Motion and Structure of Singularities in General Relativity

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(Received 1 July 1969)

A new approach to equations of motion in general relativity is presented. Unlike the usual approaches, which rely on a regular background, this approach, by means of the method of spin coefficients, is based on the properties of the null cones emanating from a singular world line. We apply it to the Robinson-Trautman metrics as well as to their charged counterparts, and obtain exact equations of motion that give an intrinsic description of the behavior of a singularity in its own space-time. Two results of this approach are the discovery of an internal structure for a singularity and the surprising appearance of the Abraham radiation-reaction term in the charged case.

I. INTRODUCTION

THIS is the first paper of a series in which a new approach to equations of motion in general relativity will be presented. The point of view adopted in this approach is that matter is to be represented by suitably defined singularities (elementary singularities) in the field, the field being the Weyl tensor. The problem then is to give a description of the motion of a singularity, intrinsic to its own space-time, with no reference to a regular background space as is usually done.[1] Such a description is provided by an approach that is based on the structure of the null cone in the neighborhood of the singularity. The application of Einstein's field equations in spin-coefficient formalism yields exact equations of motion for these singularities. A concomitant result, unsuspected by us, is the discovery of an internal structure for these elementary singularities and equations governing the time development of this structure.

In this paper we will restrict ourselves to the special case in which the elementary singularities are free, i.e., not interacting with incoming fields. This leads to the Robinson-Trautman type-II metrics,[2–4] and to the analogous case in the Einstein-Maxwell theory, namely, the charged Robinson-Trautman (RT) metrics.

In Sec. II we discuss the notation and general formalism. It will be assumed that, to a large extent, the spin-coefficient formalism[5] is known. Section III will be devoted to the analysis and interpretation of the RT metrics. Section IV will be concerned with the charged counterpart of the RT metrics. Here we get the first significant result of our approach, namely, the rigorous derivation of the well-known Abraham radiation-reaction force,[6] with no *ad hoc* assumptions and no mass renormalization.

Future papers in the series will discuss the introduction of an intrinsic dipole moment and angular momentum to the singularity and their effect on its motion. The interaction of the singularity with both incoming gravitational and electromagnetic fields will also be discussed.

II. GENERAL FORMALISM

The basis for our analysis of motion is the structure of the null cones emanating from a singular world line in space-time. With this as motivation, we consider a family of null hypersurfaces, each labeled by $u=\text{const}$, with an affine parameter r measuring "distance" along the null geodesics lying in the hypersurfaces, and two "angular" coordinates x^i ($i=2,3$) labeling the geodesics; that is, we construct a null coordinate system.[5] (At this point the null surfaces are arbitrary, but later they will be made definite, giving a geometric invariance to the work.)

We now introduce a standard null tetrad system[5] associated with this null coordinate system; that is, we define the set of vectors l^μ, n^μ, m^μ, $\bar{m}^\mu$, where l^μ (tangent to the null geodesics) and n^μ are real null vectors, m^μ and its complex conjugate $\bar{m}^\mu$ are complex null vectors, and

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1\,,$$

with all other scalar products vanishing. In the coordinate system constructed above ($x^0=u$, $x^1=r$, x^2, x^3), these vectors take the form

$$l_\mu=\delta_\mu{}^0\,,\quad l^\mu=\delta_1{}^\mu\,,\quad m^\mu=\omega\delta_1{}^\mu+\xi^i\delta_i{}^\mu\,,$$
$$n^\mu=\delta_0{}^\mu+U\delta_1{}^\mu+X^i\delta_i{}^\mu\,,\quad i=2,3. \tag{2.1}$$

The contravariant components of the metric tensor are

* Research sponsored in part by the Aerospace Research Laboratories, through the European Office of Aerospace Research, OAR, U. S. Air Force, under Contract No. F61052-69-C-0012, and through Wright-Patterson Air Force Base, under Contract. No. F33615-70-C-1081.

† Research carried out under a Rockefeller Foundation scholarship and a University of Pittsburgh travel grant.

[1] For reviews of the usual approaches to equations of motion in general relativity, see J. N. Goldberg, in *Gravitation*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 102; W. Tulczyjew, in *Proceedings of the 1965 International Conference on Relativistic Theories of Gravitation* (King's College, London, 1965), Vol. II.

[2] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **265**, 463 (1962).

[3] E. Newman and L. Tamburino, J. Math. Phys. **3**, 902 (1962).

[4] J. Foster and E. Newman, J. Math. Phys. **8**, 189 (1967).

[5] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

[6] See, e.g., F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965).

given by[7]

$$g^{\mu 0}=\delta_1{}^\mu\,,\quad g^{11}=2(U-\omega\bar{\omega})\,,$$
$$g^{1i}=X^i-(\xi^i\bar{\omega}+\bar{\xi}^i\omega)\,, \tag{2.2}$$
$$g^{ik}=-(\xi^i\bar{\xi}^k+\bar{\xi}^i\xi^k)\,.$$

The spin coefficients ρ and σ defined by[5]

$$\rho=l_{\mu;\nu}m^\mu\bar{m}^\nu=-\tfrac{1}{2}l^\mu{}_{;\mu}\,,$$
$$\sigma=l_{\mu;\nu}m^\mu m^\nu=\text{(complex shear)}$$

play a significant role in what follows.

A severely restrictive condition on the family of null surface is now imposed, namely, that near $r=0$,

$$\rho=-r^{-1}+O(r)\,,\quad \sigma=O(r)\,. \tag{2.3}$$

Geometrically, this condition means that in the neighborhood of the origin, the null surfaces behave like "cones." (In fact, in flat space, light cones are characterized by $\rho=-1/r$ and $\sigma=0$.) It also fixes the origin for r. When condition (2.3) is satisfied, and the Weyl tensor is singular at $r=0$, we call the singularity an *elementary singularity*.

The importance of condition (2.3) lies in the fact that it permits us to prove that the two-surfaces, u and r constant (that is, the cross sections of the cones), possess a metric given by

$$g_{ij}=g^0{}_{ij}r^2+O(r^3)\,,\quad g^0{}_{ij}=g^0{}_{ij}(u,x^k)\,. \tag{2.4}$$

The proof consists in first noting that from (2.2)

$$g_{ij}g^{jk}=\delta_i{}^k\,,$$

and then remembering that ξ^i satisfies[5]

$$\partial\xi^i/\partial r=\rho\xi^i+\sigma\bar{\xi}^i.$$

From this one can show that

$$\xi^i=\xi^{0i}/r+O(r)\,,$$

and, by taking the inverse of g^{ij}, obtain the result (2.4).

