Systems of Self-Gravitating Particles in General Relativity and the Concept of an Equation of State*

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A method of self-consistent fields is used to study the equilibrium configurations of a system of self-gravitating scalar bosons or spin-$\frac{1}{2}$ fermions in the ground state without using the traditional perfect-fluid approximation or equation of state. The many-particle system is described by a second-quantized free field, which in the boson case satisfies the Klein-Gordon equation in general relativity, $\nabla_\alpha\nabla^\alpha\phi=\mu^2\phi$, and in the fermion case the Dirac equation in general relativity $\gamma^\alpha\nabla_\alpha\psi=\mu\psi$ (where $\mu=mc/\hbar$). The coefficients of the metric $g_{\alpha\beta}$ are determined by the Einstein equations with a source term given by the mean value $\langle\phi|T_{\mu\nu}|\phi\rangle$ of the energy-momentum tensor operator constructed from the scalar or the spinor field. The state vector $\langle\phi|$ corresponds to the ground state of the system of many particles. In both cases, for completeness, a nonrelativistic Newtonian approximation is developed, and the corrections due to special and general relativity explicitly are pointed out. For N bosons, both in the region of validity of the Newtonian treatment (density from 10^{-80} to 10^{54} g cm^{-3}, and number of particles from 10 to 10^{40}) as well as in the relativistic region (density $\sim10^{54}$ g cm^{-3}, number of particles $\sim10^{40}$), we obtain results completely different from those of a traditional fluid analysis. The energy-momentum tensor is anisotropic. A critical mass is found for a system of $N\sim[(\text{Planck mass})/m]^2\sim10^{40}$ (for $m\sim10^{-25}$ g) self-gravitating bosons in the ground state, above which mass gravitational collapse occurs. For N fermions, the binding energy of typical particles is $G^2m^5N^{4/3}\hbar^{-2}$ and reaches a value $\sim mc^2$ for $N\sim N_{\text{crit}}\sim[(\text{Planck mass})/m]^3\sim10^{57}$ (for $m\sim10^{-24}$ g, implying mass $\sim10^{33}$ g, radius $\sim10^6$ cm, density $\sim10^{15}$ g/cm^3). For densities of this order of magnitude and greater, we have given the full self-consistent relativistic treatment. It shows that the concept of an equation of state makes sense only up to 10^{42} g/cm^3, and it confirms the Oppenheimer-Volkoff treatment in extremely good approximation. There exists a gravitational spin-orbit coupling, but its magnitude is generally negligible. The problem of an elementary scalar particle held together only by its gravitational field is meaningless in this context.

I. INTRODUCTION

THAT a system in its degenerate state, composed of a critical number of particles, will necessarily undergo gravitational collapse was first pointed out by Chandrasekhar and Landau.[1] In the intervening years, many questions have been raised and much new information has been learned about gravitational collapse.[2] Among the questions that constantly recur, none are asked more frequently than these: (i) What does one really know about the equations of state of matter at supranuclear density? (ii) What right does one have to use an equation of state at all? The first question will not be treated here and for good reasons: One knows the equations of state of "catalyzed" matter with sufficient accuracy from everyday density up to the density $\sim10^{14}$ g/cm^3 of nuclear matter, and one has an argument from causality (speed of sound < speed of light) that no allowable modifications of the equations of state at supranuclear densities can change the critical mass by more than a factor of the order of 2 from an estimated figure $M\sim M_\odot=2\times10^{33}$ g.[3]

We focus here on the second question: Can one discuss stability against gravitational collapse without mentioning an equation of state at all? Eddington[4,5] raised questions about the possibility of using an equation of state at all and also purported to derive an equation of state quite different in the relativistic domain from Chandrasekhar's standard equations of state for a degenerate ideal Fermi gas. Today one takes seriously none of his results but only his motivation. He sought some escape from the concept of the critical mass, made so vivid by the first detailed calculation of the critical mass by Chandrasekhar.[6,7] Happily, in the same period

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[1] S. Chandrasekhar, Monthly Notices Roy. Astron. Soc. **91**, 456 (1931); L. Landau, Phys. Z. USSR **1**, 285 (1932).

[2] For a complete review on the argument see Kip S. Thorne, California Institute of Technology Report, 1968 (unpublished).

[3] See, for example, C. W. Misner and H. S. Zapolsky, Phys. Rev. Letters **12**, 635 (1964). The problem of the velocity of sound larger than the velocity of light in connection with causality has recently been critically analyzed by S. A. Bludman and M. A. Ruderman, Phys. Rev. **170**, 1176 (1968); M. A. Ruderman, *ibid.* **172**, 1286 (1968).

[4] Sir A. Eddington, Monthly Notices Roy. Astron. Soc. **95**, 195 (1935).

[5] Sir A. Eddington, Monthly Notices Roy. Astron. Soc. **96**, 20 (1935).

[6] S. Chandrasekhar, Monthly Notices Roy. Astron. Soc. **95**, 207 (1935); **95**, 226 (1935).

[7] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover Publications, Inc., New York, 1957).

Dirac,[8] starting from first principles and employing the Hartree-Fock model of the atom, showed for the first time how to go straight from the physics of bound orbitals to the concept of an equation of state, as had already been done in the Fermi-Thomas atom model.[9–11]

A. Fermions

To justify the concept of an equation of state, Dirac showed that it was only necessary to change the effective potential by a small fraction of its value over one wavelength.[12] This condition is normally reasonably well satisfied in atoms containing a large number of electrons. In Sec. III, we extend the original Dirac arguments to the context of general relativity and particles moving with relativistic velocity, taking into account all the effects of the spinorial variables.

We explicitly point out how difficulties arise in our problem before one violates the condition laid down by Dirac: a potential varying slowly over one wavelength. A spin-orbit gravitational interaction starts to be quantitatively important as soon as the effective gravitational potential varies percentagewise by a significant amount over a typical distance $\sim L(m^*/m)^{4/3} \sim 10^{-8}$ cm. [We indicate by $L=(\hbar G c^{-3})^{1/2}$ the Planck length, by $m^*=(\hbar c G^{-1})^{1/2}$ the Planck mass, and by m the neutron mass.] This coupling is generally neglected in the fluid approximation. Its physical significance and order of magnitude are analyzed in Sec. III. It seems at first sight preposterous that in a system of 10- or 100-km radius the effective gravitational potential can vary significantly over 10^{-8} cm. However, in a configuration of sufficiently high central density, the rate of fall of the density is also very high (Fig. 1). Specifically, for each hundred-fold increase in the central density, the half-radius of this "central core" (Schwarzschild coordinate where the density falls to half-value) decreases by one power of 10.[13] These enormous changes in the core have practically no effect on the rest of the star; the core in this sense is almost "isolated" from the rest of the star. The outer radius and the total mass of the star are influenced less and less as the central density goes to higher and higher values[14] (Fig. 2).

Nothing, in principle, prevents the central density from being as high as 10^{42} g cm^{-3} with a radius of the central core of the order of magnitude of 10^{-8} cm. Under this condition the concept of an equation of state no longer makes sense. Naturally, it is a fantastic idealization to think of particles moving about "freely" at a density of 10^{42} g cm^{-3} and responding only to the curvature of space. Even so, the spin-orbit gravitational coupling has negligible effect on the radius and total mass of the system. In Sec. III we show the basic reason for this result: When many fermions are present, the Pauli exclusion principle forces the typical fermion into a state with very high quantum numbers. Then the JWKB approximation is applicable everywhere except in the core which is exceedingly small; outside the core, the self-consistent field method that we have used gives exactly the same results as the traditional fluid analysis, and therefore the concept of an equation of state is perfectly well justified.[15]

B. Bosons

The direct opposite is true in the case of an idealized system composed of many bosons interacting only by way of gravitational forces. Some aspects of this problem have been previously treated.[16–19] When the system is in its ground state, each individual boson is also in the ground state (one and the same state for all bosons). Their distribution of stress, except near the center, is anisotropic. Therefore, the concept of an equation of state is completely inappropriate. Figure 3 shows the stress ellipsoid at selected distances from the center. At nonrelativistic energies (few particles, weak gravitational binding), a Newtonian treatment is possible. In this regime, a simple scaling law brings out a similarity between nonrelativistic systems with different numbers of bosons N. We find

(central density)
$$\sim 0.9 G^3 N^4 m^{10} \hbar^{-6} \times 10^{-3} = 7.08\times 10^{-108} N^4 \text{ g cm}^{-3},$$

(distance from center at which the potential falls to half-value)
$$\sim 6.24 \hbar^2 G^{-1} N^{-1} m^{-3} = 7.55\times 10^{27} N^{-1} \text{ cm},$$

[8] P. A. M. Dirac, Proc. Cambridge Phil. Soc. **26**, 376 (1930).

[9] L. H. Thomas, Proc. Cambridge Phil. Soc. **23** 542 (1926).

[10] E. Fermi, Z. Physik **48**, 73 (1928).

[11] For a complete review on the argument see M. R. Brillouin, *Exposés sur la théorie des quanta, 1, 5, actualités scientifiques et Industrielles* (Hermann Cie., Paris, 1935). See also J. A. Wheeler, in *Colloques internationaux No. 170 du C.N.R.S.* (Editions du C.N.R.S., Paris, 1969), p. 154.

[12] P. A. M. Dirac, Ref. 8, rigorously demonstrated that if we neglect the fact that the momenta p do not commute with the coordinate q, we can introduce a classical description of the atom adopting an equation of state as previously done by Thomas and Fermi, Ref. 9. In this treatment Dirac neglected the contributions from the spin variables.

[13] This important scaling law was first put in evidence by H. Bondi, Proc. Roy. Soc. (London) **A281**, 39 (1964). See also B. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965), Chap. 5.

[14] These features of the curve of equilibrium mass as a function of central density were first recognized by B. K. Harrison, Phys. Rev. **137**, B1644 (1965).

[15] A completely different situation exists if one insists on extrapolating the expansion of the universe all the way back to the point that the Hubble length (particle horizon) in the Friedman-Lemaitre model is of the order of magnitude of the Compton wavelength of the particles. See, e.g., E. R. Harrison, Nature **215**, 151 (1967); P. J. E. Peebles, *ibid.* **220**, 237 (1968). In the case, it is not possible to describe the particles in a locally Minkowski coordinate frame, ignoring the second-order variation in the components of the metric tensor of the matter. The use of an equation of state for the matter is meaningless and it would be necessary, as we propose, to treat the particles as a field.

[16] R. Ruffini, thesis, University of Rome, 1967 (unpublished); Hamburg Seminar Report, 1967 (unpublished).

[17] S. Bonazzola and F. Pacini, Phys. Rev. **148**, 1269 (1966).

[18] S. Bonazzola and R. Ruffini, Bull. Am. Phys. Soc. **13**, 571 (1968).

[19] R. Ruffini, in Proceedings of the Fifth International Conference on Gravitation, Tbilisi, 1969 (to be published).

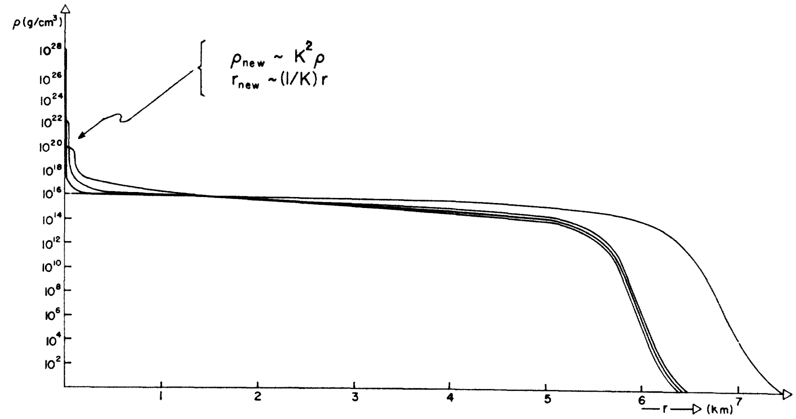

FIG. 1. Density is plotted as a function of the radial coordinate for a system of self-gravitating fermions for selected values of the central density. Near the origin there exists a very simple scaling law (Bondi scaling law). A solution for a value of the central density $K^2\rho_c$, where K^2 is a constant, is obtained from the solution of central density ρ_c, taking the value of ρ at the point r and then multiplying ρ by K^2 and r by $1/K$. Enormous changes in the core have practically no effect on the rest of the distribution.

