Angle and Phase Coordinates in Quantum Mechanics

J. ZAK*

Department of Physics, Northwestern University, Evanston, Illinois 60201

(Received 14 April 1969; revised manuscript received 26 June 1969)

A general approach to the description of an angle and phase is given on the basis of the kq representation. It is shown that an angular coordinate in quantum mechanics has to be treated as a quasicoordinate in order to avoid inconsistencies. The kq representation leads to a consistent definition of the angular-momentum–angle degree of freedom. Using the correspondence between classical and quantum mechanics for the phase of a harmonic oscillator, operators are defined that form a new quantum-mechanical representation. This representation clarifies the concept of the phase and sheds light on the general understanding of rotations in quantum mechanics.

I. INTRODUCTION

THE very important concepts of angle and phase operators have been a great challenge to understanding the fundamentals of quantum mechanics. In recent years these concepts attracted much attention in connection with coherent states in laser physics, superconductivity, and superfluidity, and also in connection with the energy time uncertainty relation. The main difficulty in treating the angle and phase coordinates is connected with their representation by a linear operator.[1] Any observable that is a function of angle and phase must by the very nature of angle and phase be periodic with the period 2π. The difficulty arises in connection with the necessity of having a linear operator which is at the same time periodic. The recently introduced kq representation[2] turns out to be very useful for treating an angular coordinate in quantum mechanics.[3] The reason for this is that the kq representation uses the concept of quasicoordinates, i.e., coordinates defined within some constant. Originally this representation was defined by using x and p so that x is measurable within the constant a, and p within the constant $2\pi/a$ (here, $\hbar=1$). Most naturally, the kq representation can be applied to an angular coordinate because the latter is defined within a constant 2π. The treatment of the angle and phase in this paper will be based on the kq representation.

In recent publications,[1,4–6] the concepts of angle and phase have been to a great extent clarified. In particular, it was shown[4] that the well-accepted description of the phase by a Hermitian operator is inconsistent, and it was suggested that the phase operator be defined in an indirect way by using Hermitian cosine and sine operators. The same operators were also used in more recent publications.[5,6]

In this paper, alternative expressions for the cosine and sine operators are given which, in addition to defining the phase operator of a harmonic oscillator, also lead to a very interesting connection between rotations in the xp plane and rotations in regular space. The cosine and sine operators [$C(\varphi)$ and $S(\varphi)$] are obtained in this paper by a proper symmetrization of the corresponding classical expressions. The properly symmetrized $C(\varphi)$ and $S(\varphi)$ operators define a new quantum-mechanical representation which gives a consistent description of the angle in the xp plane and the phase of a harmonic oscillator. The eigenfunctions of the operators $C(\varphi)$ and $S(\varphi)$ in the representation of a harmonic oscillator turn out to be the normalized Legendre polynomials. This result is of great interest and sheds light on the general understanding of rotations in quantum mechanics.

In Sec. II, a consistent definition of the angular-momentum–angle degree of freedom is given. Section III deals with an angular coordinate in the xp plane (phase plane) and the phase of a harmonic oscillator. As in Sec. II, the treatment starts with a classical description of the problem, and the operators are obtained by symmetrizing the corresponding classical expressions. In Sec. III, the main section of the paper, the new quantum-mechanical representation (the phase representation) is defined. In Sec. IV a discussion of rotations in quantum mechanics is given in light of the phase representation. A connection is established between rotations in regular space and in the phase plane. It is shown that l, the quantity that defines the eigenvalues of the square of the angular momentum, $\mathbf{l}^2=l(l+1)$, can be given the meaning of a generator of infinitesimal rotations in the phase plane, zp_z. Section V is a general discussion of the results of this paper. The Appendix gives a detailed analysis of the previous solution[4–6] for the phase of a harmonic oscillator.

II. AN ANGLE IN REGULAR SPACE

An angle α in the xy plane can be defined as follows:

$$\cos\alpha=x/(x^2+y^2)^{1/2}, \tag{1}$$

$$\sin\alpha=y/(x^2+y^2)^{1/2}. \tag{2}$$

---

* Permanent addresses: Department of Physics, Technion, Israel; Institute of Technology, Haifa, Israel.

[1] W. H. Louisell, Phys. Letters **7**, 60 (1963).

[2] J. Zak, Phys. Rev. Letters **19**, 1385 (1967); Phys. Rev. **169**, 686 (1968).

[3] J. Zak, Bull. Am. Phys. Soc. **14**, 87 (1969).

[4] L. Susskind and J. Glogower, Physics **1**, 49 (1964).

[5] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **14**, 387 (1965).

[6] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

The formulas (1) and (2) can be used in both classical and quantum mechanics, and by measuring $\cos\alpha$ and $\sin\alpha$ one can define α (within a multiple of 2π). In classical mechanics, this forms a complete definition of the angle α. In quantum mechanics one would have to show that the operators (1) and (2) form a complete system of commuting operators. This latter is needed in order to specify by means of the eigenvalues of $\cos\alpha$ and $\sin\alpha$ a complete set of eigenfunctions. In the conventional definition of the angle in quantum mechanics, α is defined as a multiplication operator. This enables one to use α for specifying a complete set of states. It leads, however, to a number of inconsistencies. One of them is connected with the commutation relation between α and the z component l_z of the angular momentum:

$$[l_z,\alpha]=-i, \tag{3}$$

where

$$l_z=xp_y-yp_x. \tag{4}$$

In (4), p_x and p_y are the x and y components of the linear momentum. In order to see the inconsistency connected with (3), let us write the matrix elements of both sides of (3) in eigenstates of l_z:

$$(m-m')(m|\alpha|m')=-i\delta_{mm'}. \tag{5}$$

Here m amd m' are eigenvalues of l_z. Relation (5) cannot possibly be true[1] for $m=m'$, if the matrix element $(m|\alpha|m')$ is well defined: The left-hand side of (5) equals zero for $m=m'$, while the right-hand side is different from zero. This contradiction follows from the assumption that α has finite matrix elements between eigenstates of l_z and from the assumption of relation (3).