We define the fundamental two-surface (F2S) metric by

$$g^0{}_{ij}=\lim_{r\to 0}g_{ij}/r^2\,.$$

It can be proved (though we will not do so here) that, if the line $r=0$ is a *regular*, timelike (null, spacelike) line, then the F2S has a positive (zero, negative), constant Gaussian curvature.

The F2S is most conveniently represented in conformally flat coordinates. Thus, its line element[7] can be written as

$$dl^2=g^0{}_{ij}dx^idx^j=(1/2P^2)d\zeta d\bar{\zeta}\,,$$

where $\zeta=x^2+ix^3$. The quantity $P=P(u,\zeta,\bar{\zeta})$ will be our fundamental variable; from it, as we shall see later, is derivable, in principle, all information concerning the motion and internal structure of singularities. If the line $r=0$ is a singular line (i.e., the Weyl tensor becomes infinite on it), then the Einstein field equations yield differential equations for the determination of P. These are the equations that will be analyzed and the ones from which equations of motion and the time development of the internal structure will be extracted.

In the coordinate system we have adopted, the only coordinate freedom[8] that remains is the renaming of the null surfaces; formally, this is given by

$$u'=f(u)\,,\quad r'=r/\dot{f}\,,\quad \zeta'=g(u,\zeta)\,. \tag{2.5}$$

[Later, $g(u,\zeta)$ will be made into ζ by demanding that directions be parallel-transferred.] It can be seen from (2.5) that the F2S is defined up to an arbitrary factor $\dot{f}^2$. It will be shown later how this factor can be normalized to unity.

Before the effects of the Einstein field equations on the F2S can be considered, it is essential for interpretive reasons to consider first the *flat-space* case of this coordinate system associated with an arbitrary timelike world line.

Let y^μ be Minkowskian coordinates and $y^\mu=\xi^\mu(u)$ be the parametric form of an arbitrary timelike world line. We now consider the coordinate transformation from $y^\mu\to x^\mu=(u,r,x^2,x^3)$ given by

$$y^\mu=\xi^\mu(u)+rl^\mu(u,\zeta,\bar{\zeta})\,,\quad \zeta=x^2+ix^3 \tag{2.6}$$

with the conditions (see Ref. 9)

$$\dot{\xi}_\mu\dot{\xi}^\mu=2\,,\quad \dot{\xi}_\mu l^\mu=1\,,\quad l_\mu l^\mu=0\,. \tag{2.7}$$

For fixed u, the vector field $l^\mu(u,\zeta,\bar{\zeta})$ sweeps out the directions of the full null cone as ζ and $\bar{\zeta}$ vary over their range. One must now specify how the directions ζ and $\bar{\zeta}$ are to be propagated as u varies. This is most easily done by setting

$$l^\mu=b^\mu/P_0\,,\quad b^\mu=b^\mu(\zeta,\bar{\zeta})\,,\quad P_0=P_0(u,\zeta,\bar{\zeta})\,. \tag{2.8}$$

Differentiation with respect to u then leads to the propagation law

$$\dot{l}^\mu=-(\dot{P}_0/P_0)l^\mu\,, \tag{2.9}$$

expressing the parallel transfer of the direction of l^μ.

The metric tensor in the x^μ coordinates,

$$g_{\mu\nu}=(\partial y^\alpha/\partial x^\mu)(\partial y^\beta/\partial x^\nu)\eta_{\alpha\beta}\,,$$

is now found to have the following components:

$$g_{\mu 1}=\delta_\mu{}^0\,,\quad g_{00}=2(1+\dot{\xi}_\alpha\dot{l}^\alpha r)\,,$$
$$g_{0i}=0\,,\qquad g_{ij}=l_{\alpha,i}l^\alpha{}_{,j}r^2\,, \tag{2.10}$$

where use has been made of (2.7) and (2.9). Note that

[7] The factor 2 in the line element is for conventional reasons and the F2S metric is really defined by $d\zeta d\bar{\zeta}/P^2$.

[8] This can be seen by considering the infinitesimal coordinate transformations that preserve (2.4) and keep the line $r=0$ unchanged.

[9] The number 2 means that instead of being the proper time, $u=\frac{1}{2}\sqrt{2}$ times the proper time. This choice of normalization is adopted for purely notational reasons, i.e., to make the line element agree with E. T. Newman and R. Penrose, Proc. Roy. Soc. (London) A305, 175 (1968).

from (2.7)–(2.9),

$$P_0=\dot{\xi}_\mu b^\mu, \quad (2.11a)$$

$$\dot{P}_0/P_0=-\dot{\xi}_\mu \dot{l}^\mu=\ddot{\xi}_\mu l^\mu. \quad (2.11b)$$

Also,

$$l_{\alpha,i}l^\alpha{}_{,j}=b_{\alpha,i}b^\alpha{}_{,j}/P_0{}^2. \quad (2.12)$$

Since P_0 and b^μ were defined in (2.8) only up to an arbitrary u-independent factor, $b_{\alpha,i}b^\alpha{}_{,j}$ can be multiplied by a u-independent factor. Then, because all two-surfaces are conformal to the plane, we can choose

$$b_{\alpha,i}b^\alpha{}_{,j}=-\tfrac{1}{2}\delta_{ij}. \quad (2.13)$$

A solution to this equation is given by

$$b^\alpha=\tfrac{1}{4}\sqrt{2}(1+\zeta\bar{\zeta},\,1-\zeta\bar{\zeta},\,\zeta+\bar{\zeta},\,(\zeta-\bar{\zeta})/i). \quad (2.14)$$

Using (2.11)–(2.13), we can now write the four-dimensional line element as

$$ds^2=2[1-(\dot{P}_0/P_0)r]du^2+2dudr-r^2d\zeta d\bar{\zeta}/2P_0{}^2. \quad (2.15)$$