(energy to remove all the particles to ∞ separation)

$$\sim 0.246 G^2 m^5 N^3 \hbar^{-2} = 8.86\times 10^{-85} N^3 \text{ ergs},$$

having chosen for m the meson mass (2.489×10^{25} g). Evidently there exists a critical value of the order $N \sim (\hbar c/G) m^{-2}$ at which the binding energy per particle becomes comparable to the rest energy. For a number of particles of this order of magnitude, the Newtonian nonrelativistic treatment fails.

We have developed a fully relativistic self-consistent treatment for the case N comparable to or greater than the N_{crit} for bosons. A detailed analysis of the system of equations obtained is given in Sec. II. The solution of the equations was carried out by computer; the particulars of the integration method are given in the Appendix. It is of great interest to know at what point the change from stability to instability takes place in the family of equilibrium configurations that we have found. One of us (R. R.) hopes to return to this question. Without waiting for this analysis to be completed, one can immediately draw one new conclusion: *There exists no equilibrium configuration for a system of more than* $N_{\text{crit}} \sim [(\textit{Planck mass})/m]^2$ *ideal self-gravitating bosons in their ground state.*

There are both great differences and at the same time great similarities between a system of ideal self-gravitating bosons and a system of ideal self-gravitating fermions. Each is characterized by its own critical mass. On the other hand, there is an enormous contrast between bosons and fermions with respect to the value of the critical number $\{N_{\text{crit}} \sim [(\text{Planck mass})/m]^3$ for fermions, $N_{\text{crit}} \sim [(\text{Planck mass})/m]^2$ for bosons$\}$ and

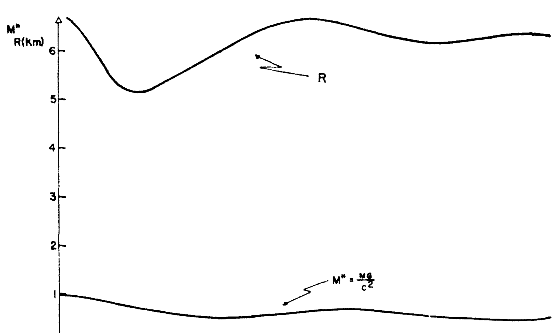

FIG. 2. Radius R and the total mass (expressed in km, $M^* = MG/c^2$) of a neutron star are plotted as a function of the central density in the range 10^{16} g cm$^{-3} \leq \rho_{\text{cent}} \leq 10^{22}$ g cm^{-3}. As the central density goes to higher and higher values, the radius R and the mass M^* are influenced less and less and approach asymptotic values $R_\infty = 6.4$ km and $M_\infty = 0.617 M_\odot$.

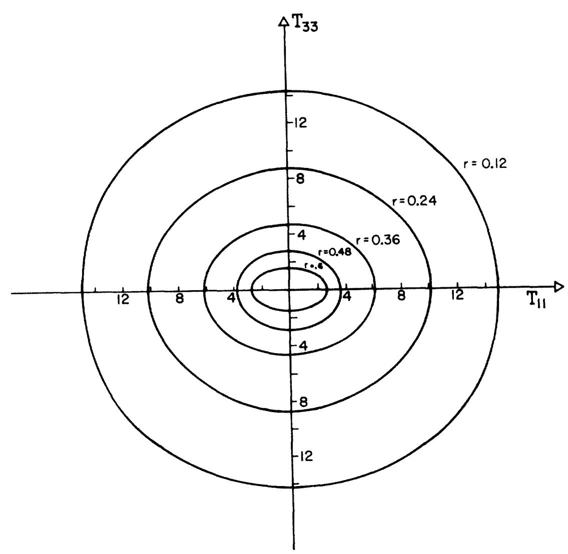

FIG. 3. Stress ellipsoid for a degenerate gas of self-gravitating bosons is plotted at selected distances from the center. From this figure it is evident how the anisotropy increases from the center (the stress is the same in all directions, $T_1{}^1 = T_2{}^2 = T_3{}^3$) to the outside ($T_1{}^1/T_2{}^2 = T_1{}^1/T_3{}^3 = 1.75$). The plot refers to a distribution with $R_{01}(0) = 1.0$ (see Sec. II). The radial coordinate is measured in units $\hbar(mc)^{-1}$, the stress tensor in units $\hbar^2(2m)^{-1}N$.

to the dimensions of the system required to reach relativistic conditions. This difference is due principally to the fact that all the N bosons are in the ground state, whereas the N fermions, according to the Pauli principle, are distributed in the N lowest energy states of the phase space.

Section II also notes that it is absolutely meaningless to consider in the present context the "problem" of one elementary particle held together only by its gravitational field.

II. BOSONS

A. Newtonian Treatment

In Newtonian theory the gravitational potential V satisfies the Poisson equation

$$\Delta V = -4\pi G\rho\,, \tag{1}$$

where ρ is the matter density and G is Newton's gravitational constant. The Schrödinger equation for a particle of mass m, in the presence of a gravitational potential V, is

$$\Delta\psi + 2m\hbar^{-2}(E+mV)\psi = 0\,. \tag{2}$$

We are interested in a system of self-gravitating bosons, all in the same quantum state. We therefore assume that the gravitation potential V satisfies equation (1) with $\rho = N\psi^*\psi m$, where N is the number of bosons and ψ is the wave function of the quantum state under consideration. The wave function is normalized to 1:

$$\int \psi^*\psi d^3x = 1\,. \tag{3}$$

We shall consider only the ground state of the system $n=1$, $l=0$ which we may assume to be spherically symmetric. Consequently, the resulting system of equations in dimensionless units is

$$\hat{r}^{-1}\frac{d^2}{d\hat{r}^2}(\hat{r}\phi) + (\hat{E}+\hat{V})\phi = 0\,, \tag{4a}$$

$$\hat{r}^{-1}\frac{d^2}{d\hat{r}^2}(\hat{r}\hat{V}) = -\phi^*\phi\,, \tag{4b}$$

$$\int \phi^*\phi\hat{r}^2 d\hat{r} = 1\,, \tag{4c}$$

where

$$r = \tfrac{1}{2}\hbar^2 m^{-3}G^{-1}N^{-1}\hat{r}\,, \tag{5a}$$

$$\psi = (2\pi)^{-1/2}(2m^3GN\hbar^{-2})^{3/2}\phi\,, \tag{5b}$$

$$V \equiv (\text{one-particle potential}) = 2\hbar^{-2}G^2N^2m^4\hat{V}\,, \tag{5c}$$

$$E \equiv (\text{one-particle energy}) = 2G^2N^2m^5\hbar^{-2}\hat{E}\,. \tag{5d}$$

We have carried out a numerical integration of the system (4) by the Runge-Kutta method. We have found the eigenvalue $\hat{E}$ by looking at the behavior of the wave function at infinity. As usual, we have determined the ground state by requiring that the eigenfunction have no nodes. The results are given in Table I and in Figs. 4 and 5. Knowing the solution of the universal system of Eq. (4) and thanks to relations (5), it is possible to obtain a solution relative to an arbitrary number of bosons simply by making appropriate scale changes. If we distinguish quantities referring to solutions with N_1 and N_2 particles by suffixes 1 and 2, respectively, we obtain the following relations:

$$E_2 = E_1(N_2/N_1)^2\,, \quad r_2 = r_1(N_1/N_2)\,, \quad \psi_2 = \psi_1(N_2/N_1)^{3/2}\,, \quad V_2 = V_1(N_2/N_1)^2\,. \tag{6}$$

These arguments could suggest that for any number of bosons in the ground state there exists always a position of equilibrium. But we must analyze whether the theory we have used always makes sense.

It is possible to divide the plot of Fig. 5 into three regions. To an increase in the particle number there corresponds: in region I an increase in the total energy

TABLE I. The dimensionless quantities ϕ, $\hat{V}$, and $\hat{r}$ relative to the equilibrium configuration of many self-gravitating bosons in their ground state ($n=1$, $l=0$) are given. These data refer to a Newtonian approximation valid for a number of particles $N \ll [(\text{Planck mass})/m]^2$, where m is the mass of the boson under consideration. The solution relative to a fixed number of bosons N is obtained from the dimensionless quantities ϕ, $\hat{V}$, and $\hat{r}$ by appropriate scale factor [see relations (5)].

$\hat{r}$	$\hat{V}$	ϕ	$\hat{r}$	$\hat{V}$	ϕ
0.00692	0.15793	0.08329	11.44081	0.08411	0.02122
1.04637	0.15667	0.08214	12.48026	0.07809	0.01714
2.08581	0.15305	0.07883	13.51971	0.07270	0.01374
3.12526	0.14741	0.07371	14.55915	0.06788	0.01095
4.16470	0.14022	0.06724	15.59859	0.06359	0.00867
5.20415	0.13203	0.05997	16.63803	0.05975	0.00683
6.24359	0.12335	0.05240	17.67747	0.05632	0.00536
7.28304	0.11460	0.04497	18.71693	0.05324	0.00419
8.32248	0.10614	0.03798	19.06340	0.05228	0.00386
9.36193	0.09816	0.03164	20.10285	0.04960	0.00300
10.40137	0.09808	0.02605	21.14230	0.04717	0.00233

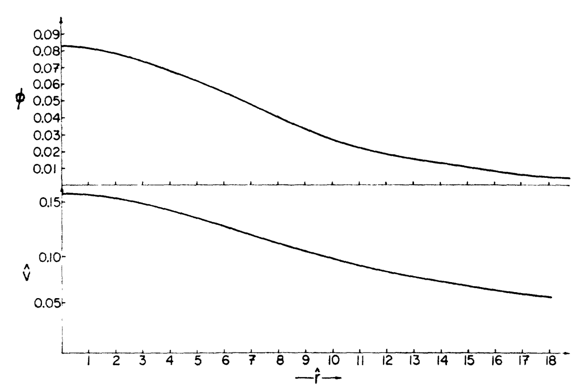

FIG. 4. Dimensionless quantities ϕ and $\hat{V}$ relative to the equilibrium configurations of many self-gravitating bosons in their ground state ($n=1$, $l=0$) in the nonrelativistic domain are given as a function of the dimensionless coordinate $\hat{r}$. The exact numerical values are given in Table I.

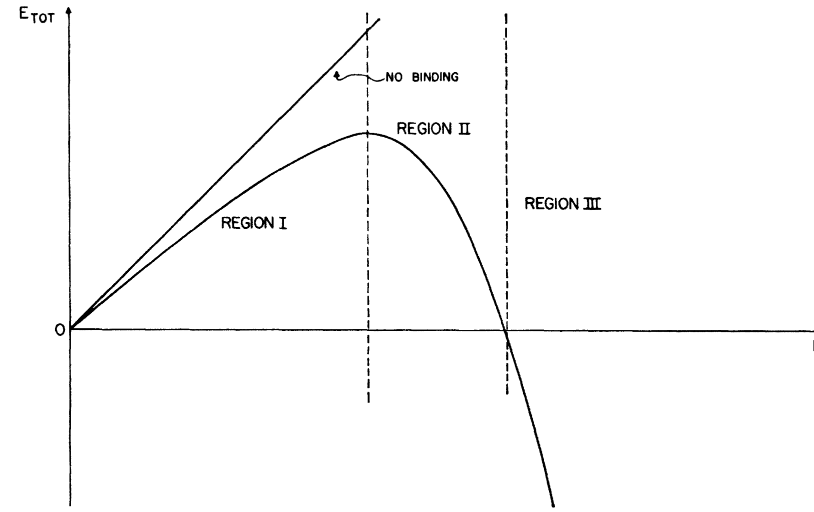

FIG. 5. Total energy E_{tot} as evaluated by numerical computation in the Newtonian approximation is plotted against the number of particles N. We can understand the qualitative behavior of this diagram by considering the formula $E_{tot} = Nmc^2 - 0.1626N^3G^2m^5\hbar^{-2}$. The maximum of the total energy corresponds to a particle number given approximately by $[(\text{Planck mass})/m]^2$. We indicate by m the mass of the elementary boson.

of the system; in region II a decrease in the total energy of the system, which nevertheless remains positive; in region III a decrease in the total energy of the system, which is now negative.

It is very important to observe that at the end of region I the gravitational energy of one particle is of the order of magnitude of the rest mass energy of the particle. Therefore, it is clear that corrections coming from special relativity must be taken into account. Moreover, no doubt exists that, in the regions II and III, the application of Newtonian gravitational theory is meaningless and one would expect important modifications from the use of general relativity.[20]

B. General Relativistic Treatment

The effect of introducing general relativity is comparatively simple as long as the particles are treated in a statistical way (no allowance for the details of particle-particle coupling). Then the gravitational problem is no more difficult than that of the Hartree-Fock atom. In both cases the interaction is universal, in the sense that one law covers all ranges of distances (in contrast, i.e., to nuclear forces). The case of ideal particles coupled gravitationally differs from the case of electric coupling in this respect, that no "interaction" ever puts in a direct appearance. Instead, thanks to the geometrical interpretation of gravitation, it is possible to treat the interaction by simply considering the field equation for free particles in a curved space where the metric is determined by the system of particles itself. This treatment has the advantage of being valid even in the region of an arbitrarily strong gravitational field.