Another inconsistency connected with α as a linear multiplication operator is the following: The eigenfunctions of l_z in the α representation, $\psi_m(\alpha)$, are periodic in α. When the operator α is applied to them, one gets $\alpha\psi_m(\alpha)$, which is not periodic. This means that α is not well defined because it removes the functions $\psi_m(\alpha)$ from the space in which they are defined.

The above inconsistencies are avoided if one defines α by means of the operators (1) and (2). It remains to show that these operators form a complete set. This can be proved by using the kq representation.[2] Originally, this representation was defined for the coordinate x and the momentum p:

$$T(a)=e^{ipa}, \tag{6}$$

$$T(b)=e^{ixb}, \tag{7}$$

where

$$ab=2\pi. \tag{8}$$

It has been shown[2] that $T(a)$ and $T(b)$ form a complete set of commuting operators. The operators $T(a)$ and $T(b)$ can be defined for any pair of conjugate coordinates; for the angular-momentum–angle coordinates they will be

$$T(2\pi)=e^{2\pi i l_z}, \tag{9}$$

$$T(1)=e^{i\alpha}, \tag{10}$$

where a was put equal to 2π and b equal to 1. This choice of a and b satisfies relation (8) and is in agreement with α's definition within a multiple of 2π. As in the case of x and p, the operators (9) and (10) form a complete set of commuting operators. The proof of this statement can be carried out in the same way as for the operators (6) and (7). One only has to replace p by l_z and x by α, and notice that p and l_z are differential operators ($p=-i\partial/\partial x$, $l_z=-i\partial/\partial\alpha$) when operating on functions of x and α, correspondingly. The differential character of l_z follows from its definition (4).[7] The operators (9) and (10) form, therefore, a complete set of commuting operators. The additional requirement of periodicity of the wave function as functions of α makes the eigenvalues of l_z integers, and the operator (9) becomes a unit operator. One is, therefore, left with the operator (10) *which by itself forms a complete set.*[8] *The operator* (10) *defines, therefore, a quantum-mechanical representation.* The operator (10) also satisfies the very important feature that should be required from any operator that is used for defining the angle α: The operator (10) is periodic in α. It is because of this feature that the kq representation is so suitable for the definition of an angular coordinate. Being built on quasicoordinates, this representation takes care of the periodic nature of the angle α in a straightforward way.

The connection between the operator (10) and the operators (1) and (2) is very simple. The operator $e^{i\alpha}$ is a unitary operator and is a sum of two commuting Hermitian operators:

$$e^{i\alpha}=\cos\alpha+i\sin\alpha, \tag{11}$$

where $\cos\alpha$ and $\sin\alpha$ are given by (1) and (2), respectively. The eigenvalues of $e^{i\alpha}$ define, therefore, the eigenvalues of $\cos\alpha$ and $\sin\alpha$ and vice versa.[8] From here it is clear that the angle α is completely and consistently defined by the eigenvalues either of $e^{i\alpha}$ or of $\cos\alpha$ and $\sin\alpha$. According to Ref. 2, the eigenvalues of $e^{i\alpha}$ are $e^{i\alpha'}$, where α' varies from 0 to 2π. The eigenfunction of $e^{i\alpha}$ corresponding to the eigenvalue $e^{i\alpha'}$ is[2]

$$\psi_{\alpha'}(\alpha)=\sum_n \delta(\alpha-\alpha'-2\pi n)=\frac{1}{2\pi}\sum_n e^{i(\alpha-\alpha')n}. \tag{12}$$

The functions (12) form a complete and orthonormal set

$$\int \psi_{\alpha'}^*(\alpha)\psi_{\alpha''}(\alpha)d\alpha=\sum_n \delta(\alpha'-\alpha''-2\pi n). \tag{13}$$

[7] D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1951).

[8] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1958).

This completes the construction of a quantum-mechanical representation based on the operator $e^{i\alpha}$ in (10).