Thus, knowing the velocity and acceleration of a world line, one can construct uniquely [from (2.11) and (2.14)] the line element (2.15); conversely, knowing (2.15), or, in particular, *knowing the F2S, i.e.,* $P_0(u,\zeta,\bar{\zeta})$, *one can determine all the properties of the world line.*

For example, the maximum value of $\dot{P}_0/P_0$ on the sphere at each u is proportional to the magnitude of the acceleration, and the direction in which this maximum occurs is the direction of the acceleration, that is,

$$\max(\dot{P}_0/P_0)=\surd(\tfrac{1}{2}\ddot{\xi}^2)\,,\quad \ddot{\xi}^2=\ddot{\xi}_\mu\ddot{\xi}^\mu$$
$$\zeta_{\max}=\text{direction of }\ddot{\xi}^\mu. \quad (2.16)$$

This is seen from the fact that, from (2.11),

$$\dot{P}_0/P_0=\ddot{\xi}_\mu l^\mu=\tfrac{1}{2}\sqrt{2}\ddot{\xi}_\mu S^\mu,\quad S^\mu=\sqrt{2}l^\mu-(\dot{\xi}^\mu/\sqrt{2})\,,$$

where S^μ is the unit, radial, spacelike vector normal to $\dot{\xi}^\mu$. Thus, when S^μ is in the direction of the acceleration, $\dot{P}_0/P_0$ has its maximum. We can therefore give a unique statement of equivalence between the acceleration $\ddot{\xi}^\mu$ and $\dot{P}_0/P_0$, writing this as

$$\ddot{\xi}^\mu\leftrightarrow\dot{P}_0/P_0. \quad (2.17)$$

A similar statement can be made for that part of the derivative of the acceleration that is normal to the velocity vector, namely,

$$\dddot{\xi}^\mu+\tfrac{1}{2}\ddot{\xi}_\alpha\ddot{\xi}^\alpha\dot{\xi}^\mu\leftrightarrow\ddot{P}_0/P_0+\tfrac{1}{2}\ddot{\xi}^2. \quad (2.18)$$

To prove this, we first note that the u derivative of $\ddot{\xi}_\mu l^\mu=\dot{P}_0/P_0$ is given by

$$\dddot{\xi}_\mu l^\mu-\ddot{\xi}_\mu l^\mu(\dot{P}_0/P_0)=\ddot{P}_0/P_0-(\dot{P}_0/P_0)^2\,,$$

or

$$\dddot{\xi}_\mu l^\mu=\ddot{P}_0/P_0.$$

By writing this in terms of S^μ, i.e.,

$$\tfrac{1}{2}\sqrt{2}(\dddot{\xi}_\mu+\tfrac{1}{2}\ddot{\xi}^2\dot{\xi}_\mu)S^\mu=\ddot{P}_0/P_0+\tfrac{1}{2}\ddot{\xi}^2\,,$$

we obtain (2.18).

An alternative means of obtaining the equivalences (2.17) and (2.18) is to show that $\dot{P}_0/P_0$ and $\ddot{P}_0/P_0+\frac{1}{2}\ddot{\xi}^2$ can both be expressed in terms of $l=1$ spherical harmonics, and then to exploit the well-known relationship between the components of three-dimensional vectors and the coefficients of Y_{1m}. (See Appendix A.)

The equivalences can now be used to give an alternative expression for equations of motion, e.g.:

(a) Free-particle or geodesic motion,

$$m\ddot{\xi}^\mu=0\leftrightarrow m\dot{P}_0/P_0=0. \quad (2.19)$$

(b) Motion with the Abraham radiation-reaction force,[6]

$$\sqrt{2}m\ddot{\xi}^\mu=\tfrac{2}{3}e^2(\dddot{\xi}^\mu+\tfrac{1}{2}\ddot{\xi}^2\dot{\xi}^\mu)\leftrightarrow\sqrt{2}m\dot{P}_0/P_0$$
$$=\tfrac{2}{3}e^2(\ddot{P}_0/P_0+\tfrac{1}{2}\ddot{\xi}^2)\,, \quad (2.20)$$

where u, it must be remembered, is not the proper time.

(c) Motion with a general force,

$$\sqrt{2}m\ddot{\xi}^\mu=F^\mu\leftrightarrow\sqrt{2}m(\dot{P}_0/P_0)=F(u,\zeta,\bar{\zeta})\,, \quad (2.21)$$

where F is expressible as an $l=1$ spherical harmonic.

Actually, these equivalences hold more generally, applying as well to regular timelike world lines in an arbitrary Riemannian space, where a similar coordinate system can be constructed at least locally.

There is an interesting by-product of this alternative description of acceleration. It is possible, by means of Doppler-shift observations of light emanating from the world line, to measure $\dot{P}_0/P_0$ directly (at least in principle) and thus measure the acceleration of the line.

Consider a family of observers completely surrounding the world line and let their velocity field be $v^\mu(\tau,\zeta,\bar{\zeta})$, where τ is the proper time along an observer trajectory and $(\zeta,\bar{\zeta})$ gives the angular position of the observer. The ratio of source frequency to observed frequency is given by

$$\sqrt{2}f_s/f_0=d\tau/du=\dot{\xi}_\mu l^\mu/v_\mu l^\mu=1/v_\mu l^\mu.$$

Differentiating this with respect to τ, we get

$$\sqrt{2}(d/d\tau)f_s/f_0=-(d\tau/du)^2(d/d\tau)(v_\mu l^\mu)$$
$$=-(d\tau/du)^2(v_{\mu;\nu}v^\nu l^\mu+l_{\mu;\nu}v^\mu v^\nu).$$

If the observers are all moving along geodesics, then

$$\sqrt{2}(d/d\tau)f_s/f_0=-(d\tau/du)^2 l_{\mu;\nu}v^\mu v^\nu.$$

Since $l_\mu=\delta_\mu{}^0$ in our coordinate system, we get

$$\sqrt{2}(d/d\tau)f_s/f_0=(d\tau/du)^2\Gamma^0{}_{\mu\nu}v^\mu v^\nu$$
$$=\Gamma^0{}_{\mu\nu}(dx^\mu/du)dx^\nu/du$$
$$=-\tfrac{1}{2}g_{\mu\nu,1}(dx^\mu/du)dx^\nu/du$$
$$=(\dot{P}_0/P_0)+(r/2P_0{}^2)(d\zeta/du)d\bar{\zeta}/du.$$

Now, $d\zeta/du$ measures angular velocity and hence behaves as r^{-1}. Therefore, for observers at infinity,

$$\sqrt{2}(d/d\tau)f_s/f_0=\dot{P}_0/P_0. \quad (2.22)$$

III. ANALYSIS OF RT SOLUTIONS

In Sec. II we developed an alternative to the usual mode of describing motion in both flat space and an arbitrary Riemannian space. We now show how this new mode of description can be applied when an elementary singularity exists in the field.

