -\sqrt{2}C & \\ & & & 0 & 0 & \\ \cline{4-6} & & & -1 & 0 & \\ & & & 0 & -1 & \\ \hline & -1 & 0 & & & \\ & 0 & -1 & & & \\ \cline{1-3} & 0 & 0 & & & \\ & -C & 0 & & & \\ & 0 & -\sqrt{2}C & & & \\ & \sqrt{2}C & 0 & & & \\ & 0 & C & & & \\ & 0 & 0 & & & \end{array}\right|,$$

$$\beta_1=\left|\begin{array}{c|cc|cc|c}
 & & & \sqrt{2}C & 0 & \\
 & & & 0 & 0 & \\
 & & & 0 & C & \\
 & & & -C & 0 & \\
 & & & 0 & 0 & \\
 & & & 0 & -\sqrt{2}C & \\
\cline{4-6}
 & & & 0 & -1 & \\
 & & & -1 & 0 & \\
\hline
 & 0 & 1 & & & \\
 & 1 & 0 & & & \\
\cline{1-3}
 & -\sqrt{2}C & 0 & & & \\
 & 0 & -C & & & \\
 & 0 & 0 & & & \\
 & 0 & 0 & & & \\
 & C & 0 & & & \\
 & 0 & \sqrt{2}C & & &
\end{array}\right|,$$

$$\beta_2=\left|\begin{array}{c|cc|cc|c}
 & & & \sqrt{2}iC & 0 & \\
 & & & 0 & 0 & \\
 & & & 0 & iC & \\
 & & & iC & 0 & \\
 & & & 0 & 0 & \\
 & & & 0 & \sqrt{2}iC & \\
\cline{4-6}
 & & & 0 & -i & \\
 & & & i & 0 & \\
\hline
 & 0 & i & & & \\
 & -i & 0 & & & \\
\cline{1-3}
 & -\sqrt{2}iC & 0 & & & \\
 & 0 & -iC & & & \\
 & 0 & 0 & & & \\
 & 0 & 0 & & & \\
 & -iC & 0 & & & \\
 & 0 & -\sqrt{2}iC & & &
\end{array}\right|,$$

$$\beta_3=\left|\begin{array}{c|cc|cc|c}
 & & & 0 & 0 & \\
 & & & -\sqrt{2}C & 0 & \\
 & & & -C & 0 & \\
 & & & 0 & -C & \\
 & & & 0 & -\sqrt{2}C & \\
 & & & 0 & 0 & \\
\cline{4-6}
 & & & -1 & 0 & \\
 & & & 0 & 1 & \\
\hline
 & 1 & 0 & & & \\
 & 0 & -1 & & & \\
\cline{1-3}
 & 0 & 0 & & & \\
 & C & 0 & & & \\
 & 0 & \sqrt{2}C & & & \\
 & \sqrt{2}C & 0 & & & \\
 & 0 & C & & & \\
 & 0 & 0 & & &
\end{array}\right|,$$

$$\eta=\left(\begin{array}{c|c|c|c} & & & \begin{matrix}1&0&0&0&0&0\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{matrix} \\ \hline & & \begin{matrix}-1&0\\0&-1\end{matrix} & \\ \hline & \begin{matrix}-1&0\\0&-1\end{matrix} & & \\ \hline \begin{matrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{matrix} & & & \end{array}\right).$$

We also have $\eta=\eta^\dagger=\eta^{-1}$.

IV. CONCLUSIONS

Once the requirement that β_0 be diagonalizable is dropped, we can construct a whole hierarchy of spin-$\frac{1}{2}$ equations. This makes it clear that it is also possible to construct a hierarchy of higher-spin equations. In fact, one simply starts with one of the representations given in (3) that describes a higher-spin equation and one stacks on top of this the higher spins. These are then eliminated by requiring their corresponding submatrices or blocks belonging to $\beta_0{}^2$ to be nilpotent. Thus, the physical principles spelled out before for an equation of the form

$$(\beta_\mu p^\mu - m)\psi=0\,,$$

namely, form-invariance, irreducibility, Lagrangian, and pure spin and mass, do not suffice to determine such an equation uniquely.

In the case of spin $\frac{1}{2}$, all of the resultant equations, except Dirac's, lead to a nonrenormalizable electrodynamics and, therefore, in the presence of interactions are not equivalent to the Dirac equation. Furthermore, since all of these equations have a different number of spinor components, interactions between two different types of such spin-$\frac{1}{2}$ particles cannot be of the Yukawa type and are likely to be of the current-current type. This automatically leads to a conservation law implying that the type of spin-$\frac{1}{2}$ particle involved is conserved. It is tempting to speculate that this may have something to do with the fact that the μ-neutrino differs from the electron-neutrino and that muons and electrons are described by different spin-$\frac{1}{2}$ fields. At present, however, there seems to be no justification for this. A first test would be to calculate the g factor for a particle described by one of these other spin-$\frac{1}{2}$ fields.

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