A number of remarks can be made on the commutation relation for the angular-momentum–angle coordinates. As was already pointed out, the usual relation [relation (3)] is contradictory and cannot be used. From relations (1), (2), and (4) it follows that

$$[l_z, \cos\alpha] = i \sin\alpha, \tag{14}$$

$$[l_z, \sin\alpha] = -i \cos\alpha, \tag{15}$$

$$[\sin\alpha, \cos\alpha] = 0. \tag{16}$$

Since $\cos\alpha$ and $\sin\alpha$ define the angle α, their uncertainties will define the uncertainty in α. The commutation relations (14) and (15) have already been suggested[1,5] as a means of avoiding the inconsistencies contained in relation (3). The uncertainty relations that follow from (14) and (15) are[1]

$$\Delta l_z \Delta(\cos\alpha) \geq \tfrac{1}{2}\langle \sin\alpha \rangle, \tag{17}$$

$$\Delta l_z \Delta(\sin\alpha) \geq \tfrac{1}{2}\langle \cos\alpha \rangle. \tag{18}$$

For states that are well localized around α_0, namely, $\langle \sin\alpha \rangle \approx \sin\alpha_0$, $\langle \cos\alpha \rangle \approx \cos\alpha_0$, $\Delta(\cos\alpha) \approx \sin\alpha_0 \Delta\alpha$, and $\Delta(\sin\alpha) \approx \cos\alpha_0 \Delta\alpha$, relations (17) and (18) lead to the conventional uncertainty relation[6]

$$\Delta l_z \Delta\alpha \geq \tfrac{1}{2}. \tag{19}$$

It is to be noticed that in none of the relations [relations (1), (2), and (10)–(15)] defining the angle α, does α itself appear. In all these relations, the angle appears through either $e^{i\alpha}$ or $\cos\alpha$ and $\sin\alpha$. The same is also true about the eigenfunctions of l_z:

$$\psi_m(\alpha) = e^{im\alpha}/\sqrt{(2\pi)}. \tag{20}$$

We see therefore that the kq representation leads to a consistent definition of the angular coordinate: It leads to an operator [operator (10)] that forms a complete system of commuting operators,[8] it is periodic in α, and its eigenfunctions (12) form a complete set of functions. These requirements should be satisfied by any operator used for defining the angle α in quantum mechanics.

It is important to point out that the definition of the angle α [relations (1) and (2)] holds in both classical and quantum mechanics. As will be seen later, this is connected with the fact that relations (1) and (2) do not contain noncommuting operators [this is not true for the definition of the phase angle, Eqs. (29) and (30)]. The classical Poisson brackets for l_z, $\cos\alpha$, and $\sin\alpha$ are obtained from relations (14)–(16) according to the general rule[8] by dividing the right-hand sides of these relations by i ($\hbar$ is assumed equal to 1). There is, therefore, the regular correspondence between classical and quantum mechanics[8] in the definition of the angle in regular space. As will be seen in Sec. III, this correspondence is violated in the definition of the phase.

For future reference, let us write down the results of this section in the language of "second quantization." It is possible to define operators $b^\dagger$ and b that in some sense resemble properties of the creation and annihilation operators

$$b^\dagger = e^{i\alpha}, \tag{21}$$

$$b = e^{-i\alpha}. \tag{22}$$

The operator (21) is the same as in (10). It can be easily checked that

$$b^\dagger |m\rangle = |m+1\rangle, \tag{23}$$

$$b|m\rangle = |m-1\rangle, \tag{24}$$

where $|m\rangle$ is an eigenstate of l_z corresponding to the eigenvalue m. As seen from (23) and (24), the operators $b^\dagger$ and b behave like creation and annihilation operators correspondingly. However,

$$b^\dagger b = bb^\dagger = 1, \tag{25}$$

which means that

$$[b, b^\dagger] = bb^\dagger - b^\dagger b = 0. \tag{26}$$

For real annihilation and creation operators, the commutator does not vanish[8] [see also relation (46), in Sec. III]. It can also be verified that

$$[l_z, b^\dagger] = b^\dagger, \tag{27}$$

$$[l_z, b] = -b. \tag{28}$$

The relation (26) is equivalent to (16), while either (27) or (28) is equivalent to the relations (14) and (15) together. This second-quantization notation will also be used in the next section.

III. AN ANGLE IN PHASE PLANE

An angle in the xp plane (phase plane) can be defined in a way similar to the definition given by relations (1) and (2) in regular space. Here, x is the coordinate of a particle and p its momentum. In order to have the same dimensions in both directions, $m\omega x$ and p will be used as coordinates (m is the mass of the particle and ω has the frequency dimension). In analogy with (1) and (2), an angle φ in the $(m\omega x, p)$ plane is defined as follows:

$$\cos\varphi = m\omega x / [(m\omega x)^2 + p^2]^{1/2}, \tag{29}$$

$$\sin\varphi = -p / [(m\omega x)^2 + p^2]^{1/2}. \tag{30}$$

The reason for the minus sign in (30) will become clear later. Unlike in regular space where the definitions (1) and (2) were valid both classically and quantum-mechanically, the formulas (29) and (30) are meaningful only in classical mechanics. The reason for this is that (29) and (30) contain noncommuting quantities (x and p), and the meaning of the division sign is not clear unless properly described. Before going to the quantum-mechanical definition, let us first complete the classical case. It is known[9] that in classical mechanics one can define a quantity P,

$$P=(2m\omega)^{-1}[(m\omega x)^2+p^2], \tag{31}$$

which is conjugate to the angle φ in the phase plane—i.e., the classical Poisson brackets for P and φ are