In particular, we consider here the special case when (2.3) becomes

$$\rho=-r^{-1}, \quad \sigma=0. \tag{3.1}$$

This condition leads uniquely to the class of metrics known as the RT solutions,[2] which we now summarize. (Actually, we are only considering the type-II RT metrics.) The line element for the RT solutions can be written as

$$ds^2=2\left(K-\frac{\dot{P}}{P}r-\frac{M(u)}{r}\right)du^2+2dudr-\frac{r^2}{2P^2}d\zeta d\bar{\zeta}, \tag{3.2}$$

where $K=\eth\eth^*\ln P$ is the Gaussian curvature of the F2S ($\eth\eth^*\equiv 4P^2\partial^2/\partial\zeta\partial\bar{\zeta}$),[10] and $P(u,\zeta,\bar{\zeta})$ and $M(u)$ satisfy the equation

$$\dot{M}-3M\dot{P}/P=\eth\eth^*K. \tag{3.3}$$

From this equation we will extract the time dependence of M, the equations of motion, and the time development of the internal degrees of freedom (still to be defined). In all our generalizations from (3.1) to (2.3) in a general $R_{\mu\nu}=0$ and in the Einstein-Maxwell theory, Eq. (3.3) is modified by the addition of extra terms representing the interactions of the singularity with gravitational and electromagnetic background fields.

We recall that P is still defined up to an arbitrary function of u. A conventional choice of this factor is to make the surface area of the F2S constant, i.e.,

$$\frac{d}{du}\int\frac{dx^2dx^3}{P^2}=0.$$

This is equivalent to setting $\dot{M}=0$ in (3.3). Here, however, we shall make a different choice of this factor.

We impose the important regularity restriction that $P=P_0H$, where P_0 is defined from (2.11a) and (2.14), and H is a regular function on the sphere. We then use the arbitrary factor in the definition of P to set the $l=0$ part of H equal to one and to incorporate the $l=1$ part into the definition of P_0. The latter can be done because any variation in P_0 is such that $\delta P_0/P_0$ is an $l=1$ harmonic. This is seen by varying $\eth_0\eth_0^*\ln P_0=1$,[11] which yields

$$\eth_0\eth_0^*\delta P_0/P_0=-2\delta P_0/P_0,$$

thus showing that $\delta P_0/P_0$ satisfies the eigenvalue equation for an $l=1$ spherical harmonic (see Appendix A). Hence the regularity condition can be written

$$P=P_0(1+I), \tag{3.4}$$

where I is expandable in terms of $l\geqslant 2$ spherical harmonics. (It is very likely that the additional condition $I>-1$ should also be imposed to keep the F2S a deformed sphere.) *I is interpreted to represent internal degrees of freedom.*

When (3.4) is substituted into (3.3), we get

$$\dot{M}-3M\dot{P}_0/P_0-3M\dot{I}/(1+I)=(1+I)^2\eth_0\eth_0^*K, \tag{3.5a}$$

where

$$\begin{aligned}K&=(1+I)^2\eth_0\eth_0^*\ln[P_0(1+I)]\\&=(1+I)^2+(1+I)\eth_0\eth_0^*I-\eth_0I\cdot\eth_0^*I.\end{aligned} \tag{3.5b}$$

If we carry out the differentiation of K in (3.5), we can write the equation in the form

$$\begin{aligned}\dot{M}-3M\dot{P}_0/P_0-3M\dot{I}/(1+I)&\\=(1+I)^3(\eth_0\eth_0^*\eth_0\eth_0^*I+2\eth_0\eth_0^*I)&\\-(1+I)^2\eth_0^2I\cdot\eth_0^{*2}I.&\end{aligned} \tag{3.6}$$

This equation can in principle (though not in practice, because of its extreme nonlinearity) be expanded in spherical harmonics such that the $l=0$ part gives the time dependence of M (which is related to the "mass"), the $l=1$ part gives an equation of the form (2.21) for $\dot{P}_0/P_0$, and the $l\geqslant 2$ parts give the time dependence of the internal degrees of freedom.

In particular, if $I=0$, then $\dot{M}=0$, and $\dot{P}_0/P_0=0$, i.e., the singularity is unaccelerated with constant M. This solution gives the Schwarzschild metric, with the Schwarzschild mass m_s related to M by $M=2\sqrt{2}m_s$.

Although a great deal of effort has been spent attempting to find exact solutions (other than Schwarzschild) to (3.3) or (3.5a), not one that satisfies (3.4) has so far been found. It thus appears as if solutions must be investigated by approximation methods.[4]

The linearization of (3.6) yields

$$\dot{M}-3M(\dot{P}_0/P_0)-3M\dot{I}=\eth_0\eth_0^*\eth_0\eth_0^*I+2\eth_0\eth_0^*I. \tag{3.7}$$

If I is assumed to have a definite l value, i.e., satisfy $\eth_0\eth_0^*I=-l(l+1)I$, then

$$\dot{M}-3M(\dot{P}_0/P_0)-3M\dot{I}=l(l+1)[l(l+1)-2]I, \tag{3.8}$$

from which it follows that

$$\begin{aligned}&\dot{M}=0, \quad M(\dot{P}_0/P_0)=0,\\&\dot{I}=-(1/3M)l(l+1)[l(l+1)-2]I,\end{aligned} \tag{3.9}$$

or

$$I=I_0\exp\{-[l(l+1)/3M][l(l+1)-2]u\}. \tag{3.10}$$

Thus, to first order, M is conserved, there is no acceleration, and the internal degrees of freedom decay exponentially. A brief look at the nonlinear terms shows that when they are taken into account, M will no longer be constant, $\dot{P}_0/P_0$ will not be zero in general (except

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[10] For properties of $\eth$ and $\eth^*$, see E. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966); J. Goldberg *et al.*, *ibid.* **8**, 2155 (1967). Because of typographical difficulties, the $\bar{\eth}$ of earlier references appear here as $\eth^*$.