We shall consider scalar bosons described by the curved-space[21] Klein-Gordon equation

$$\begin{aligned} \nabla_\alpha \nabla^\alpha \phi + \mu^2 \phi &= 0, \\ \nabla_\alpha \nabla^\alpha \phi^* + \mu^2 \phi^* &= 0, \end{aligned} \tag{7}$$

where $\mu = mc/\hbar$ and ∇_α and ∇^α are, respectively, the operators of covariant and contravariant differentiation. This equation can be derived from the Lagrangian[22]

$$\mathcal{L} = -\hbar^2(2m)^{-1}(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - \mu^2\phi^*\phi). \tag{8}$$

In the usual way, we can derive the following conserved quantities: the symmetric energy-momentum tensor

$$T_{\mu\nu} = 2(|g|)^{-1/2}\left(\frac{\partial}{\partial x^\alpha}\frac{\partial(\sqrt{|g|})\mathcal{L}}{\partial(\partial g^{\mu\nu}/\partial x^\alpha)} - \frac{\partial(\sqrt{|g|})\mathcal{L}}{\partial g^{\mu\nu}}\right), \tag{9}$$

and the current vector

$$J^\mu = i\hbar^{-1}c\{[\partial\mathcal{L}/\partial(\partial_\mu\phi^*)]\phi^* - [\partial\mathcal{L}/\partial(\partial_\mu\phi)]\phi\}, \tag{10}$$

where $g = \det g_{\alpha\beta}$.

We only wish to consider spherically symmetric distributions of equilibrium. Therefore, we may express the metric in Schwarzschild coordinates ($x^0 = ct$, $x^1 = r$,

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[20] Using the Newtonian gravitational theory, many authors have examined and have tried to give physical meaning to the facts that (a) the total energy of a system of many self-gravitating particles could decrease even if the number of particles increases and (b) the total energy of the system could be negative. See, for example, F. Pacini, Ann. Astrophys. **29**, 193 (1966). In a very simple and intuitively meaningful way based on a semiclassical approach J. M. Levy-Leblond and P. Thurnauer [Am. J. Phys. **34**, 110 (1966)] indicate that the nonlinearity of gravitational interaction should eliminate the existence of systems of self-gravitating particles with negative total energy. In the next paragraph we develop a *fully* relativistic treatment that explicitly shows how both effects (a) and (b) completely disappear. Therefore, these effects resulted from the use, in a highly relativistic region, of a nonrelativistic treatment; and it is clear that they are physically meaningless.

[21] We use a metric with signature $+---$; Greek indices run from 0 to 3, Latin small indices run from 1 to 3; Latin capital indices are used for spinors and run from 1 to 4. For the signs of the Einstein equation we follow L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962).

[22] See, for example, A. Trautman, Bull. Acad. Polon. Sci. **4**, 675 (1965); see also Ref. 21.

$x^2=\theta$, $x^3=\varphi$),

$$ds^2=B(r)c^2dt^2-A(r)dr^2-r^2(\sin^2\theta d\varphi^2+d\theta^2). \quad (11)$$

In this system, Eq. (7) becomes

$$(|g|)^{-1/2}\partial_i[g^{ik}(|g|)^{1/2}\partial_k\phi]+B^{-1}\partial_0^2\phi+\mu^2\phi=0. \quad (12)$$

It is possible to make a separation of variables in Eq. (12) by setting

$$\phi(r,\theta,\varphi,t)=R(r)Y_l^m(\theta,\varphi)e^{-i(E/\hbar)t}, \quad (13)$$

where $Y_l^m(\theta,\varphi)$ is a spherical harmonic. The function R must satisfy the equation

$$R_{ln}''+(2/r+\tfrac{1}{2}B'/B-\tfrac{1}{2}A'/A)R_{ln}' +A[E_{nl}^2B^{-1}\hbar^{-2}c^{-2}-\mu^2-l(l+1)A^{-1}r^{-2}]R_{ln}=0, \quad (14)$$

where the prime denotes differentiation with respect to r.[23]

The most general bound solution of Eq. (7) can be expressed in the following way:

$$\phi(r,\theta,\varphi,t)=\sum_{nlm} c_{nlm}R_{nl}Y_m^le^{-i(E_{nl}/\hbar)t} +\sum_{nlm} b_{nlm}R_{nl}Y_m^{l*}e^{i(E_{nl}/\hbar)t}. \quad (15)$$

Since we are considering a neutral field, ϕ is real and therefore $\phi=\phi^*$ and

$$b_{nlm}=c_{nlm}^*. \quad (16)$$

In the formalism of second quantization, ϕ is an operator and can be separated into two components,

$$\phi^+=\sum_{nlm} \mu_{lmn}^+R_{nl}Y_m^l(\theta,\psi)e^{-i(E_{nl}/\hbar)t}, \quad (17)$$

$$\phi^-=\sum_{nlm} \mu_{lmn}^-R_{nl}Y_m^{l*}(\theta,\psi)e^{+i(E_{nl}/\hbar)t}, \quad (18)$$

so that

$$\phi=\phi^++\phi^-.$$

μ_{lmn}^+ and μ_{lmn}^- are, respectively, the creation and annihilation operators for a particle with angular momentum $\hbar l$, azimuthal momentum $\hbar m$, and energy E_{nl}. These operators satisfy the commutation rules

$$[\mu_{lmn}^+, \mu_{l'm'n'}^-]=\delta_{ll'}\delta_{mm'}\delta_{nn'}, \quad (19a)$$

$$[\mu_{lmn}^+, \mu_{l'm'n'}^+]=[\mu_{lmn}^-, \mu_{l'm'n'}^-]=0. \quad (19b)$$

From the operator ϕ, it is possible to construct the energy-momentum tensor operator $T_{\mu\nu}$ and the current vector operator J^μ. We consider a state $|Q\rangle$ for which all the N particles are in the ground state

$$|Q\rangle=|N,0,0,0,\rangle.$$

We compute the mean values of the components of the operators $T_{\mu\nu}$ and J^μ for this state. We obtain

$$\langle Q|T_0{}^0|Q\rangle=-\tfrac{1}{2}\hbar^2m^{-1}N \times\{[B^{-1}E_{01}{}^2/(\hbar^2c^2)+\mu^2]R_{01}{}^2+A^{-1}R_{01}'^2\}, \quad (20a)$$

$$\langle Q|T_1{}^1|Q\rangle=\tfrac{1}{2}\hbar^2m^{-1}N \times\{[B^{-1}E_{01}{}^2/(\hbar^2c^2)-\mu^2]R_{01}{}^2+A^{-1}R_{01}'^2\}, \quad (20b)$$

$$\langle Q|T_2{}^2|Q\rangle=\langle Q|T_3{}^3|Q\rangle=\tfrac{1}{2}\hbar^2m^{-1}N \times\{[B^{-1}E_{01}{}^2/(\hbar^2c^2)-\mu^2]R_{01}{}^2-R_{01}'^2A^{-1}\}, \quad (20c)$$

$$\langle Q|T_0{}^i|Q\rangle=0, \quad (20d)$$

where E_{01} and R_{01} are, respectively, the eigenvalue and the radial part of the eigenfunction of the ground state ($n=1$, $l=0$). The mean value of the component J^0 of the current vector is

$$\langle Q|J^0|Q\rangle=E_{01}Nm^{-1}c^{-2}R_{01}{}^2B^{-1}. \quad (21)$$

From the expressions (20a) and (20b) and from the Einstein equations in Schwarzschild coordinates, we obtain

$$A'/(A^2r)+(1/r^2)(1-1/A) =\epsilon\{[B^{-1}E_{01}{}^2/(\hbar^2c^2)+\mu^2]R_{01}{}^2+A^{-1}R_{01}'^2\}, \quad (22a)$$

$$B'/(ABr)-(1/r^2)(1-1/A) =\epsilon\{[B^{-1}E_{01}{}^2/(\hbar^2c^2)-\mu^2]R_{01}{}^2+A^{-1}R_{01}'^2\}, \quad (22b)$$

where

$$\epsilon=4\pi Gc^{-4}\hbar^2m^{-1}N,$$

and these together with the Eq. (14) form a closed self-consistent system. The other equations,

$$G_2{}^2=kT_2{}^2 \quad\text{and}\quad G_3{}^3=kT_3{}^3,$$

are consequences of Eq. (22) because of the Bianchi identities

$$\nabla_\alpha G^\alpha{}_\beta=0$$

and of the relation

$$T_{\nu;\mu}{}^\mu=0.$$

The normalization condition

$$\int(\sqrt{-g})\langle J^0\rangle d^3x=N \quad (23)$$

is, explicitly,

$$4\pi\int E_{01}m^{-1}c^{-2}R_{01}{}^2B^{-1/2}A^{1/2}r^2dr=1. \quad (24)$$

The initial conditions and the boundary conditions are

$$R_{01}(0)=\text{const}, \quad (25a)$$

$$R_{01}'(0)=0, \quad (25b)$$

$$A(0)=1, \quad (25c)$$

$$B(\infty)=1. \quad (25d)$$

It is possible to show[16,17] that the conditions (25) are consistent with the system of equations (14) and (22).

We have put the system into dimensionless units obtaining the following expressions:

$$B'/(AB\hat{r})-(1/\hat{r}^2)(1-1/A) =\hat{\epsilon}[(B^{-1}\hat{E}_{01}{}^2-1)\hat{R}_{01}{}^2+A^{-1}R_{01}'^2], \quad (26a)$$

[23] It is possible to demonstrate that the spectrum of eigenvalues of bound states is discrete; see Refs. 16 and 19.

$$A'/(A^2\hat{r})+(1/\hat{r}^2)(1-1/A)$$
$$=\hat{\epsilon}[(B^{-1}\hat{E}_{01}{}^2+1)\hat{R}_{01}{}^2+A^{-1}\hat{R}_{01}'^2], \quad (26b)$$

$$\hat{R}_{01}''+(2/\hat{r}+B'/2B-A'/2A)\hat{R}_{01}'$$
$$+A(\hat{E}_{01}{}^2B^{-1}-1)\hat{R}_{01}=0, \quad (26c)$$

$$\int_0^\infty \hat{R}_{01}{}^2B^{-1/2}A^{1/2}\hat{r}^2d\hat{r}=1, \quad (26d)$$

where now the prime denotes differentiation with respect to $\hat{r}$, and

$$\hat{r}=r\mu, \quad (27a)$$

$$\hat{E}_{01}=E_{01}/mc^2, \quad (27b)$$

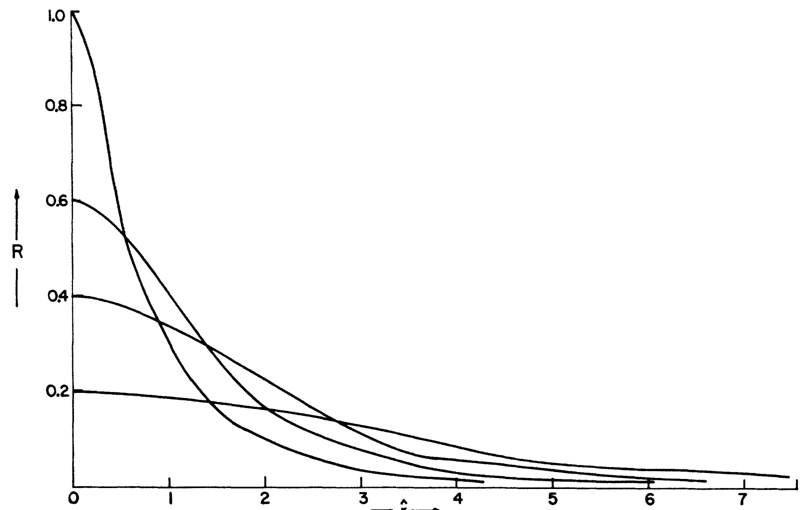
$$\hat{R}=R(4\pi\hat{E}_{01}\mu^{-3})^{1/2}, \quad (27c)$$

$$\hat{\epsilon}=\epsilon\mu^3(4\pi\hat{E}_{01})^{-1}=L^2\mu^2N/\hat{E}_{01}, \quad (27d)$$

L being the Planck length $L=(\hbar G/c^3)^{1/2}$.