$$\{P,\varphi\}=-1. \tag{32}$$

One can also verify that

$$\{P, \cos\varphi\}=\sin\varphi, \tag{33}$$

$$\{P, \sin\varphi\}=-\cos\varphi, \tag{34}$$

$$\{\cos\varphi, \sin\varphi\}=0. \tag{35}$$

Relations (33)–(35) are the same as the relations satisfied by the classical quantities l_z, $\cos\alpha$, and $\sin\alpha$, correspondingly [the latter relations can be obtained as was mentioned before from (14)–(16) by dividing the right-hand sides by i]. It is for this reason that sine in (30) was defined with a minus sign. Classically, there is, therefore, a complete analogy between the angle–angular-momentum commutation relations in regular space and φ-P commutation relations in the phase plane. This analogy goes even further. It is known that the component of the angular momentum in the z direction, l_z, is a generator of an infinitesimal rotation in the xy plane. Similarly, one can check that P given by (31) is a generator of infinitesimal rotations in the $(m\omega x,p)$ plane. Indeed, the infinitesimal transformation caused by P is[9]

$$\delta p=(m\omega x)d\varphi, \tag{36}$$

$$\delta(m\omega x)=pd\varphi, \tag{37}$$

where the infinitesimal parameter was denoted by $d\varphi$. The transformation (36), (37) expresses a rotation by $d\varphi$ in the $(m\omega x,p)$ plane. The analogy between l_z and α on one side, and P and φ on the other, is therefore complete.

It is very interesting to note that the quantities P and φ are closely related to the Hamiltonian and phase of a harmonic oscillator. Indeed, the Hamiltonian of a harmonic oscillator is

$$H=p^2/2m+\tfrac{1}{2}m\omega^2x^2. \tag{38}$$

The solution of (38) can be given by

$$m\omega x=(2m\omega P)^{1/2}\cos\varphi, \tag{39}$$

$$p=-(2m\omega P)^{1/2}\sin\varphi, \tag{40}$$

where P is given by (31) and has the meaning of H/ω, and φ is the phase of the harmonic oscillator. Relations (39) and (40) coincide with the definition of φ in (29) and (30), and we see, therefore, that the angle φ in phase space is in classical mechanics also the phase of a harmonic oscillator. Since very many problems in classical physics and in particular in quantum mechanics are represented by the harmonic oscillator, the investigation of the angle in the phase plane is of fundamental importance.

The treatment of an angle in the phase plane in quantum mechanics is connected with some difficulties which are caused by the noncommutativity of x and p. One difficulty (which is not a fundamental one) is due to the symmetrization procedure for defining operators from the classical expressions (29)–(31). There is no problem with finding the operator P from (31):

$$P=\tfrac{1}{2}(aa^\dagger+a^\dagger a)=n+\tfrac{1}{2}, \tag{41}$$

where n is the number operator for a harmonic oscillator[8] and a, $a^\dagger$ are annihilation and creation operators:

$$a=[(2m\omega)^{-1/2}](m\omega x+ip), \tag{42}$$

$$a^\dagger=[(2m\omega)^{-1/2}](m\omega x-ip). \tag{43}$$

The operators a and $a^\dagger$ have the following properties[8]:

$$a|n\rangle=n^{1/2}|n-1\rangle, \tag{44}$$

$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle, \tag{45}$$

$$[a,a^\dagger]=1. \tag{46}$$

They are real annihilation and creation operators and differ therefore from b and $b^\dagger$ defined in (21) and (22).

The problem, however, is not so straightforward in regard to symmetrizing the expressions for $\cos\varphi$ and $\sin\varphi$. In a recent paper by Lerner,[10] it was shown that there is no unique way for symmetrizing these classical expressions. In order to clarify this, let us define the following expressions:

$$e^{i\varphi}=\cos\varphi+i\sin\varphi=a^*/P^{1/2}, \tag{47}$$

$$e^{-i\varphi}=\cos\varphi-i\sin\varphi=a/P^{1/2}, \tag{48}$$

where a and a^* are the classical expressions for the operators a and $a^\dagger$ in (42) and (43), correspondingly, and P is given by (31). In Refs. 4–6, the operator corresponding to the classical expression (47), for example, is defined as

$$e^{i\varphi}=a^\dagger(n+1)^{-1/2}. \tag{49}$$

Expression (49) corresponds to replacing P by $(n+1)$ in (47). However, we know that the symmetrized expression for P is given by (41) and equals $n+\frac{1}{2}$. Thus, taking $P=n+1$ in (49) corresponds to the assumption that P is given by $aa^\dagger$ and not by the symmetrized expression (41). The denominator in (49) is a "little bit" bigger than the correct symmetrized expression. Clearly, one could choose for P the expression $a^\dagger a=n$, and then the denominator in (45) would be $n^{1/2}$, which is a "little bit" smaller than the symmetrized expression. As shown in Ref. 10, many different choices are possible. The problem is, therefore, to find symmetrized ex-

[9] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co.. Inc., Reading, Mass., 1950).

[10] E. Lerner, Nuovo Cimento **56B**, 183 (1968).

pressions for $C(\varphi)$ and $S(\varphi)$ that lead to reasonable physical results.