[11] In our notation here, $\eth$ applies to an arbitrary two-surface, while $\eth_0$ applies to the unit sphere, i.e., $\eth_0\eta=2P_0^{1-s}\partial(P_0^s\eta)/\partial\zeta$ for any spin-weight s quantity η.

when I contains only even spherical harmonics), and I will continue to decay exponentially. This leads us to conjecture, with no proof in sight, that all solutions of (3.5a) will decay towards Schwarzschild.

The discussion of the concept of mass has been intentionally avoided until now because it appears to be quite complicated. There seem to be at least four candidates for the title *mass*, all involving M times some function of u. In each case the function is such that all the different "masses" agree with the Schwarzschild mass m_s in the static case.

If we let the area of the F2S be

$$A(u)=\int\frac{dx^2dx^3}{P^2},$$

then four possible definitions of mass are the following.

(1) $m_i=MA^2/32\sqrt{2}\pi^2$, which appears to be a good candidate for the inertial mass of the singularity because of its appearance in the ponderomotive law (see Sec. IV).

(2) $m_B=(M/8\sqrt{2}\pi)\int\frac{1}{2}(1+\zeta\bar{\zeta})dx^2dx^3/P^3$, which appears to be equivalent to the Bondi mass,[12,13] since it has the property that $\dot{m}_B<0$.

(3)[2] $m_c=MA^{3/2}/2\sqrt{2}(4\pi)^{3/2}$, which satisfies the conservation law $\dot{m}_c=0$ (valid for the RT solutions).

(4) $m_e=MA/8\sqrt{2}\pi$, which has nothing to recommend it other than its analogy with the definition of charge (see Sec. IV).

The proof of the conservation law $\dot{m}_c=0$ is straightforward. Differentiation of m_c yields

$$\dot{m}_c=\frac{A^{1/2}}{2\sqrt{2}(4\pi)^{3/2}}\int\frac{dx^2dx^3}{P^2}\left(\dot{M}-3M\frac{\dot{P}}{P}\right).$$

Then from (3.3) we get

$$\dot{m}_c=\frac{A^{1/2}}{2\sqrt{2}(4\pi)^{3/2}}\int dx^2dx^3\frac{\partial}{\partial\zeta}\frac{\partial K}{\partial\bar{\zeta}}=0.$$

To prove that $\dot{m}_B<0$, we set $P=P_0'W$, where $P_0'=\frac{1}{2}(1+\zeta\bar{\zeta})$, and W is a positive quantity. Equation (3.3) can then be written as

$$\dot{M}-3M(\dot{W}/W)=W^3ð_0'ð_0'^*V-W^2|ð_0'^2W|^2,$$

where $V=ð_0'ð_0'^*W+2W$, and $ð_0'$ is defined with respect to P_0'. Dividing this by $8\sqrt{2}\pi W^3P_0'^2$ and integrating over the F2S, we obtain

$$\frac{1}{8\sqrt{2}\pi}\frac{d}{du}\int\frac{MP_0'dx^2dx^3}{P^3}=\dot{m}_B=-\frac{1}{8\sqrt{2}\pi}\int\frac{dx^2dx^3}{WP_0'^2}|ð_0'^2W|^2.$$

Since the last integral is always positive, it follows that $\dot{m}_B<0$.

[12] H. Bondi *et al.*, Proc. Roy. Soc. (London) **A269**, 21 (1962).

[13] L. Derry, R. Isaacson, and J. Winicour, Phys. Rev. **185**, 1647 (1969). The total energy of the RT solutions defined by these authors using a different formalism appears to be the same as m_B.

IV. ANALYSIS OF RT MAXWELL SOLUTIONS

In this section we will present and analyze a class of solutions of the Einstein-Maxwell equations that are characterized by the coincidence of a principal null direction of the Maxwell field and a doubly degenerate principal null direction of the Weyl tensor, these null directions being hypersurface-orthogonal. These solutions are the Einstein-Maxwell analogs of the RT metrics. It can easily be shown that for these solutions

$$\rho=-r^{-1},\quad \sigma=0.$$

Therefore, an F2S is present, and if the Weyl tensor is singular at $r=0$, there exists an elementary singularity.

After a relatively simple calculation, if we use the spin-coefficient formalism and impose coordinate conditions identical to those used for the RT solutions, we obtain the following metric and Maxwell field (see Appendix B for details):

$$ds^2=2\left(K-\frac{\dot{P}}{P}r+\frac{\psi_2^0}{r}+\frac{E^2}{r^2}\right)du^2+2dudr-\frac{r^2}{2P^2}d\zeta d\bar{\zeta},\quad (4.1)$$

$$\phi_0=0,\quad \phi_1=\frac{E(u)}{r_2},\quad \phi_2=\frac{\phi_2^0(u,\zeta,\bar{\zeta})}{r},\quad (4.2)$$

where $K=ðð^*\ln P$ is the Gaussian curvature of the F2S, and $E(u)$, $\psi_2^0(u,\zeta,\bar{\zeta})$, and $\phi_2^0(u,\zeta,\bar{\zeta})$ satisfy the equations

$$\dot{\psi}_2^0-3\psi_2^0\dot{P}/P=-ðð^*K+\phi_2^0\bar{\phi}_2^0,\quad (4.3)$$

$$\dot{E}-2E\dot{P}/P=-ð\phi_2^0,\quad (4.4)$$

$$ðð^*\psi_2^0=-2E(\dot{E}-2E\dot{P}/P).\quad (4.5)$$

{Note: If we make the change of variable $\phi_2^0=ð^*R$, (4.4) becomes

$$ðð^*R=-[\dot{E}-2E(\dot{P}/P)],\quad (4.4')$$

which looks very similar to (4.5).}

We now make the same assumption as was made in the RT discussion, namely,

$$P=P_0(1+I),\quad I>-1$$

and again consider the different spherical harmonic terms in (4.3) as yielding the equations of motion and the time dependence of the internal degrees of freedom.