We have carried out a numerical integration of the system (26) for different values of the radial function R_{01} at the origin. We have plotted some results in Figs. 6–8. Particulars of the integration method are described in the Appendix.

FIG. 6. Radial function R_{01} is plotted as a function of $\hat{r}$ (dimensionless) for selected values of $R_{01}(0)$ at the origin.



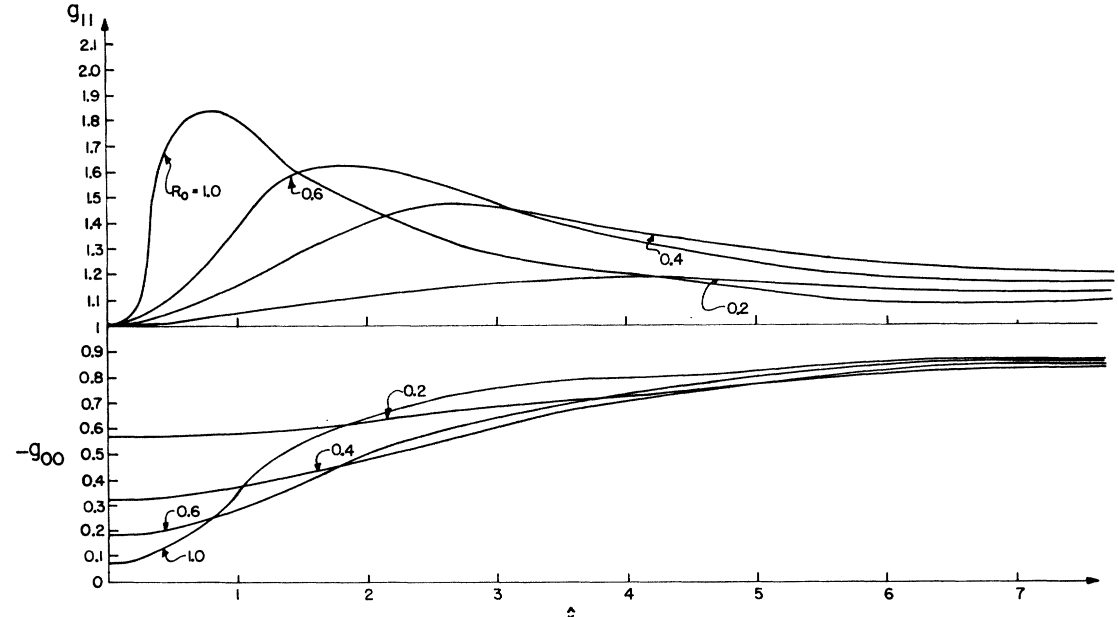


FIG. 7. Coefficients g_{11} and g_{00} of the metric are plotted as functions of $\hat{r}$ (dimensionless) for selected values of the radial function R_{01} at the origin. To an increase of the central density $[\rho_c\sim R_{01}{}^2(0)]$ corresponds an increase in the maximum of g_{11} and a decrease in the minimum of g_{00}.

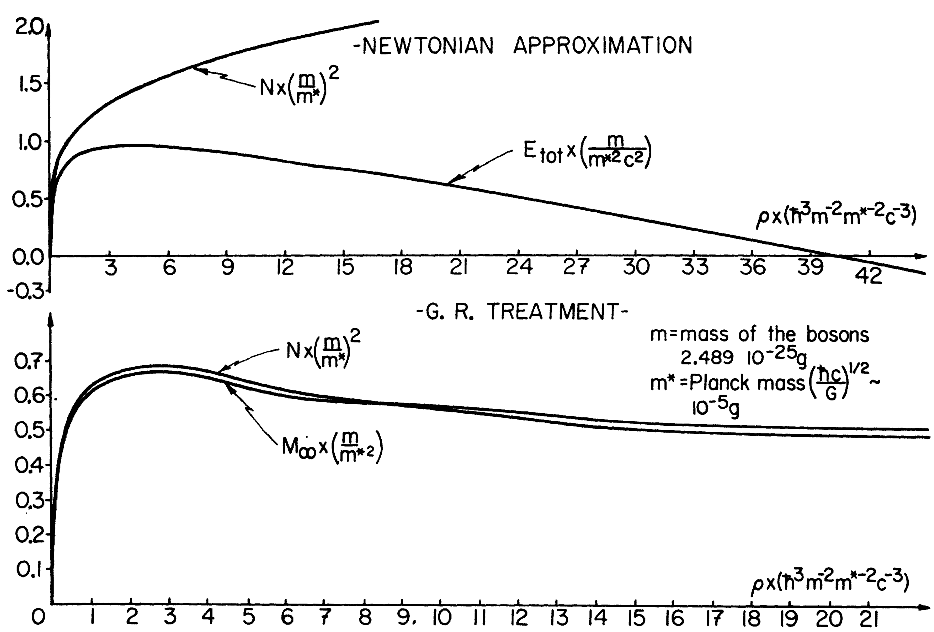


FIG. 8. Mass at infinity multiplied by $(m\times m^{*-2})$ and the total number of particles multiplied by $(m/m^*)^2$ (m^* = Planck mass $\sim 10^{-5}$ g and m = mass of the single boson $= 2.689\times 10^{-25}$ g) as obtained from the general relativistic treatment are plotted as a function of the central density. We have adopted a particular scale to focus our attention on the extreme relativistic region $\{N\sim[(\text{Planck mass})/m]^2\}$ where the contributions of general and special relativity are more important. For a direct comparison, we also show the corresponding quantity obtained in a Newtonian approximation. For a number of particles $N<[(\text{Planck mass})/m]^2$, the general relativistic treatment approaches asymptotically the Newtonian approximation. From this figure it is clear that in the full relativistic treatment, to an increase (decrease) in the particle number, there always corresponds an increase (decrease) of the mass at infinity. This result eliminates one of the strongest difficulties of the Newtonian approximation, where, for sufficiently high density, an increase in the particle number corresponds to a decrease in the total energy of the system (in the Newtonian approximation this last quantity, divided by c^2, takes the place of the mass at infinity of general relativity). The mass at infinity always stays positive and, at least to the accuracy of our numerical computations, seems to approach an asymptotic positive value when the central density goes to infinity. The total number of particles in the general relativistic treatment reaches a maximum value $N_{\rm crit}$, otherwise nonexistent in the Newtonian approximation; in this way the concept of a critical mass is introduced and the presence of the gravitational collapse, also in the case of the bosons, seems unavoidable. We notice that, in the asymptotic region, increasing the central density results in the curve of the mass at infinity crossing the curve of the total number of particles, suggesting the existence of gravitationally unbound states.

The introduction of special relativity (Klein-Gordon equation) and general relativity eliminates completely some difficulties present in the nonrelativistic Newtonian approximation; i.e., the regions II and III of Fig. 6 have disappeared. An increase (decrease) in the number of particles *always* corresponds to an increase (decrease) in the mass at infinity. (See Fig. 8.)

On the other hand, the relativistic treatment introduces the concept of critical mass. The mass at infinity and the number of particles expressed as functions of the central density (see Fig. 8) reach a maximum $M_{\rm crit}\sim 0.311\times 10^{-9}/m$ g, $N_{\rm crit}\sim 3.01\times 10^{-10}/m^2$, corresponding to a central density $\rho_{\rm crit}\sim 5.26\times 10^{97}m^2$ g/cm^3, where m is the boson's mass in g. Both quantities reach their peak values at the same value of the central density (or the same value of any other appropriate parameter). After this maximum they decrease monotonically for an arbitrary increase of the central density.[24] We give some numerical values in Table II.

Imagine bosons of one or another mass, and out of each kind of boson imagine a system put together composed of very many identical particles. For each kind of boson there will be a different critical mass. When the mass m of the particle goes to zero, the critical number of particles $N_{\rm crit}$ goes to infinity. So does the critical mass $M_{\rm crit}$. For the case of distributions endowed with the critical mass, the Schwarzschild radial coordinate r, at which g_{11} reaches the maximum, also goes to infinity as m goes to zero. Simultaneously, the central density ρ goes to zero.

One can treat a system of many bosons at constant temperature T as a fluid with an equation of state $p=p(\rho)$ derived from quantum statics in flat space. Can we extend this treatment to $T=0$ (ground state)? No. We run into difficulties because the pressure is proportional to $T^{5/2}$ and therefore vanishes in the limit $T\to 0$. Proceeding to this limit, we would never obtain any of the configurations of equilibrium that we have found. It is clear that the approximation of treating the system as a perfect fluid is completely inadequate at

[24] This asymptotic behavior can be studied even from the analytic point of view. The system of equations (26) admits in fact a scaling law between solutions relative to different numbers of particles in the asymptotic regions $[\rho(0)\to\infty]$. The existence of such a scaling law was pointed out to one of us (R. R.) by J. A. Wheeler.

TABLE II. In (A) are given some numerical results relative to the Newtonian nonrelativistic treatment of a system of many self-gravitating bosons in the ground state ($n=1$, $l=0$). The mass of the bosons has been chosen to be $m=2.689\times10^{-25}$ g. The value of the radial coordinate for which the potential has one-half of its value at the origin has been defined to be the radius of the distribution. The total mass of the system has been computed neglecting the binding energy. From these numerical values, the presence of a scaling law clearly appears in the nonrelativistic treatment. It is also clear that the density at which such a quantum-gravitational bounded state takes place is strongly dependent upon the number of particles under consideration. In (B) numerical results for the extreme relativistic region are given. $R_{01}(0)$ is the value of the radial part of the wave function at the origin. The mass at infinity has been computed from the asymptotic behavior of g_{11} and g_{00} at infinity and the value is given in units $(\hbar c G^{-1} m^{-1})$. The eigenvalue E_{01} has been determined by requiring that the radial function R_{01} goes to zero at infinity and is measured in units of mc^2, where m is the boson mass. The value of the radial coordinate r corresponding to the maximum of g_{11} has been defined to be the radius of the distribution (units $\hbar m^{-1}c^{-1}$). The minimum of g_{00} is attained at the origin and its value is fixed in agreement with the requirement $g_{00}(\infty)=1$. The number of particles determined by the integral $\int\langle J^0\rangle(-g)^{1/2}d^3x=N$ is measured in units $L^{-2}m^{-2}$, where $L=(\hbar c/G)^{1/2}$.

(A) Nonrelativistic Newtonian region

Number of particles	Radius (cm)	Mass of the system (g)	Binding energy over total energy	Density (g cm^{-3})	Gravitational potential at $r=0$ (cm^2 sec^{-2})
10^{10}	7.55×10^{17}	2.489×10^{-15}	1.060×10^{-62}	1.047×10^{-62}	3.089×10^{-42}
10^{20}	7.55×10^{7}	2.489×10^{-5}	1.060×10^{-42}	1.047×10^{-47}	3.089×10^{-22}
10^{30}	7.55×10^{-3}	2.489×10^{5}	1.060×10^{-22}	$1.047\times10^{+18}$	3.089×10^{-2}

(B) Relativistic region

$R_{01}(0)$	Number of particles $\times(m^2m^{*-2})$	Radius$\times(\hbar^{-1}mc)$	$M_\infty\times(mm^{*-2})$	Eigenvalue $E_{01}=E_{01}(mc^2)^{-1}$	Max of g_{11}	Min of g_{44}
0.2	0.6389	5.100	0.6207	0.9403	1.236	0.5771
0.4	0.6225	2.992	0.6086	0.8993	1.452	0.3207
0.6	0.5163	2.072	0.5249	0.8783	1.632	0.1687

$T=0$ in a system of this kind. It is essential to allow, as we have, for the fact that all the particles fall into the iowest quantum state, a state which moreover carries lts own characteristic distribution of pressure, stress, and density. Moreover, the pressure is anisotropic and very different from zero.

The anisotropy is due to the factor $A^{-1}R'^2$ which appears with different sign in the component $T_1{}^1$ and in the components $T_2{}^2$ and $T_3{}^3$ of the tensor momentum energy. In the distribution that we have considered, all the particles are in the same ground state $n=1$, $l=0$ and they are limited to a region of the order of the de Broglie wavelength $\hbar p^{-1}$. For a number of particles $N\sim[(\text{Planck mass})/m]^2$, we have $p\sim mc$. Referred to $\hbar/mc$, the inhomogeneity of the effective gravitational potential $(G\rho c^{-2})$ is of the order $G\rho\hbar^2m^{-2}c^{-4}\sim1$. Under such conditions the gravitational disturbance in the energy-momentum tensor of the system of particles and the anisotropy due to the tide-producing force are indeed expected to be very large. However, if the N bosons are equally distributed in excited states, the radial distribution $R(r)$ of the system can have an absolutely negligible derivative with respect to r. In this case, the quantum gravitational bound state for the system still exists, but the anisotropy in the energy-momentum tensor disappears.