In this paper we choose the following properly symmetrized operators for the classical expressions (47) and (48):

$$E(\varphi)=a^{\dagger}f(n), \quad (50)$$

$$E(-\varphi)=f(n)a, \quad (51)$$

where

$$f(n)=(n+1)^{1/2}/(n+\tfrac{1}{2})^{1/2}(n+\tfrac{3}{2})^{1/2}. \quad (52)$$

The operator $E(\varphi)$ corresponds to the classical expression $e^{i\varphi}$ [cf. Eq. (47)] and $E(-\varphi)$ corresponds to $e^{-i\varphi}$. The reason one does not denote the operators (50) and (51) by $e^{i\varphi}$ and $e^{-i\varphi}$, correspondingly, is because these operators are not unitary.[4] The choice of $f(n)$ in the definitions (50) and (51) is based partly on the symmetrization of the expressions (47) and (48) and partly on a heuristic argument that is justified by the results given below. In previous work,[4–6] the function $f(n)$ according to (49) is [call it $\bar{f}(n)$]:

$$\bar{f}(n)=(n+1)^{-1/2}. \quad (53)$$

For large n, the functions in (52) and (53) coincide, whereas for small n, they are, of course, different.

In a similar way one can also define operators $C(\varphi)$ and $S(\varphi)$ that correspond to $\cos\varphi$ and $\sin\varphi$ in (29) and (30), respectively. One has

$$C(\varphi)=\tfrac{1}{2}[a^{\dagger}f(n)+f(n)a], \quad (54)$$

$$S(\varphi)=(2i)^{-1}[a^{\dagger}f(n)-f(n)a]. \quad (55)$$

From this definition it is clear that $C(\varphi)$ and $S(\varphi)$ are constructed from $E(\varphi)$ and $E(-\varphi)$ as if the latter were exponential operators. Now we come to another difficulty that follows from the noncommutativity of x and p: The operators $C(\varphi)$ and $S(\varphi)$, as can be verified [see relation (71) in Sec. V], do not commute. As was mentioned before, an angle in regular space or in the phase plane can be defined by measuring the cosine and the sine of the angle. In the case of the regular space this could be done both in classical and quantum mechanics [$\cos\alpha$ and $\sin\alpha$ can be measured simultaneously, since they commute; see relation (16)]. Classically, there is a complete analogy between α and φ. Quantum-mechanically, this analogy is broken. There exist operators ($\cos\alpha$ and $\sin\alpha$) which measure the angle α in regular space, while the corresponding operators in the phase plane cannot be measured together. One thus comes to the conclusion that a coordinate perfectly well-defined classically, i.e., the angle in the phase plane and the phase of a harmonic oscillator, is not an observable in quantum mechanics.

Since the operators $C(\varphi)$ and $S(\varphi)$ do not commute, the question can be asked whether each of them separately defines a quantum-mechanical representation. Let us show that the answer is affirmative. To do so we write down the eigenvalue equation for $C(\varphi)$ [the treatment of $S(\varphi)$ is similar]

$$C(\varphi)|\cos\varphi'\rangle=\cos\varphi'|\cos\varphi'\rangle. \quad (56)$$

One can expand the vectors in (56) in the eigenvectors $|n\rangle$ of the harmonic oscillator[4]

$$|\cos\varphi'\rangle=\sum_{n=0}^{\infty}C_n(\varphi')|n\rangle. \quad (57)$$

From (56) and (57) and from the definition (54) of $C(\varphi)$, one has

$$\left(\frac{n+\frac{3}{2}}{n-\frac{1}{2}}\right)^{1/2} nC_{n-1}+(n+1)C_{n+1} -2\cos\varphi'\,[(n+\tfrac{1}{2})(n+\tfrac{3}{2})]^{1/2}C_n=0. \quad (58)$$

One can easily verify that relation (58) is exactly the recursion formula for normalized Legendre polynomials.[11] The eigenstates of $C(\varphi)$ are therefore

$$|\cos\varphi'\rangle=\sum_{n=0}^{\infty}\varphi_n(\cos\varphi')|n\rangle, \quad (59)$$

where the eigenvalues $\cos\varphi'$ assume values from -1 to 1 and $\varphi_n(\cos\varphi')$ is a normalized Legendre polynomial[11]:

$$\varphi_n(z)=(n+\tfrac{1}{2})^{1/2}(1/2^n n!)(d^n/dz^n)(z^2-1)^n. \quad (60)$$

Since the Legendre polynomials form a complete and orthonormal set, the operator $C(\varphi)$ defines a quantum-mechanical representation: It is a Hermitian operator with eigenfunctions that form a complete set. The eigenvalues of $C(\varphi)$ form a continuous spectrum from -1 to 1 (an independent proof of the spectrum is provided by Lerner's results[10]), and they can therefore be denoted by $\cos\varphi'$. The angle φ' varies from 0 to π. It defines an angle measured from the positive direction of the x axis (in three-dimensional space this corresponds to a polar angle).

Similar results are obtained for the operator $S(\varphi)$. Its eigenfunctions are

$$|\cos\varphi''\rangle=\sum_{n=0}^{\infty}(-1)^n\varphi_n(\cos\varphi'')|n\rangle. \quad (61)$$

The eigenvalues of $S(\varphi)$ range from -1 to 1 and they can therefore be denoted by $\cos\varphi''$ with φ'' varying from 0 to π and defining an angle measured from the negative direction of the p axis. There are two reasons for such a "strange" notation [to denote the eigenvalues of the sine operator $S(\varphi)$ by $\cos\varphi''$]: First, it is denoted by cosine in order to give the meaning of the angle that is measured by the operator $S(\varphi)$, namely, it measures the angle from the $-p$ axis in the xp plane [the minus direction of the p axis is chosen in correspondence with the classical definition (30)]; secondly, it is denoted

[11] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966).

by φ'' in order to show that $S(\varphi)$ measures an angle different from the one measured by $C(\varphi)$. In the classical case, $\cos\varphi$ and $\sin\varphi$ measure together the angle φ. As was mentioned before, in quantum mechanics this is impossible because $C(\varphi)$ and $S(\varphi)$ do not commute. What was shown here is that each of these operators separately defines an angle: $C(\varphi)$ defines an angle measured from the x axis, and $S(\varphi)$ an angle measured from the $-p$ axis.