Although Eqs. (4.3), (4.4) [or (4.4′)], and (4.5) are extremely complicated (the only known exact solution is the Reissner-Nordström[14] solution), and we know of no systematic approximation method, nevertheless a fair amount of information can be extracted.

[14] This is obtained explicitly by setting $E=\sqrt{2}e$, $\psi_2^0=-2\sqrt{2}m$, $I=0$, $\phi_2^0=0$, $P_0=\frac{1}{2}(1+\zeta\bar{\zeta})$, $K=1$, where e and m are, respectively, the charge and the mass.

First, note that by multiplying (4.4) by P^{-2} and integrating over the F2S, one shows

$$\frac{d}{du}(EA)=\frac{d}{du}\int E\frac{dx^2dx^3}{P^2}=\int\left(\dot{E}-2\frac{\dot{P}}{P}E\right)\frac{dx^2dx^3}{P^2}=0\,, \tag{4.6}$$

where $A(u)$ is the area of the F2S. Thus, EA is conserved and can be taken as proportional to the charge e,

$$e=EA/4\sqrt{2}\pi\,, \tag{4.7}$$

and $E(u)$ can be considered as essentially the charge per unit area.

A useful way of looking at Eqs. (4.3)–(4.5) is to pretend that one can solve the last two for $\psi_2{}^0$ and $\phi_2{}^0$ as functions of P (or P_0 and I) and then to substitute these into the first to determine P_0 and I. There seems to be little hope of doing this exactly, but one can do it partially and obtain a very startling result.

If (4.4′) and (4.5) are rewritten in terms of P_0 and I, and if we make the substitutions

$$\psi_2{}^0=-M(u)-2E^2\dot{P}_0/P_0+2E^2S \tag{4.8}$$

and

$$R=-E\dot{P}_0/P_0+ET\,, \tag{4.9}$$

the two equations become

$$(1+I)^2\eth_0\eth_0{}^*S=-2(2I+I^2)\dot{P}_0/P_0+2\dot{I}/(1+I)-\dot{E}/E\,, \tag{4.10}$$

$$(1+I)^2\eth_0\eth_0{}^*T=-2(2I+I^2)\dot{P}_0/P_0+2\dot{I}/(1+I)-\dot{E}/E\,. \tag{4.11}$$

The important point to be noted is that both S and T vanish when $I=0$; i.e., (4.8) and (4.9) separate out the I-independent parts of $\psi_2{}^0$ and R (or $\phi_2{}^0$). [In fact, if $I=0$, (4.4) and (4.5) can be solved exactly to yield (4.8) and (4.9), with $S=0$ and $T=0$.] If we now substitute (4.8) and (4.9) into (4.3), we obtain (after some regrouping of terms)

$$\dot{M}+2E^2\left(\frac{\ddot{P}_0}{P_0}+\tfrac{1}{2}\xi^2\right)-3M\frac{\dot{P}_0}{P_0}-9E^2\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{6}\xi^2\right]=\mathfrak{F}\,, \tag{4.12}$$

where

$$\begin{aligned}\mathfrak{F}={}&2E^2\left(\dot{S}-3S\frac{\dot{P}_0}{P_0}\right)+3\frac{\dot{I}}{1+I}\left(M+2E^2\frac{\dot{P}_0}{P_0}-2E^2S\right)\\&+E^2(2I+I^2)\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{2}\xi^2\right]-4E\dot{E}\left(\frac{\dot{P}_0}{P_0}-S\right)\\&+(1+I)^2\left[\eth_0\eth_0{}^*K+E^2\left(\eth_0T\cdot\eth_0{}^*\frac{\dot{P}_0}{P_0}+\eth_0{}^*T\cdot\eth_0\frac{\dot{P}_0}{P_0}-\eth_0T\cdot\eth_0{}^*T\right)\right],\end{aligned} \tag{4.13}$$

and K is that given in (3.5b). Because of the vanishing of S, T, and $K-1$ when $I=0$, we have $\mathfrak{F}=0$ when $I=0$. Since $\mathfrak{F}$ can in principle be decomposed into a sum of different spherical harmonic terms, $\mathfrak{F}_l$, the different l terms in (4.12) can be separated out, thus yielding

$$\dot{M}=\mathfrak{F}_0\,, \tag{4.14a}$$

$$3M\frac{\dot{P}_0}{P_0}-2E^2\left(\frac{\ddot{P}_0}{P_0}+\tfrac{1}{2}\xi^2\right)=\mathfrak{F}_1\,, \tag{4.14b}$$

$$-9E^2\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{6}\xi^2\right]=\mathfrak{F}_2\,, \tag{4.14c}$$

$$0=\mathfrak{F}_l\quad(l\geqslant 3)\,. \tag{4.14d}$$

Then, if (4.14b) is multiplied by $A^2/3\times 32\pi^2$, it becomes

$$\sqrt{2}m_i\dot{P}_0/P_0-\tfrac{2}{3}e^2(\ddot{P}_0/P_0+\tfrac{1}{2}\xi^2)=F\,, \tag{4.15}$$

where

$$F=\frac{A^2}{3\times 32\pi^2}\mathfrak{F}_1\,,\quad e=\frac{EA}{4\sqrt{2}\pi}\,,\quad m_i=\frac{MA^2}{32\sqrt{2}\pi^2}\,.$$

In the static case, e and m_i reduce to the charge and mass, respectively, of the Reissner-Nordström metric. Equation (4.15) is our main result: a rigorous derivation of the Abraham radiation-reaction term (2.20), with no model or mass renormalization needed. (In a future paper it will be shown that, if the solution is generalized to an arbitrary solution of the Einstein-Maxwell equations, with an elementary singularity and F2S, the same result plus the Lorentz force law is obtained.)

A second result follows from (4.14c): if $I=0$, then $\dot{P}_0/P_0=0$; hence, if there is acceleration, then the internal degrees of freedom *must* be excited.