C. Possible Generalization of the Method

In the preceding paragraph, we have studied the problem of a system of bosons in the ground state. It would be interesting to study the corresponding problem for a distribution function

$$\langle 0,0,\cdots,N_{nlm},\cdots,0|\,, \tag{28}$$

i.e., all the bosons in the same excited state, and to examine the dependence of the critical mass upon the quantum numbers n, l, m of that state.

We would have to compute the mean value of the energy-momentum tensor corresponding to this distribution. The radial function would satisfy Eq. (14), and in the mean value of the energy momentum tensor some quantities depending on n, l, and m would be present, e.g.,

$$\langle T_1{}^1\rangle=A^{-1}R_{nl}'^2+(B^{-1}E_{nl}^2-\mu^2)R_{nl}^2-[l(l+1)/r^2A]R_{nl}^2. \tag{29}$$

The computation of the mean value is completely analogous to the calculation (20) for the ground state. The number of zeros in the radial function is equal to the difference $n-l-1$. This number was zero for the self-gravitating system in its ground state. For the general excited state this difference will be large, and the radial function will have many nodes. However, there is another case where again the number of nodes is zero, namely, large n, but with l also large and equal to $n-1$. A simple analysis shows that we are dealing here with waves running round in a thin "active region" or "spherical zone of activity." It is interesting to see that if we write the equations in the limit $\mu\to0$, we obtain after some simple approximations Wheeler's equation for geons—not, however, electromagnetic geons (built on a field of spin 1) nor gravitational geons (spin-2 field) but geons built on a scalar field of spin 0.

A further generalization to distributions of the form

$$\langle N_{100},N_{200},N_{210},\cdots,0| \tag{30}$$

and a corresponding examination of the critical mass would be possible.

In this case we would have a number of radial equations equal to the number of different values of l and in the energy-momentum tensor a sum of contributions belonging to all the different values of n, l, and m for which $N_{nlm} \neq 0$, e.g.,

$$\langle T_1{}^1 \rangle = \sum_{lmn} c_{lmn} \langle T^1{}_{1lmn} \rangle , \qquad (31)$$

where the c_{lmn} are some suitable normalization factors. The computation of the mean values would be done using the commutation rules (19).

Down to how small a number of particles does it make sense to use the statistical treatment which we have given for a system of N ideal self-gravitating bosons in their ground state? For bosons of any familiar mass value, the statistical treatment in the small-N limit is Newtonian. It gives for the binding energy of the N-boson system

$$E_{\rm bind} = 0.1626 N^3 G^2 m^5 \hbar^{-2} , \qquad (32)$$

and for the two-boson system

$$E_{\rm bind} = 1.3008 G^2 m^5 \hbar^{-2} . \qquad (33)$$

On the other hand, the exact treatment of the ideal two-boson system follows from the standard theory of the hydrogen atom, when we insert (a) for the mass the reduced mass $\frac{1}{2}m$ of the two-boson system and (b) for the coefficient e^2 of $1/r^2$ in the expression for the force the Newtonian value Gm^2; thus,

$$E_{\rm bind} = 0.25 G^2 m^5 \hbar^{-2} . \qquad (34)$$

Comparing (33) and (34), we see that the statistical treatment gives a value for the binding ~ 5.2 times greater than the correct value. The discrepancy will be of the same order whenever we go to the full relativistic treatment and we consider the case of a small number of particles of appropriately larger mass ($\sim 10^{-5}$ g). The reason for the error is clear: The quantity $\langle J^0 \rangle$ does not represent a real density of particles but only a density of probability. Therefore, the metric is computed in correspondence not to the time-changing momentary distribution of matter but to a probability distribution. Thus it would appear that the treatment developed here, while valid for a system of a large number of particles, is a poor approximation for a single particle as well as for a system of only two or three particles (large fluctuation away from any average density, correction for center of mass, etc.). In fact, a system of equations equivalent to the system (26) and substantially equivalent to the equations published in Refs. 16 and 17 has been recently analyzed in connection with the problem of one particle or a few particles by Feinblum and McKinley,[25,26] and by Kaup.[27] Moreover, in our opinion there is not the slightest reason to believe that the considerations on a relativistic many-boson system given in this paper have any relevance whatever to the quite different problem of the internal structure of a single boson.[28,29]

III. FERMIONS

To bring out the effects that we are looking for with maximum clarity, we restrict attention here and in the following to an idealized system of fermions: particles which interact with each other exclusively by gravitational forces (no electric forces, no nuclear forces), and which are treated as stable (no β decay, no other elementary particle transformations). A collection of neutrons (in the first few minutes, before β decay can occur) is the closest approximation we have today to such a system. However, it should be emphasized that well-known effects come into play for neutrons at sufficiently high densities, which make a neutron star depart from the ideal system under consideration here in respects which are important and which are still not sufficiently well understood to be neglected in a detailed analysis.[30]

As we have pointed out in the Introduction, for a system of many self-gravitating fermions, a "spike" in the density at the origin (Fig. 1) forces the effective potential to change substantially over dimensions of the order $r \sim L(m^*/m)^{4/3}$. Then the concept of equation of state breaks down.

We make the analysis first in the framework of Newtonian mechanics: The potential seems to be perfectly regular for the self-gravitating system of fermions (Fig. 9). In this figure the gravitational potential (χ/x), expressed in appropriate dimensionless units, is plotted as a function of the distance from the center, also in appropriate dimensionless units. Both quantities are taken from the tables of Emden[31] for a polytrope of index $n = \frac{3}{2}$; that is a function which satisfies the

[25] D. A. Feinblum and W. A. McKinley, Phys. Rev. **168**, 1445 (1968).

[26] Independently from the objection of the applicability of our method to the one-particle system that we have just explained, a serious difficulty comes from the development proposed by Feinblum and McKinley in the neighborhood of the origin [Eqs. (18)–(20), and Sec. 7 in Ref. 25]. This development implies that either (a) $g_{00}(0) = \infty$ or (b) $g_{11}(0) = 0$, or both. We know that for any physically acceptable distribution of matter with $p > 0$, it must always happen that $g_{00}'/g_{00} > 0$, and this shows that the condition (a) is not compatible with the requirement of Minkowskian space-time at infinity. On the other hand, the condition (b) is not compatible with the standard expression for the space part of a metric, regular at the origin, which is expressed in Schwarzschild coordinates; namely, that $g_{11} = -1/(1 - \int_0^r T_0{}^0 r^2 dr/r)$. Moreover, with the expression proposed in Ref. 25, the scalar of curvature and the invariants of curvature are singular near the origin.

[27] D. J. Kaup, Phys. Rev. **172**, 1332 (1968).

[28] We understand from M. J. Stakvilevicius and Y. P. Terletsky by private communication to one of us (R. R.) that they have a somewhat different point of view than ours on this question.

[29] In the neighborhood of one elementary particle of 10^{-5} g and 10^{-33} cm, there is an extremely strong vacuum polarization due to the gravitational field of the particle. This argument has been developed by one of us (R. Ruffini, Ref. 19).

[30] Strong interaction between the particles, creation of new particles, gravitational vacuum fluctuations, velocity of sound greater than c, etc.

[31] R. Emden, *Gaskugeln* (Verlag von Teubner, Berlin, 1907).

equation

$$d^2\chi/dx^2=-\chi^{3/2}/\sqrt{x}. \tag{35}$$

The function χ satisfies the normalization condition

$$\int_0^{x_{\max}} x^{1/2}\chi^{3/2}dx=1. \tag{36}$$

The connection between this polytrope and the system of N fermions is well known. The usual radial coordinate r in the Newtonian system is connected with the dimensionless coordinate x by the equation

$$r=N^{-1/3}bx, \tag{37}$$

where the unit of length b has the value

$$b=\tfrac{1}{2}(\tfrac{3}{4}\pi)^{2/3}\hbar^2m^{-3}G^{-1}.$$

The value of the gravitational potential (relative to the gravitational potential on the surface of the system as standard of reference) is

$$\mathcal{G}=GNm\chi/r, \tag{38}$$

where G is the Newtonian gravitational constant.

The kinetic energy (KE) of a fermion at the point r is connected with the Fermi momentum and the potential $\mathcal{G}$ by the equation

$$(\text{KE})_{\max}=p_F^2(2m)^{-1}=m\mathcal{G}. \tag{39}$$

The mass density of particles is

$$\rho(r)=(8\pi/3)m\hbar^{-3}p_F^3. \tag{40}$$

This is the source term in the Newtonian equation for the gravitational potential

$$\Delta\mathcal{G}=-4\pi G\rho. \tag{41}$$

From Eqs. (37) and (38) it follows immediately that the density distribution for an arbitrary particle number can be obtained directly from the graph of χ/x as function of x in Fig. 9.

The smoothness of the dimensionless potential plotted in Fig. 9 shows that there is no "spike" in the potential for a system containing a reasonable number of particles. However, with increaseing N the whole scale shrinks. Automatically, what was a potential without a "spike" becomes a potential which is everywhere a "spike." Then the statistical treatment fails.

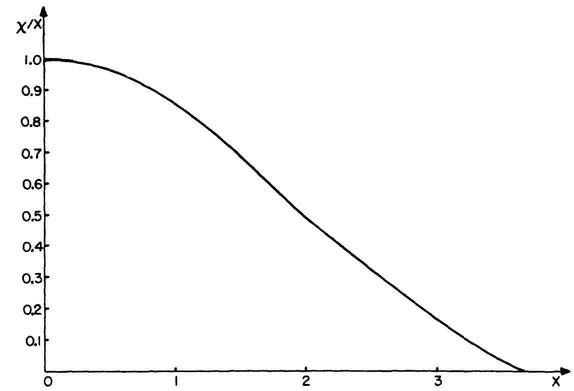


FIG. 9. Newtonian gravitational potential of a system of self-gravitating fermions in degenerate state is plotted as a function of the radius in appropriate dimensionless unit [$\mathcal{G}=GNmb^{-1}(\chi/x)$, $r=N^{-1/3}bx$, where G is the Newtonian gravitational constant, $b=\frac{1}{2}(\frac{3}{4}\pi)^{2/3}\hbar^2m^{-3}G^{-1}$, and m is the mass of the fermion].

Long before one arrives at this critical value of N, however, the bulk of the fermions have been promoted to relativistic energies (last entry in Table III). The nonrelativistic treatment fails. The Newtonian nonrelativistic regime ends when the Fermi KE, largest at the center of the system, attains a value of the order of mc^2; thus

$$(\text{KE})_{\text{Fermi}}=m\mathcal{G}\sim mc^2, \tag{42}$$

$$GN_{\text{rel}}^{4/3}m^2b^{-1}(\chi/x)_0\sim mc^2. \tag{43}$$

From this equation we find

$$N_{\text{rel}}\sim m^{*3}/m^3(\sim 10^{57}\text{ for }m=m_N=1.6\times10^{-24}\text{ g}). \tag{44}$$

Here $m^*=(\hbar c/G)^{1/2}=2.2\times10^{-5}$ g is the Planck mass and $(\chi/x)_0$ is the dimensionless measure of the gravitational potential at the center.

For values of $N\geq N_{\text{rel}}$, the Newtonian treatment has to be modified in two ways: (a) The nonrelativistic relation $E_F=p_F^2(2m)^{-1}$ between the Fermi energy and the Fermi momentum must be replaced by the relativistic one, and (b) the Newtonian theory of gravity must be corrected to general relativity, because the dimensions of the system are becoming comparable to the Schwarzschild radius.

TABLE III. Properties of an ideal system of self-gravitating fermions in the Newtonian regime. The mass of the ideal neutral fermions considered is $m=1.6\times10^{-24}$ g. Here the "radius" is the distance at which the Fermi kinetic energy falls to half-value; $(\text{KE}/mc^2)_{r=0}$ is the kinetic energy of the particle at the center in units mc^2. ρ_0 is the central density, M is the total mass (neglecting the negative mass of gravitational binding), and r_{Schw} is the gravitational radius of the system endowed with this mass. From the last line (10^{54} particles) it is evident how the effects of special and general relativity are manifested simultaneously.