As was mentioned before, in previous work[4–6] a different definition of the cosine and sine operators was given. For example, the cosine operator was defined as follows:

$$\cos\phi=\tfrac{1}{2}[a^\dagger \bar{f}(n)+\bar{f}(n)a], \tag{62}$$

with $\bar{f}(n)$ given by (53). The eigenstate of this operator corresponding to the eigenvalue $\cos\theta$ was found[4] to be

$$\sum_n \sin(n+1)\theta|n\rangle. \tag{63}$$

In the semiclassical limit (large n), the symmetrized expression (54) and definition (62) coincide. In the quantum limit (small n), however, they are different. For example, $f(0)=2/\sqrt{3}$, while $\bar{f}(0)=1$. This slight change, however, makes a very big difference. Unlike solution (63), the coefficients in the eigenstates (59) of $C(\varphi)$ are the Legendre polynomials. As will be shown in Sec. IV, this leads to a profound connection between rotations in regular space and in phase space.

Having established that each of the operators $C(\varphi)$ and $S(\varphi)$ forms a quantum-mechanical representation, it is interesting to find out the connection of the latter with other well-known representations. Relations (59) and (61) give the eigenfunctions of $C(\varphi)$ and $S(\varphi)$, respectively, expressed in the eigenfunctions of a harmonic oscillator. From (59), for example, one can find $|n\rangle$ expressed in the eigenfunctions of $C(\varphi)$:

$$|n\rangle=\int d(\cos\varphi)\varphi_n(\cos\varphi)|\cos\varphi\rangle. \tag{64}$$

From the expressions (54) and (55) it is very simple to write $C(\varphi)$ and $S(\varphi)$ in the n representation. For example,

$$\langle n|C(\varphi)|n'\rangle=\tfrac{1}{2}[\delta_{n',n-1}n^{1/2}f(n-1)+\delta_{n',n+1}(n+1)^{1/2}f(n)]. \tag{65}$$

A very interesting result follows from Eq. (59): The Legendre polynomial $\varphi_n(\cos\varphi)$ is the eigenfunction of a harmonic oscillator (for the state n) in the $\cos\varphi$ representation. Since $\varphi_n(\cos\varphi)$ is a Legendre polynomial, it satisfies the following equation[11]:

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\varphi_n(\varphi)}{d\varphi}\right)+n(n+1)\varphi_n(\varphi)=0. \tag{66}$$

From (66) it follows that the operator $n(n+1)$ in the $\cos\varphi$ representation is as follows:

$$n(n+1)=-\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d}{d\varphi}\right). \tag{67}$$

While the expression (67) for $n(n+1)$ is very simple, the expression for n itself is not simple at all. It should be mentioned here that if n and φ were conjugate coordinates (as they are in classical mechanics), then n would be given by $-i\partial/\partial\varphi$.

One can also express the connection between the $\cos\varphi$ representation and, for example, the x representation. Of particular interest is the expression for the eigenfunction (59) in the x representation. One has

$$\langle x|\cos\varphi\rangle=\sum_{n=0}^{\infty}\varphi_n(\cos\varphi)\psi_n(x), \tag{68}$$

where $\psi_n(x)$ are the Hermite functions. Relation (68) gives a definition of a new function, $\langle x|\cos\varphi\rangle$, as a series of products of well-known functions. From the definition (68) it is clear that the functions $\langle x|\cos\varphi\rangle$ form a complete and orthonormal set of functions.

The operators $C(\varphi)$ and $S(\varphi)$ in (54) and (55) and the number operator n for a harmonic oscillator [see relation (41)] satisfy the following commutation relations:

$$[n,C(\varphi)]=iS(\varphi), \tag{69}$$

$$[n,S(\varphi)]=-iC(\varphi), \tag{70}$$

$$[C(\varphi),S(\varphi)]=\frac{1}{2i}\left(\frac{n^2}{n^2-\frac{1}{4}}-\frac{(n+1)^2}{(n+1)^2-\frac{1}{4}}\right). \tag{71}$$

The first two relations are the same as for the operators l_z, $\cos\alpha$, and $\sin\alpha$ in (14) and (15). Relation (71) differs, however, from the corresponding relation for $\cos\alpha$ and $\sin\alpha$ [relation (16)]. Because of the noncommutativity of $C(\varphi)$ and $S(\varphi)$, these operators cannot be used for defining the phase of a harmonic oscillator. This can also be expressed in a different way: From the definitions (50), (51), (54), and (55) it is clear that

$$E(\varphi)=C(\varphi)+iS(\varphi), \tag{72}$$

$$E(-\varphi)=C(\varphi)-iS(\varphi). \tag{73}$$

That it is impossible to define an operator for the phase follows also from the nonunitarity[4] of $E(\varphi)$. For a unitary $E(\varphi)$, one can write $E(\varphi)=e^{i\varphi}$ with φ a Hermitian operator, and this would lead to a definition of φ, the phase. However, since $E(\varphi)$ is not unitary:

$$E(\varphi)E(\varphi)^\dagger=E(\varphi)E(-\varphi)=n^2/(n^2-\tfrac{1}{4}), \tag{74}$$

it cannot be used for defining φ.