It appears as if little more can be done with the exact equations. The linearization of (4.10) and (4.11) yields

$$\eth_0\eth_0{}^*S=2\dot{I}\,,\quad \eth_0\eth_0{}^*T=2\dot{I}\,,\quad \dot{E}=0\,, \tag{4.16}$$

while (4.12) becomes

$$\dot{M}+2E^2(\ddot{P}_0/P_0+\tfrac{1}{2}\xi^2)-3M\dot{P}_0/P_0=2E^2\dot{S}+3M\dot{I}+\eth_0\eth_0{}^*\eth_0\eth_0{}^*I+2\eth_0\eth_0{}^*I\,. \tag{4.17}$$

If we now assume (with no loss of generality because of the linearity) that I has a definite l value, i.e., $\eth_0\eth_0{}^*I=-\lambda I$, $\lambda=l(l+1)$, then

$$S=-2\dot{I}/\lambda\,,\quad T=-2\dot{I}/\lambda\,, \tag{4.18}$$

and (4.17) breaks up into

$$\dot{M}=0\,,\quad 3M\dot{P}_0/P_0-2E^2(\ddot{P}_0/P_0+\tfrac{1}{2}\xi^2)=0\,, \tag{4.19}$$

$$(4E^2/\lambda)\ddot{I}-3M\dot{I}-\lambda(\lambda-2)I=0\,. \tag{4.20}$$

The general solution of (4.20) has exponential growth, which though unpleasant, is not necessarily catastrophic. The unpleasantness is due to the fact that the Reissner-Nordström solution is unstable (at least initially) to small perturbations I and to small accelerations [runaway solutions of (4.19)]. It is not at all clear what effect, if any, the nonlinear terms have in stabilizing or reversing this behavior. It is our hope that these nonlinear terms will actually have a stabilizing effect on the solutions.

Assuming that the dynamical situation is physically reasonable, one can look at both the gravitational and electromagnetic radiation fields, which are, respectively (see Appendix B),

$$\psi_4{}^0 = -\eth^*\eth^*\frac{\dot{P}}{P} = -\eth_0{}^*\left[(1+I)^2\eth_0{}^*\left(\frac{\dot{P}_0}{P_0}+\frac{\dot{I}}{1+I}\right)\right] \approx -\eth_0{}^*\eth_0{}^*\dot{I}\,,$$

$$\phi_2{}^0 = \eth^* R = (1+I)\eth_0{}^* R = (1+I)\eth_0{}^*(ET - E\dot{P}_0/P_0) \approx E\eth_0{}^* T - E\eth_0{}^*\dot{P}_0/P_0\,.$$

The last term in the $\phi_2{}^0$ is precisely the Lienard-Wiechert radiation field; the first term is what one would expect from an electric 2^l-pole field ($l \geqslant 2$). The same remark applies to the $\psi_4{}^0$. This suggests then the interpretation of I, the internal structure, as representing some type of distribution from which both the gravitational and electric multipole moments can be calculated.

As a last remark, we point out that the method, discussed at the end of Sec. II, of measuring $\dot{P}_0/P_0$ (and hence $\ddot{\xi}^\mu$) applies just as well to both the RT solutions and their charged counterparts, with two provisos: (1) It will be $\dot{P}/P$ that will be measured; and (2) the frequency ratio $f_s/f_0 = d\tau/\sqrt{2}du$ will no longer be the ratio of two proper times, but rather the ratio of a proper time to our coordinate "time" u. The hypothesis is that u is a physically significant coordinate, which plays the role of the proper time at the singularity.

V. SUMMARY AND CONCLUSIONS

We have presented what we believe is a novel approach to the theory of equations of motion in general relativity, in which motion is analyzed in terms of the structure and "time" dependence of the family of null cones emanating from a special class of singularities. Several surprising results have appeared, which at the beginning of the work we had no reason to suspect. The first of these was the discovery of an internal structure for the singularity, which responds in a unique way to the acceleration, and from which one can calculate the gravitational and electric multipole moments. The second was the appearance of the Abraham radiation-reaction term, with the correct numerical coefficient and with no mass renormalization. These results are reminiscent of the early *ad hoc* models in classical electron theory of an extended particle,[6] for in our approach it is as though the internal structure makes the singularity behave as if it were a finite-sized body.

There are several directions in which this work can be generalized while preserving condition (2.3).

(1) One can allow incoming fields to couple to the RT type of solutions. The analysis, which is rigorous but done only in the vicinity of the singularity, is nearing completion. Unique equations of motion are again obtained (with geodesic motion in the test-particle limit), the internal degrees of freedom being driven by the incoming field (which is in turn modified by the presence of the singularity). In the presence of an incoming Maxwell field, we obtain, in addition to the radiation-reaction force, the Lorentz force law.

(2) In the work presented here and in the above generalization, the Weyl tensor has r^{-3} singularities. By allowing r^{-4} singularities, further degrees of freedom are introduced, which in linear theory correspond to the dipole moment and spin or angular momentum. This generalization would thus permit the discussion of "spinning" singularities. It should be noted that if r^{-5} singularities in the Weyl tensor are introduced, condition (2.3) would be violated. Hence, it is not possible to generalize the method to higher-order singularities.

(3) A further type of generalization we are investigating is to change condition (2.3) into

$$\rho = -[r + iJ(u,\zeta,\bar{\zeta})]^{-1} + O(r)\,, \quad \sigma = O(r)\,.$$

This would be analogous to generalizing from the Schwarzschild to the Kerr metric. Presumably, this would introduce angular degrees of freedom, which so far have been missing from our analysis.

Although it is far too early to tell, we would like to speculate that the approach presented here will lead to a meaningful classical theory of elementary particles, which in some sense can then be quantized and in turn shed light on elementary-particle physics.

ACKNOWLEDGMENTS

We wish to express our thanks to Professor F. Pirani and to King's College, London, where much of this work was done. We would also like to thank Professor Geroch, Professor Penrose, and Professor Pirani for many stimulating discussions.

APPENDIX A

We give here, essentially without proof, a set of useful relations that were needed in simplifying many of the equations in the text.