N	Radius (cm)	$(\text{KE}/mc^2)_{r=0}$	ρ_0 (g/cm³)	M (g)	r_{Schw} (cm)
10^3	6.34×10^{23}	3.901×10^{-73}	1.172×10^{-90}	1.675×10^{-21}	12.4×10^{-50}
10^{21}	6.34×10^{17}	3.901×10^{-49}	1.172×10^{-54}	1.675×10^{-3}	12.4×10^{-32}
10^{39}	6.34×10^{11}	3.901×10^{-25}	1.172×10^{-18}	1.675×10^{15}	12.4×10^{-14}
10^{57}	6.34×10^{5}	3.901	1.172×10^{18}	1.675×10^{33}	12.4×10^{4}

Historically, Landau[32] (1932) and Chandrasekhar[33] (1931) considered the effect of special relativity before Oppenheimer and Volkoff[34] added the effects of general relativity. In the meantime, the properties of an idealized system of fermions have been studied in considerably more detail.[35] As the central density goes higher and higher, a localized "spike" indeed develops in the gravitational potential.[36] Ultimately, it becomes so sharp that in the region of the spike the concept of the equation of state therefore breaks down.

In the following section we trace out in detail the properties of the region of the spike, the connections between the theory of many self-gravitating particles and the concept of the equation of state, and finally the modification which comes about in the region of the spike.

We found that the modifications in the region of the spike are qualitatively extremely important; however, we believe that they cause no more trouble in the theory of the neutron star than the corresponding troubles caused for the theory of the atom and for the same reason: The volume of the effective region is negligible compared to the volume of the entire system. On the other hand, we show explicitly that outside the region of the "spike" the application of the equation of state is perfectly legitimate and coincides with the treatment of many fermions with a self-consistent field method.

A. Formalism of the Relativistic Treatment

We apply the formalism of Sec. II to the case of fermions. We assume a familiarity with the spinor formalism in a differentiable manifold. Nevertheless, it is useful to recall a few definitions.[37]

It simplifies the problem to adopt a system of isotropic coordinates. Assuming a spherical symmetric and static distribution, the metric in this system of coordinates is

$$ds^2=B(r)c^2dt^2-A(r)[(dx^1)^2+(dx^2)^2+(dx^3)^2], \quad (45)$$

where

$$r^2=x^2+y^2+z^2, \qquad x^1=x, \quad x^2=y, \quad x^3=z. \quad (46)$$

The Dirac matrices must satisfy the relation

$$\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2g_{\alpha\beta}I, \quad (47)$$

where I is the unit matrix.

In the Majorana representation and with the metric (45), we obtain for the γ_α the following expressions:

$$\gamma_1=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix}A^{1/2}, \quad (48a)$$

$$\gamma_2=\begin{vmatrix}0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{vmatrix}A^{1/2}, \quad (48b)$$

$$\gamma_3=\begin{vmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}A^{1/2}, \quad (48c)$$

$$\gamma_0=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}B^{1/2}. \quad (48d)$$

We will also use the Pauli representation; we have the relation

$$\gamma_{\text{Pauli}}=Q^{-1}\gamma_{\text{Majorana}}Q, \quad (49a)$$

where Q is the unitary matrix

$$Q=(1/\sqrt{2})\begin{vmatrix}1&1&i&i\\-i&i&-1&1\\-i&i&1&-1\\1&1&-i&-i\end{vmatrix}. \quad (49b)$$

The Dirac equation is

$$\gamma^{\alpha A}{}_B\nabla_\alpha\psi^B=\mu\psi^A, \quad (50)$$

with ψ^A indicating a contravariant spinor of four components. The covariant derivative ∇_α is

$$\nabla_\alpha\psi^A=\partial_\alpha\psi^A+\sigma_\alpha{}^A{}_B\psi^B, \quad (51)$$

$$\nabla_\alpha\psi_A=\partial_\alpha\psi_A-\sigma_\alpha{}^B{}_A\psi_B, \quad (52)$$

where

$$\sigma_\alpha{}^A{}_B=-\tfrac{1}{4}\Gamma_\alpha{}^\delta{}_\beta\gamma_\delta{}^A{}_C\gamma^{\beta C}{}_B. \quad (53)$$

From (51)–(53) we obtain

$$\nabla_\alpha\gamma_\beta{}^A{}_B=\partial_\alpha\gamma_\beta{}^A{}_B-\Gamma_{\beta\alpha}{}^\delta\gamma_\delta{}^A{}_B+\sigma_\alpha{}^A{}_C\gamma_\beta{}^C{}_B-\sigma_\alpha{}^D{}_B\gamma_\beta{}^A{}_D=0. \quad (54)$$

We define the covariant and antisymmetric two-spinor fundamental form

$$\omega_{AB}=-\omega_{BA}, \quad (55)$$

with components

$$\omega_{14}=-i(-g)^{1/4}, \quad \omega_{23}=-i(-g)^{1/4}, \qquad \omega_{42}=\omega_{13}=\omega_{24}=\omega_{34}=0, \quad (56)$$

for which we have

$$\nabla_\alpha\omega_{AB}=0.$$

This fundamental form is used to raise and lower the spinorial indices

$$\psi_A=\omega_{AB}\psi^B. \quad (57)$$

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[32] See Ref. 1.

[33] See Refs. 1, 6, and 7.

[34] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 455 (1939).

[35] See, for example, B. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, Ref. 13.

[36] See Fig. 1.

[37] We will follow the formalism of A. Lichnerowicz, *Propagateur et quantification en relativité générale* (Gauthier-Villars, Paris, 1966); see also A. Lichnerowicz, Bull. Soc. Math. France, 1966; Ann. Inst. Henri Poincaré **1**, No. 3 (1964).

B. Dirac Equation in a Given Isotropic Metric

If we write Eq. (50) using the metric (45), we obtain the equation

$$\gamma^{\alpha A}{}_B\partial\alpha(A^{7/8}B^{3/8}\psi^B)=\mu\psi^A A^{7/8}B^{3/8}, \tag{58}$$

which in the Pauli representation is

$$\begin{aligned}
&A^{1/2}(B^{-1/2}E\hbar^{-1}c^{-1}+\mu)P^1\\
&\qquad+(\partial_1P^4-i\partial_2P^3-\partial_3P^4)=0,\\
&A^{1/2}(B^{-1/2}E\hbar^{-1}c^{-1}+\mu)P^2\\
&\qquad+(\partial_1P^3+i\partial_2P^3-\partial_3P^4)=0,\\
&A^{1/2}(-B^{-1/2}E\hbar^{-1}c^{-1}+\mu)P^3\\
&\qquad+(\partial_1P^2-i\partial_2P^2+\partial_3P^1)=0,\\
&A^{1/2}(-B^{-1/2}E\hbar^{-1}c^{-1}+\mu)P^4\\
&\qquad+(\partial_1P^1+i\partial_2P^1-\partial_3P^2)=0.
\end{aligned}\tag{59}$$

Here we have put

$$\psi^A(x^1,x^2,x^3,x^0)=P^A(x^i)e^{iE\hbar^{-1}c^{-1}x^0}A^{-7/8}B^{-3/8}. \tag{60}$$

Let us introduce a polar system of coordinates; just as in flat space, it is possible to separate the radial part of the function P^A from the angular part. We obtain the following complete set of solutions.

For $k>0$, $E>0$:

$$\begin{aligned}
&P_{\mathrm{I}}{}^1{}_{kmn}=[(k-1+m)/(2k-1)]^{1/2}F_{kn}Y_{k-1}{}^{m-1},\\
&P_{\mathrm{I}}{}^2{}_{kmn}=[(k-m)/(2k-1)]^{1/2}F_{kn}Y_{k-1}{}^{m},\\
&P_{\mathrm{I}}{}^3{}_{kmn}=-i[(k-m+1)/(2k+1)]^{1/2}G_{kn}Y_k{}^{m-1},\\
&P_{\mathrm{I}}{}^4{}_{kmn}=i[(k+m)/(2k+1)]^{1/2}G_{kn}Y_k{}^{m}.
\end{aligned}\tag{61a}$$

For $k<0$, $E>0$:

$$\begin{aligned}
&P_{\mathrm{II}}{}^1{}_{-kmn}=[(-k-m+1)/(-2k+1)]^{1/2}\\
&\qquad\times F_{kn}Y_{-k}{}^{m-1},\\
&P_{\mathrm{II}}{}^2{}_{-kmn}=-[(-k+m)/(-2k+1)]^{1/2}\\
&\qquad\times F_{kn}Y_{-k}{}^{m},\\
&P_{\mathrm{II}}{}^3{}_{-kmn}=-i[(-k+m-1)/(-2k-1)]^{1/2}\\
&\qquad\times G_{kn}Y_{-k-1}{}^{m-1},\\
&P_{\mathrm{II}}{}^4{}_{-kmn}=-i[(-k-m)/(-2k-1)]^{1/2}\\
&\qquad\times G_{kn}Y_{-k-1}{}^{m}.
\end{aligned}\tag{61b}$$

For $k>0$, $E<0$:

$$\begin{aligned}
&P_{\mathrm{III}}{}^1{}_{kmn}=[(k-1+m)/(2k-\tfrac{1}{2})]^{1/2}G_{-kn}Y_{k-1}{}^{m-1},\\
&P_{\mathrm{III}}{}^2{}_{kmn}=[(k-m)/(2k-1)]^{1/2}G_{-kn}Y_{k-1}{}^{m},\\
&P_{\mathrm{III}}{}^3{}_{kmn}=-i[(k-m+1)/(2k+\tfrac{1}{2})]^{1/2}F_{-kn}Y_k{}^{m-1},\\
&P_{\mathrm{III}}{}^4{}_{kmn}=i[(k+m)/(2k+1)]^{1/2}F_{kn}Y_{-k}{}^{m}.
\end{aligned}\tag{61c}$$

For $k<0$, $E<0$:

$$\begin{aligned}
&P_{\mathrm{IV}}{}^1{}_{-kmn}=[(-k-m+1)/(-2k+1)]^{1/2}\\
&\qquad\times G_{-kn}Y_{-k}{}^{m-1},\\
&P_{\mathrm{IV}}{}^2{}_{-kmn}=-[(-k+m)/(-2k+1)]^{1/2}\\
&\qquad\times G_{-kn}Y_{-k}{}^{m},\\
&P_{\mathrm{IV}}{}^3{}_{-kmn}=-i[(-k+m-1)/(-2k+1)]^{1/2}\\
&\qquad\times F_{-kn}Y_{-k-1}{}^{m-1},\\
&P_{\mathrm{IV}}{}^4{}_{-kmn}=-i[(-k+m)/(-2k+1)]^{1/2}\\
&\qquad\times F_{-kn}Y_{-k-1}{}^{m},
\end{aligned}\tag{61d}$$

where F and G are the radial functions. The angular part is described by the spherical harmonic functions $Y_k{}^m(\theta,\varphi)$; m and k are the integer quantum numbers corresponding, respectively, to the observables

$$J_z+\tfrac{1}{2}\quad\text{and}\quad K=\boldsymbol{\sigma}\cdot\mathbf{L},$$

where $\boldsymbol{\sigma}$ is the spin momentum and $\mathbf{L}$ the orbital angular momentum. The functions G and F must satisfy the system of equations

$$\begin{aligned}
&dF_{kn}/dr+F_{kn}(1-k)/r\\
&\qquad=(E_{kn}\hbar^{-1}c^{-1}B^{-1/2}+\mu)A^{1/2}G_{kn},\\
&dG_{kn}/dr+G_{kn}(1+k)/r\\
&\qquad=(-E_{kn}\hbar^{-1}c^{-1}B^{-1/2}+\mu)A^{1/2}F_{kn},
\end{aligned}\tag{62}$$

where E_{kn}, F_{kn}, and G_{kn} are the eigenvalues and the eigenfunctions corresponding to a given k. It is possible to demonstrate that the spectrum of eigenvalues is discrete.[38]

C. Einstein Equations

So far the treatment applies as well to charged particles as to neutral ones. However, we are interested only in neutral particles (ideal system of self-gravitating neutrons). Therefore, we ask that the field function verifies the condition

$$\psi^A=c\psi^A. \tag{63}$$

Here c is the charge-conjugation operator. To make this condition take its simplest form, we now go to the Majorana representation. There the operation of charge conjugation has the form

$$c\psi^A=\psi^{A*},$$

where ψ^{A*} is the complex conjugate of ψ^A. Thus we demand that

$$\psi^A=\psi^{A*}. \tag{64}$$

It is easy to see that in the Majorana representation we have from (49)

$$\begin{aligned}
&\psi_{\mathrm{I}}{}^1{}_{kmn}=\tfrac{1}{2}(P_I{}^1+iP_I{}^2+iP_I{}^3+P_I{}^4)_{kmn}\\
&\qquad\times e^{i(E_{nk}\hbar^{-1}c^{-1})x^0}A^{-7/8}B^{-3/8},\\
&\psi_{\mathrm{I}}{}^2{}_{kmn}=\tfrac{1}{2}(P_I{}^1+iP_I{}^2-iP_I{}^3+P_I{}^4)_{kmn}\\
&\qquad\times e^{i(E_{nk}\hbar^{-1}c^{-1})x^0}A^{-7/8}B^{-3/8},\\
&\psi_{\mathrm{I}}{}^3{}_{kmn}=\tfrac{1}{2}(-iP_I{}^1-P_I{}^2+P_I{}^3-iP_I{}^4)_{kmn}\\
&\qquad\times e^{i(E_{nk}\hbar^{-1}c^{-1})x^0}A^{-7/8}B^{-3/8},\\
&\psi_{\mathrm{I}}{}^4{}_{kmn}=\tfrac{1}{2}(iP_I{}^4+P_I{}^2-P_I{}^3-iP_I{}^4)_{kmn}\\
&\qquad\times e^{i(E_{nk}\hbar^{-1}c^{-1})x^0}A^{-7/8}B^{-3/8}.
\end{aligned}\tag{65}$$

In the same way it is possible to obtain $\psi_{\mathrm{II}}{}^A$, $\psi_{\mathrm{III}}{}^A$, and $\psi_{\mathrm{IV}}{}^A$. For simplicity in the following, we indicate with ↑ the eigenfunctions with $k>0$ and with ↓ the eigenfunction with $k<0$. It is now possible to write the following expressions:

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[38] A system of equations equivalent to (62) has been found for the motion of a neutrino in a spherically symmetrical gravitational field. See D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).