Having the commutation relation (71), one can find the uncertainty relation for $C(\varphi)$ and $S(\varphi)$,

$$\Delta C(\varphi)\Delta S(\varphi)\geq\tfrac{1}{2}\langle g\rangle, \tag{75}$$

where

$$g=-\frac{1}{2i}\left(\frac{n^2}{n^2-\frac{1}{4}}-\frac{(n+1)^2}{(n+1)^2-\frac{1}{4}}\right). \quad (76)$$

The commutation relation for the cosine and sine operators was discussed before,[5] and a similar discussion can be carried out for the new operators $C(\varphi)$ and $S(\varphi)$. For example, for large n, g goes to zero and the commutator for $C(\varphi)$ and $S(\varphi)$ vanishes. In this limit, the phase φ can be well defined either by $C(\varphi)$ and $S(\varphi)$ or by the operator $E(\varphi)$ in (72).

IV. CONNECTION BETWEEN ROTATIONS IN REGULAR SPACE AND xp PLANE

As is well known, in regular space the l_z component and the square of the angular momentum l^2 have as their eigenfunctions spherical harmonics, $Y_{lm}(\theta,\Phi)$,[12] where m is the eigenvalue of l_z [see Eq. (20)] and $l(l+1)$ gives the eigenvalue of $\mathbf{l}^2$:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}(\theta,\Phi)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}(\theta,\Phi)}{\partial\Phi^2}+l(l+1)Y_{lm}(\theta\Phi)=0. \quad (77)$$

For $m=0$, Y_{l0} does not depend on Φ and Eq. (77) goes over into Eq. (66) with n replaced by l. This means that for $m=0$, the operator $\mathbf{l}^2$ is given by expression (67) with φ replaced by θ:

$$\mathbf{l}^2=-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right). \quad (78)$$

Since the eigenvalues of $\mathbf{l}^2$ are $l(l+1)$, there is a close analogy between Eqs. (67) and (78). To see this analogy better, let us come back to the definition of φ and θ. Although the angles φ [in Eq. (67)] and θ [in Eq. (77)] were introduced in different ways, it is easy to show that they can be given the same meaning. Originally, φ was defined as an angle in the xp plane, while θ is the polar angle in regular space. However, in the latter case, when $m=0$, the only meaning that θ has is that of an angle measured from the z axis. It is possible therefore to add to the z axis a p_z axis, to define in such a way a zp_z plane, and to assign to θ the meaning of an angle measured from the z axis in the zp_z plane. Such a definition gives θ the same meaning that φ (the phase angle) has. Having established this possible meaning of θ, one can go on and find a very far reaching connection between l and n. Usually, in rotations in regular space, $\mathbf{l}^2$ has the meaning of an operator, while l appears as a number so that $l(l+1)$ gives the eigenvalues of $\mathbf{l}^2$. From the identity of Eqs. (67) and (78), from the similar meaning of φ and θ, and from the fact that $\mathbf{l}^2=l(l+1)$, it follows that l can be given the meaning of an operator, i.e., it can be given by the same operator in the zp_z plane as n is given in the xp plane. From (41) and (31) one has

$$n=\tfrac{1}{2}mw[(m\omega x)^2+p^2]-\tfrac{1}{2}. \quad (79)$$

Therefore,

$$l=\tfrac{1}{2}mw[(m\omega x)^2+p_z^2]-\tfrac{1}{2}. \quad (80)$$

It is easy to check that $l(l+1)$, with l defined according to (80), leads back to the operator $\mathbf{l}^2$ in (78). To see this, one can write the definition of $\cos\theta$ in a way similar to that given in (54), with n replaced by l and with x replaced by z and p by p_z in the definition of a and $a^\dagger$ [relations (42) and (43)]. Since the operator n defined by (79) leads to relation (67), the operator (80) will lead to the right-hand side of (78) and, therefore, for $m=0$, the operator $\mathbf{l}^2$ can be given by $l(l+1)$ with l defined by (80). This shows that not only $\mathbf{l}^2$ but also l has the meaning of an operator.

The result (80) has a very simple interpretation. The function $Y_{l0}(\theta)$ in Eq. (77) can be looked at as a state for one degree of freedom (the azimuth and the absolute value of the radius vector do not appear in this function). This degree of freedom is described by the angle θ which is measured from the z axis. Since $Y_{l0}(\theta)$ satisfies Eq. (66) with n replaced by l, it is clear that for a one-dimensional motion l and θ can be given the same meaning as n and φ have for rotations in the phase plane; i.e., l can be given the meaning of a generator of infinitesimal rotations in the zp_z plane (in complete analogy with n), while $\cos\theta$ can be given by the expression (54) in the zp_z plane.