One can easily prove that P_0, defined by (2.11a) and (2.14), satisfies the equation

$$\eth_0\eth_0{}^* \ln P_0 = 1\,, \qquad \text{(A1)}$$

where $\eth_0\eth_0{}^* = 4P_0{}^2\partial^2/\partial\zeta\partial\bar{\zeta}$. This simply states that the Gaussian curvature of the unit sphere is equal to 1. If this equation is now differentiated with respect to u, one obtains

$$\eth_0\eth_0{}^*\dot{P}_0/P_0 = -2\dot{P}_0/P_0\,. \qquad \text{(A2)}$$

In the flat-space null coordinate system described in Sec. II, we can choose a set of four linearly independent vectors $(\xi^\mu, l^\mu, \partial l^\mu/\partial\zeta, \partial l^\mu/\partial\bar{\zeta})$, in terms of which any Minkowskian tensor can be expressed. In particular, the Minkowski metric $\eta_{\mu\nu}$ can be written

$$\eta_{\mu\nu}=l_\mu\xi_\nu+l_\nu\xi_\mu-2l_\mu l_\nu-(\eth_0 l_\mu\eth_0^* l_\nu+\eth_0^* l_\mu\eth_0 l_\nu). \quad (A3)$$

When this is multiplied with $\ddot{\xi}^\mu\ddot{\xi}^\nu$, the following relation is obtained:

$$\eth_0\frac{\dot{P}_0}{P_0}\eth_0^*\frac{\dot{P}_0}{P_0}=-\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{2}\ddot{\xi}^2\right],\quad \ddot{\xi}^2=\ddot{\xi}_\mu\ddot{\xi}^\mu. \quad (A4)$$

With the help of (A2) and (A4), one can now show that

$$\eth_0\eth_0^*\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{6}\ddot{\xi}^2\right]=-6\left[\left(\frac{\dot{P}_0}{P_0}\right)^2+\tfrac{1}{6}\ddot{\xi}^2\right]. \quad (A5)$$

By differentiation of (A2) with respect to u, one can also prove that

$$\eth_0\eth_0^*\left(\frac{\ddot{P}_0}{P_0}+\tfrac{1}{2}\ddot{\xi}^2\right)=-2\left(\frac{\ddot{P}_0}{P_0}+\tfrac{1}{2}\ddot{\xi}^2\right). \quad (A6)$$

Comparison of (A2), (A5), and (A6) with the eigenvalue equation for spin-weight zero functions, namely,

$$\eth_0\eth_0^*\eta=-l(l+1)\eta, \quad (A7)$$

thus shows that $\dot{P}_0/P_0$ and $\ddot{P}_0/P_0+\frac{1}{2}\ddot{\xi}^2$ are both $l=1$ quantities, while $(\dot{P}_0/P_0)^2+\frac{1}{6}\ddot{\xi}^2$ is an $l=2$ quantity.

The formulas (A2), (A5), and (A6) can be generalized considerably. Details of this generalization will be presented in a future paper.

APPENDIX B

For completeness, we present here all the spin-coefficient quantities for the charged RT solutions.

Spin coefficients:

$$\begin{aligned}&\sigma=\tau=\lambda=\kappa=\pi=\epsilon=0,\\&\rho=-r^{-1},\quad \alpha=\alpha^0/r,\quad \beta=-\bar{\alpha}^0/r,\\&\gamma=\gamma^0-\tfrac{1}{2}\psi_2{}^0/r^2-|\phi_1{}^0|^2/r^3,\\&\mu=\mu^0/r-\psi_2{}^0/r^2-|\phi_1{}^0|^2/r^3,\\&\nu=\nu^0-\psi_3{}^0/r-2\bar{\phi}_1{}^0\phi_2{}^0/r^2.\end{aligned} \quad (B1)$$

Tetrad and metric variables:

$$\begin{aligned}&\omega=X^i=0,\\&\xi^i=-(P,iP)/r,\\&U=U^0-2\gamma^0 r-\psi_2{}^0/r-|\phi_1{}^0|^2/r^2.\end{aligned} \quad (B2)$$

Tetrad components of the Weyl tensor:

$$\begin{aligned}&\psi_0=\psi_1=0,\\&\psi_2=\psi_2{}^0/r^3+2|\phi_1{}^0|^2/r^4,\\&\psi_3=\psi_3{}^0/r^2+3\bar{\phi}_1{}^0\phi_2{}^0/r^3,\\&\psi_4=\psi_4{}^0/r+\eth^*\psi_3{}^0/r^2+2\bar{\phi}_1{}^0\eth^*\phi_2{}^0/r^3.\end{aligned} \quad (B3)$$

Tetrad components of the Maxwell tensor:

$$\begin{aligned}&\phi_0=0,\\&\phi_1=\phi_1{}^0/r^2,\\&\phi_2=\phi_2{}^0/r.\end{aligned} \quad (B4)$$

Relationships among "constants":

$$\begin{aligned}&\alpha^0=-\tfrac{1}{2}\eth\ln P, &&U^0=\mu^0=-K=-\eth\eth^*\ln P,\\&\gamma^0=-\tfrac{1}{2}\dot{P}/P, &&\nu^0=-2\eth^*\gamma^0,\\&\psi_2{}^0=\bar{\psi}_2{}^0, &&\phi_1{}^0=E(u)=\bar{E}(u),\\&\psi_3{}^0=\eth^*K, &&\psi_4{}^0=2\eth^{*2}\gamma^0.\end{aligned} \quad (B5)$$

Differential equations:

$$\begin{aligned}&\dot{\phi}_1{}^0+4\gamma^0\phi_1{}^0=-\eth\phi_2{}^0,\\&\eth\psi_2{}^0=2\phi_1{}^0\bar{\phi}_2{}^0,\\&\dot{\psi}_2{}^0+6\gamma^0\psi_2{}^0=-\eth\psi_3{}^0+\phi_2{}^0\bar{\phi}_2{}^0.\end{aligned} \quad (B6)$$

Components of the metric tensor:

$$g^{\mu0}=\delta_1{}^\mu,\quad g^{11}=-2\left(K-\frac{\dot{P}}{P}r+\frac{\psi_2{}^0}{r}+\frac{|\phi_1{}^0|^2}{r^2}\right),$$

$$g^{mn}=-(2P^2/r^2)\delta^{mn}.$$

In the calculations leading to the above results, we have taken the magnetic monopole moment to be zero and assumed that the Weyl tensor has no "wire" singularities (i.e., angular singularities).