(i) the symmetric energy-momentum tensor

$$T_{\alpha\beta}=\tfrac{1}{4}(\nabla_\alpha\bar{\psi}\gamma_\beta\psi+\nabla_\beta\bar{\psi}\gamma_\alpha\psi-\bar{\psi}\gamma_\beta\nabla_\alpha\psi-\bar{\psi}\gamma_\alpha\nabla_\beta\psi)\hbar c; \quad (66)$$

(ii) the current vector

$$J^\alpha=i\bar{\psi}\gamma^\alpha\psi; \quad (67)$$

(iii) the spin tensor

$$S^{\rho\alpha\beta}=\bar{\psi}\gamma^\rho(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)\psi. \quad (68)$$

With $\bar{\psi}$ we indicate the covariant spin that is obtained from the contravariant spinor ψ by means of the lowering operator (57); e.g.,

$$\bar{\psi}=\omega\psi^*.$$

In the formalism of second quantization, the wave function ψ is an operator acting on the state vector $\langle Q|$. The $T_{\alpha\beta}$, J_α, and S are also operators and their mean values are computed for a state vector $\langle Q|$ remembering the antisymmetric fermion commutation rules. In the minimum-energy state of the system, the lowest $\frac{1}{2}N$ cells of phase space are occupied. Consequently, we have for the mean values of $\langle Q|T_0{}^0|Q\rangle$ the following expressions:

$$\begin{aligned}\langle Q|T_0{}^0\uparrow|Q\rangle&=\sum_{k>0,n}B^{-1/2}E_{kn}\times|k|(G_{kn}{}^2+F_{kn}{}^2)A^{-1}B^{-1/2},\\ \langle Q|T_0{}^0\downarrow|Q\rangle&=\sum_{k<0,n}B^{-1/2}E_{kn}\times|k|(G_{kn}{}^2+F_{kn}{}^2)A^{-1}B^{-1/2},\end{aligned} \quad (69)$$

and for the mean value of the spatial trace:

$$\begin{aligned}\langle Q|T_i{}^i\uparrow|Q\rangle&=-\langle Q|T_0{}^0\uparrow|Q\rangle+mc^2\sum_{k>0,n}|k|(F_{kn}{}^2-G_{kn}{}^2)A^{-1}B^{-1/2},\\ \langle Q|T_i{}^i\downarrow|Q\rangle&=-\langle Q|T_0{}^0\downarrow|Q\rangle+mc^2\sum_{k<0,n}|k|(F_{kn}{}^2-G_{kn}{}^2)A^{-1}B^{-1/2}.\end{aligned} \quad (70)$$

For the projection of T_{ik} on a unit vector t^i normal to a radial unit vector, we have

$$\begin{aligned}\langle Q|T_{il}\uparrow t^it^l|Q\rangle&=F_{kn}G_{kn}|k|A^{-1}B^{-1/2}r^{-1},\\ \langle Q|T_{il}\downarrow t^it^l|Q\rangle&=F_{kn}G_{kn}|k|A^{-1}B^{-1/2}r^{-1}.\end{aligned} \quad (71)$$

We define, as in the boson case, the probability density ρ by means of the zero component of the vector J^μ:

$$\begin{aligned}\sum_{k>0,n}\rho_{kn}\uparrow&=\sum_{k>0,n}|k|(G_{kn}{}^2+F_{kn}{}^2)A^{-1}B^{-1/2}=J^0\uparrow(g_{00})^{1/2},\\ \sum_{k<0,n}\rho_{kn}\downarrow&=\sum_{k<0,n}|k|(G_{kn}{}^2+F_{kn}{}^2)A^{-1}B^{-1/2}=J^0\downarrow(g_{00})^{1/2}.\end{aligned} \quad (72)$$

It is possible now to write the Einstein equations for the system of N fermions:

$$\begin{aligned}R_0{}^0-\tfrac{1}{2}R&=8\pi Gc^{-4}(\langle T_0{}^0\uparrow\rangle+\langle T_0{}^0\downarrow\rangle),\\ R_i{}^i-\tfrac{3}{2}R&=8\pi Gc^{-4}(\langle T_i{}^i\uparrow\rangle+\langle T_i{}^i\downarrow\rangle).\end{aligned} \quad (73)$$

These equations, with Eq. (62) and the normalization condition

$$\int_0^\infty J^0\uparrow(\sqrt{-g})d^3x=\int_0^\infty J^0\downarrow(\sqrt{-g})d^3x=\tfrac{1}{2}N$$

determine the distribution of the N fermions in the lowest state of energy. The boundary conditions are the same as in the boson case.

D. Possible Approximations

In the ground state of the N-boson system, all the particles are in the same quantum state. How different is the N-fermion system! The Pauli principle forces all the particles into different quantum states. To solve exactly the problem of 10^{57} self-gravitating fermions, it would be necessary to compute 10^{57} eigenvalues and 10^{57} eigenfunctions. Therefore, it is necessary to develop some approximation method.

From (72) we can write the expressions (69) in the following form:

$$\begin{aligned}\langle T_0{}^0\uparrow\rangle&=\sum_{k>0,n}B^{-1/2}E_{kn}\rho_{kn\uparrow},\\ \langle T_0{}^0\downarrow\rangle&=\sum_{k<0,n}B^{-1/2}E_{kn}\rho_{kn\downarrow}.\end{aligned} \quad (74)$$

In the system of equations (62), we can eliminate G_{kn}, obtaining a second-order equation

$$F_{kn}''+(\alpha'/\alpha)F_{kn}'+F_{kn}[(E_{kn}{}^2\hbar^{-2}c^{-2}B^{-1}-\mu^2)A+\alpha'\alpha^{-1}kr^{-1}-k(k-1)r^{-2}]=0, \quad (75)$$

where

$$\alpha\approx(E_{kn}\hbar^{-1}c^{-1}B^{-1/2}+\mu)A^{1/2} \quad (76)$$

and

$$\alpha'\alpha^{-1}=\tfrac{1}{2}A'A^{-1}-\tfrac{1}{2}B'E_{kn}B^{-3/2}(E_{kn}B^{-1/2}+mc^2)^{-1}. \quad (77)$$

We now fix attention on high quantum numbers and we suppose that $E_{kn}B^{-1/2}\hbar^{-1}c^{-1}$ is greater than μ inside the distribution. Then we can write

$$\alpha'\alpha^{-1}=\tfrac{1}{2}(A'A^{-1}-B'B^{-1}). \quad (78)$$

If we put $\phi=\alpha^{-1}F$, we have for Eq. (75) the following new expression:

$$\phi''+\{[E_{kn}{}^2B^{-1}(\hbar c)^{-2}-\mu^2]A-k(k-1)r^{-2}+\alpha'\alpha^{-1}kr^{-1}+\tfrac{1}{2}(\alpha''/\alpha)-\tfrac{3}{4}(\alpha'/\alpha)^2\}\phi=0. \quad (79)$$

We apply the JWKB method. We write

$$\phi=f(r)e^{iS(r)/\hbar}, \quad (80)$$

We obtain the following exact equation[39] for S:

$$-(S')^2+(E_{kn}{}^2c^{-2}B^{-1}-m^2c^2)A-k(k-1)r^{-2}\hbar^2+[\tfrac{1}{2}(\alpha''/\alpha)-\tfrac{3}{4}(\alpha'/\alpha)^2]\hbar^2+\alpha'\alpha^{-1}kr^{-1}\hbar^2+\hbar^2\times[\tfrac{3}{4}(S''/S')^2-\tfrac{1}{2}(S'''/S')]=0. \quad (81)$$

In the case where all the quantities with $\hbar^2$ in front *are small in comparison with the others*, with the exception of $k(k-1)/r^2$ (high quantum numbers), Eq. (81) becomes

$$c^2[k(k-1)\hbar^2r^{-2}A^{-1}+A^{-1}S'^2+m^2c^2]=E_{kn}{}^2B^{-1}. \quad (82)$$

The quantity $A^{-1/2}S'$ is the projection in the radial direction of the momentum p of a particle with total energy E_{kn}, i.e.,

$$A^{-1/2}S'=\rho^ip_i,$$

where ρ^i is a unit radial vector of components

$$\rho^i=[(-g^{11})^{1/2}x^1r^{-1},\ (-g^{22})^{1/2}x^2r^{-1},\ (-g^{33})^{1/2}x^3r^{-1}];$$
$$\rho_i\rho^i=-1.$$

The quantity $E_{kn}B^{-1/2}$ is the energy of the particle and the quantity $[k(k-1)]^{1/2}\hbar A^{-1/2}r^{-1}$ is the magnitude of the projection of the momentum in a plane normal to the radial direction. We see that (82) reduces with this notation to the familiar relation between momentum and energy,

$$w=EB^{-1/2}=c(p^2+m^2c^2)^{1/2}. \quad (83)$$

We neglect in this approximation the following quantities:

$$\alpha'\alpha^{-1}kr^{-1}A^{-1}\hbar^2, \quad (84a)$$

$$A^{-1}[\tfrac{1}{2}(\alpha''/\alpha)-\tfrac{3}{4}(\alpha'/\alpha)^2]\hbar^2, \quad (84b)$$

and as usual we neglect

$$[\tfrac{3}{4}(S''/S')^2-\tfrac{1}{2}(S'''/S')]\hbar^2. \quad (84c)$$

The expressions (84a) and (84b) contain the interaction between a fermion and the metric. We consider in detail the expression (84a).

For high quantum numbers, the projection of the momentum in the plane orthogonal to the radius can be written

$$p_\theta=k\hbar r^{-1}A^{-1/2}. \quad (85)$$

We put

$$\mp j=p_\theta E^{-1}B^{1/2}=k\hbar cE^{-1}B^{1/2}r^{-1}A^{-1/2},\quad 0\leq j<1. \quad (86)$$

The $\mp$ are a consequence of the fact that k can have positive and negative values. From (81) we obtain

$$P^2=c^{-2}w^2(1-m^2c^4w^{-2}\mp j\alpha'c\alpha^{-1}A^{-1/2}\hbar w^{-1}). \quad (87)$$

The $j\alpha'c\alpha^{-1}A^{-1/2}\hbar w^{-1}$ represent a gravitational spin-orbit interaction. The particle with parallel spin modifies its binding energy in opposite sign from the particles with antiparallel spin.

For $\alpha'\alpha^{-1}cA^{-1/2}\hbar w^{-1}>j^{-1}m^2c^4w^{-2}$, the P^α would be timelike. In this limit, certainly different phenomena take place and other effects must be considered (quantization of gravitational field, interaction fermion gravition, etc.). We will now give an order of magnitude for the interaction between the spin and the angular momentum. We have from the relation (87) that this interaction is important when

$$\alpha^{-1}\alpha'\hbar cA^{-1/2}w\sim m^2c^4. \quad (88)$$

To evaluate this quantity we consider the Oppenheimer-Volkoff analytic solution for an infinite central density.[40] Using this solution, we can evaluate in Eq. (88) the factor $\alpha'\alpha^{-1}$ as well as give an approximate value for w. We must only remember that $\beta'\beta^{-1}=-p'(p+\rho)$ and the behavior of the equation of state and of the density $\rho(r)$ near the origin.[41] We see that condition (88) is satisfied when the density is $\rho\sim c^2G^{-1}L^{-2}(m/m^*)^{8/3}\sim10^{+42}$ g cm^{-3} and the central core has dimensions $r\sim L(m^*/m)^{4/3}\sim10^{-8}$ cm (having chosen for m the neutron mass, and m^* and L being the Planck mass and the Planck length).