V. DISCUSSION

In textbooks on quantum mechanics one can usually find the statement that observables in quantum mechanics and their commutation relations can "easily" be constructed from their classical counterparts. In particular, the quantum commutation relations for two observables can be obtained from their classical Poisson bracket by multiplying the latter by $i\hbar$. This rule, although in general correct is not without exception, as shown in this paper. The quasicoordinate approach to the angular coordinate shows that there is a very simple correspondence between the classical and quantum-mechanical definition of the angle in regular space. However, with respect to the angle in the xp plane, the analogy between classical and quantum mechanics is broken. As is shown in this paper, one can define classically an angle in the phase plane [Eqs. (29) and (30)] which has no quantum-mechanical analog. This angle also has the meaning of the phase of a harmonic oscillator and is an example of a well-defined classical quantity that cannot be expressed by a Hermitian operator in quantum mechanics. The above-mentioned

[12] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Co., Reading, Mass., 1958).

rule (its formulation is never very clear) does not seem, therefore, to be universal.

The quantum-mechanical treatment of an angular coordinate in the xp plane leads, as was shown in this paper, to a new representation. Although this representation was developed on the x and p coordinates, it is clear that similar results can be obtained for any pair of conjugate coordinates, because the only information that was used about x and p is their commutation relation. As an example one can mention the kinetic momentum for an electron in a magnetic field. It is known[2] that the components of the kinetic momentum in the plane perpendicular to the magnetic field form conjugate coordinates. The new representation (the phase-angle representation) introduced in this paper is therefore of a very general nature.

In addition to having possible practical applications, the new representation sheds light on the general understanding of rotations in quantum mechanics. Of particular interest is the connection between the generator of rotations in the xp plane and rotations in regular space. This connection is expressed by the fact that l, which defines the eigenvalues of the square of the angular momentum, $\mathbf{l}^2$ [$\mathbf{l}^2=l(l+1)$], can be given the meaning of a generator for rotations in the zp_z plane [see formula (80)].

Finally, one should remark about the importance of the new representation in a number of fields of physics. We have discussed the phase-angle representation in connection with the definition of phase for a harmonic oscillator. Since the phase plays a dominant role in coherent phenomena like laser physics, superconductivity, and superfluidity, one would expect the new representation to be of wide use in these fields of physics.

APPENDIX

A discussion is given here of the eigenvalue equation for the cosine operator defined in previous work[4] [Eq. (62)]:

$$\cos\phi=\tfrac{1}{2}[a^\dagger(n+1)^{-1/2}+(n+1)^{-1/2}a]. \qquad \text{(A1)}$$

The eigenvalue equation for this operator

$$\cos\phi|\lambda\rangle=\lambda|\lambda\rangle, \qquad \text{(A2)}$$

when written in the n representation (the harmonic oscillator representation), becomes[4] [for details see derivation of Eq. (58)]

$$C_n+C_{n+2}=2\lambda C_{n+1}, \qquad \text{(A3)}$$

with the boundary condition

$$C_1=2\lambda C_0. \qquad \text{(A4)}$$

It is known[11] that equation (A3) with the boundary condition (A4) is satisfied by Chebyshev polynomials of the second kind,

$$U_n(\cos\theta)=\sin(n+1)\theta/\sin\theta. \qquad \text{(A5)}$$

The solution (A5) corresponds to the eigenvalue $\lambda=\cos\theta$ in (A3) and to $C_0=1$ in (A4). The Chebyshev polynomials of the second kind, $U_n(x)$, are known to form a complete and orthonormal set of functions in the interval $(-1, +1)$ with the weight function $(1-x^2)^{1/2}$:

$$\int_{-1}^{1} U_n(x)U_{n'}(x)(1-x^2)^{1/2}dx=\tfrac{1}{2}\pi\delta_{nn'}, \qquad \text{(A6)}$$

$$\sum_{n=0}^{\infty} U_n(x)U_n(x')(1-x^2)^{1/2}=\delta(x-x'). \qquad \text{(A7)}$$

In (A6) and (A7), $\cos\theta$ of (A5) is replaced by x. For an angle θ varying in a plane, the weight function will be $\sin^2\theta$, because, for example, the integral (A6) can be written

$$\int_0^{\pi} \frac{\sin(n+1)\theta}{\sin\theta}\,\frac{\sin(n'+1)\theta}{\sin\theta}\,\sin^2\theta d\theta=\tfrac{1}{2}\pi\delta_{nn'}. \qquad \text{(A8)}$$

The Chebyshev polynomials of the second kind give, therefore, a consistent solution of Eqs. (A3) and (A4) with an eigenvalue spectrum ($\lambda=\cos\theta$) ranging from -1 to $+1$. Formally everything is correct: The problem is, however, that physically the functions $U_n(\cos\theta)$ cannot be interpreted as wave functions because for their normalization a weight function $\sin\theta$ is needed (the physical variable is $\cos\theta$). This means that $U_n(\cos\theta)$ do not have the meaning of probability distribution. In Ref. 4 the solutions of Eqs. (A3) and (A4) were chosen as

$$\sin(n+1)\theta. \qquad \text{(A9)}$$

This corresponds to the choice $C_0=\sin\theta$ in (A4). Such a choice is possible as long as $\sin\theta\neq 0$. When $\sin\theta=0$, we have $C_0=0$, and $C_1=0$ [from Eq. (A4)], and this results in a trivially zero solution. This can also be seen from (A9): $\sin\theta=0$ means $\theta=0, \pi$ and for these values of the angle the solution (A9) is trivially zero. This is clearly not physical because the operator $\cos\phi$ [Eq. (A1)] of Ref. 4 is used for defining the phase which supposedly has nonvanishing eigenstates for all possible values of the angle.