E. Case of a Weak Field

It is clear that in Eq. (69) the low quantum numbers give a negligible contribution to the mean values $\langle T_0{}^0\uparrow\rangle$ and $\langle T_0{}^0\downarrow\rangle$. Therefore we can limit our attention only to high quantum numbers and, in this limit, the summations can be replaced with integrals. From expressions (69), (72), (74), and (83), if we express the differential density of presence in the momentum space, we obtain for the energy density of our configuration the following expression:

$$\langle T_0{}^0\rangle=h^{-3}\pi^{-2}\int_0^{p_F}c(p^2+m^2c^2)^{1/2}p^2dp, \quad (89)$$

where p_F is the Fermi momentum that is related to the density of particles ρ by the expression

$$\rho=h^{-3}\pi^{-2}\int_0^{p_F}p^2dp=\tfrac{1}{3}p_F{}^3h^{-3}\pi^{-2}. \quad (90)$$

To compute the mean value of the spatial trace, we must evaluate the quantity $\sum_{kn}|k|(F_{kn}{}^2-G_{kn}{}^2)$ present in the expressions (70). Following the approximations adopted in Sec. III D, we can write

$$G_{kn}=g_{kn}e^{i(S/\hbar)}\quad\text{and}\quad F_{kn}=f_{kn}e^{i(S/\hbar)}. \quad (91)$$

We obtain for Eq. (64) the expression

$$g_{kn}[(\hbar c)^{-1}E_{kn}B^{-1/2}+\mu]A^{1/2}=(f_{kn}'-iS'\hbar^{-1}f_{kn}-kr^{-1}f_{kn}). \quad (92)$$

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[39] See, for example, A. Messiah, *Mécanique Quantique* (Dunod Cie., Paris, 1962), pp. 194–202. Clearly this treatment has been generalized in our case to a curved background space.

[40] See Ref. 34.

[41] We know that in the relativistic regime $w\sim cp$ and

$$\int_0^{p_F}8\pi p^3h^{-3}c^{-1}dp=2\pi p_F{}^4h^{-3}c^{-1}=\rho.$$

On the other hand, we have $\alpha'\alpha^{-1}\sim4\pi Gc^{-2}\rho r$, and $G\rho c^{-2}\sim r^{-2}$ from the Oppenheimer-Volkoff solution.

We multiply Eq. (92) by the complex conjugate, and, neglecting the quantity f_{kn}', we have

$$g_{kn}g_{kn}^*[(\hbar c)^{-1}E_{kn}B^{-1/2}+\mu]^2A = (k^2r^{-2}+\hbar^{-2}S'^2)f_{kn}f_{kn}^{*2}. \quad (93)$$

Remembering the expression (82), we can write (93) as follows:

$$|g_{kn}|^2[(\hbar c)^{-1}E_{kn}B^{-1/2}+\mu] = [(\hbar c)^{-1}E_{kn}B^{-1/2}-\mu]|f_{kn}|^2. \quad (94)$$

On the other hand, we know from (72) that the probability density can be expressed in the following way:

$$\rho_{kn}=|k|(|g_{kn}|^2+|f_{kn}|^2)A^{-1}B^{-1/2}. \quad (95)$$

We can therefore express $|f_{kn}|^2$ and $|g_{kn}|^2$ as a function of ρ_{kn}, and we obtain

$$\begin{aligned}|k||f_{kn}|^2&=\tfrac{1}{2}AB^{1/2}\rho_{kn}(EB^{-1/2}+mc^2)E^{-1}B^{1/2},\\ |k||g_{kn}|^2&=\tfrac{1}{2}AB^{1/2}\rho_{kn}(EB^{-1/2}-mc^2)E^{-1}B^{1/2}.\end{aligned} \quad (96)$$

Remembering Eqs. (70), we can write

$$\begin{aligned}\langle T_i{}^i\uparrow\rangle&=-\langle T_0{}^0\uparrow\rangle+\sum_{kn}\rho_{kn}m^2c^4E_{kn}{}^{-1}B^{1/2},\\ \langle T_i{}^i\downarrow\rangle&=-\langle T_0{}^0\downarrow\rangle+\sum_{kn}\rho_{kn}m^2c^4E_{kn}{}^{-1}B^{1/2}.\end{aligned} \quad (97)$$

We can, as before for the $\langle T_0{}^0\rangle$, transform the summation into integrals and express the differential density $d\rho$ in the momentum space, obtaining for the trace the expression

$$\langle T_i{}^i\rangle=\hbar^{-3}\pi^{-2}\times\int_0^{p_F}[c(p^2+m^2c^2)^{1/2}-E^{-1}m^2c^4B^{1/2}]p^2dp. \quad (98)$$

After the computation of the integrals, we obtain

$$\langle T_0{}^0\rangle=c(8\pi^2\hbar^3)^{-1}[p_F(2p_F{}^2+m^2c^2)(p_F{}^2+m^2c^2)^{1/2}-(mc)^4\sinh^{-1}(p_Fm^{-1}c^{-1})], \quad (99)$$

$$\langle T_i{}^i\rangle=-3c(8\pi^2\hbar^3)^{-1}[p_F(\tfrac{2}{3}p_F{}^2-m^2c^2)(p_F{}^2+m^2c^2)^{1/2}+(mc)^4\sinh^{-1}(p_Fm^{-1}c^{-1})]. \quad (100)$$

These expressions for the source of the Einstein equations are exactly the same as those used by Oppenheimer and Volkoff. We have also shown that the pressure is isotropic. We have, in fact, found the relation

$$\langle T_{ik}t^it^k\rangle=\tfrac{2}{3}\langle T_i{}^i\rangle, \quad (101)$$

where t^i is a unit vector normal to a radial unit vector. It is possible to verify that the relation (101) follows from the approximation previously adopted and from (71).

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APPENDIX

The system (26) appears to be a system of nonlinear differential equations of first order in the functions A and B and of second order in R. All three functions are present in each of the three equations. The system is solved by determinining the eigenvalue $\hat{E}_{01}$ which permits the boundary and initial conditions (25) and the integral condition (24) to be satisfied.

If we have a solution of (26) that satisfies the boundary and initial conditions (25) but with the integral (24) having a value $I\neq 1$, then this will be equivalent to a new solution normalized to 1 with

$$A^*=A,\quad \hat{R}^*=R/\sqrt{I},\quad N^*=NI,$$
$$B^*=B,\quad \hat{\epsilon}^*=\epsilon I,$$

where the asterisk indicates the new solution. Another very useful property that we have used during the integration of the system is that g_{00} can be defined up to an arbitrary constant factor. In other words, we can integrate the system and find the eigenvalue independently of the boundary condition (25d), and then divide g_{00} by an appropriate factor so that $g_{00}(\infty)=1$.

We have integrated the system in two completely different ways. In analogy with the usual method adopted for similar problems in atomic and molecular physics, we have used an iterative method of computation. We have expressed A' and B' as functions of A, B, and R; R'' has been expressed as functions of R', A', B', A, and B. We start from flat space $A=B=1$ and

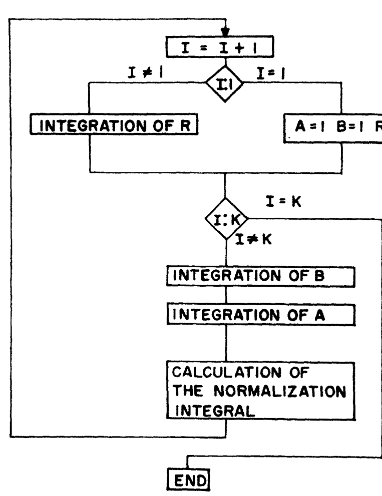

FIG. 10. Scheme of the iterative program relative to the numerical solution of the relativistic equations of many self-gravitating bosons in their ground state. The index I indicates the cycle number; the index K is determined by the number of iterations necessary to obtain a given accuracy.

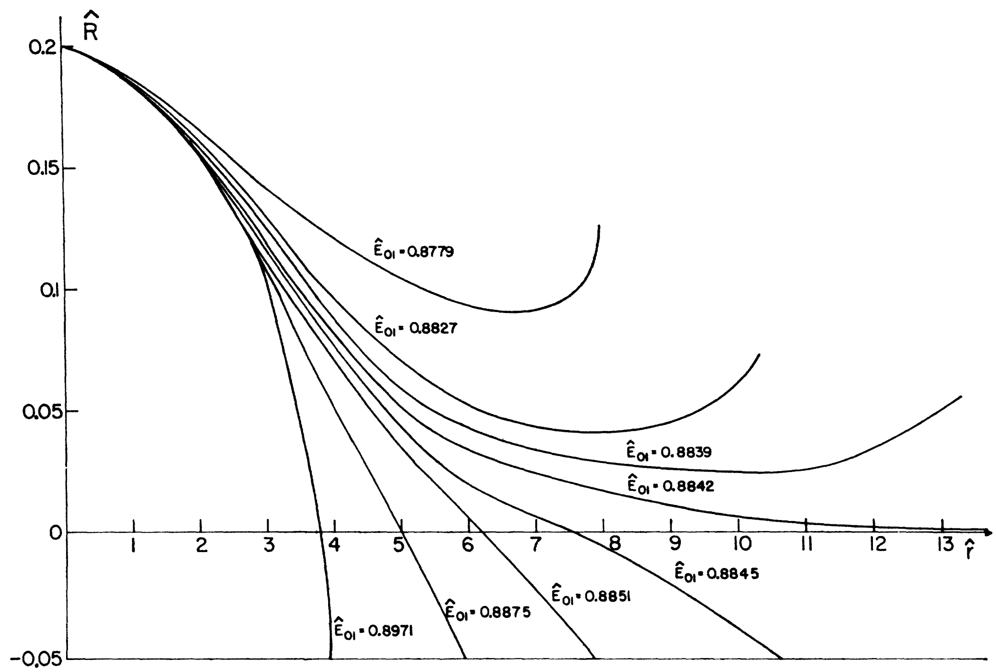

FIG. 11. Radial functions $\hat{R}_{01}$ relative to the ground-state distribution of N self-gravitating bosons are plotted as a function of $\hat{r}$ (dimensionless) for different values of $\hat{E}_{01}$. The eigenvalue is $\hat{E}_{01}=0.8842$.

from a given initial radial distribution $R(r)$. We compute new values for A and B. We put these new values in the radial equation and integrate, obtaining a new radial function. Starting from these values for A, B, and R, we start a new cycle (see Fig. 10).

For any reasonable choice of the initial function R, the procedure converges rapidly. Within five cycles, we found

$$|G_i/G_{i+1}-1|<10^{-6},$$

where G_i and G_{i+1} stand for the three functions A, B, and R, evaluated for the ith and $(i+1)$th cycles. This program was extremely accurate, but five cycles at 3 min per cycle is too long to be practical. For this reason we have developed a new program based on the Runge-Kutta method (for particulars see Ref. 18) using the preceding program only for comparison or for improving the accuracy of some results.

The method of integration is completely different from the former. We fix some value for A, B, and R at the origin and a random value for the eigenvalue $\hat{E}_{01}$. We solve all three equations simultaneously and we extend the solutions, starting from the origin, by successive intervals $\Delta r=h$.

If the value of $\hat{E}_{01}$ is correct, the radial function R decreases exponentially, reaching the value zero at infinity. If it is too small, then at a certain value of r the derivative R' changes sign; thereafter R increases and goes to $+\infty$ as r goes to $+\infty$. Moreover, if $\hat{E}_{01}$ is too large, then at a certain value of r the radial function R will change sign; as r increases further towards infinity, the function R will go to $-\infty$. The program starts the integration at the origin and extends the solution to the point where either $\hat{R}'>0$ or $\hat{R}<0$.

A subprogram optimizes the choice of a new eigenvalue and the integration starts again from the origin. The computation is stopped at the asymptotic region $R<10^{-10}$, $g_{11}\sim g_{00}\sim 1$. Some illuminating diagrams are sketched in Fig. 11.

From the asymptotic form of g_{11}, we have computed, in the usual way, the value of the mass at infinity, and from the maximum of g_{11}, we have determined the "effective radius" of the distribution:

$$\text{effective radius}=\int_0^{r(\max g_{11})}(\sqrt{g_{11}})dr.$$

The computation carried out with this second program (Runge-Kutta) is in perfect agreement with the computation of the first program (iterative method).