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Generally Covariant Integral Formulation of Einstein's Field Equations

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As a basis for a generally covariant theory of Mach's principle, we express Einstein's field equations in integral form. The nonlinearity of these equations is reflected in the kernel of the integral representation, which is a functional of the metric tensor. The functional dependence is so constructed that, subject to supplementary conditions, the kernel may be regarded as remaining unchanged to the first order when a small change in the source produces a corresponding change in the potential. To obtain this kernel, a linear differential operator is derived by varying a particular form of Einstein's field equations. The elementary solution corresponding to this linear operator provides the kernel of an approximate integral representation which becomes exact in the limit of vanishing variations. This representation is in a certain sense unique. Our discussion is confined to a normal neighborhood of the field point.

1. INTRODUCTION

THIS paper is a sequel to a previous one[1] in which an attempt was made to construct a theory of inertia which would satisfy Mach's principle. The basic idea of that theory was to express the gravitational-inertial potential as an integral over the distribution of matter, thus ensuring that the whole of the potential was due to physical sources. In the interests of simplicity, the potential was assumed to be a four-vector, but it was recognized that a satisfactory theory would have to be based on a potential which is a symmetrical second-rank tensor. The development of such a theory was promised in that paper, and this is the question to which we now return.

The first problem which must be resolved is the relation such a Machian theory of inertia would bear to general relativity, where the inertial properties of matter are described by means of a tensor potential $g^{\mu\nu}$ satisfying Einstein's field equations. It is now well understood that these equations are the only wavelike equations for a tensor potential whose sources are conserved in the presence of the field. It would therefore seem natural to express the Machian inertial theory in terms of a *generally covariant integral formulation* of Einstein's equations—provided that such a formulation could be derived in spite of the nonlinearity of these equations (this nonlinearity being enforced by the requirement that the sources be conserved). By expressing the inertial theory in this manner, the boundary conditions would be incorporated directly into the theory. This might permit a rigorous statement of Mach's principle.

However, prior to the placing of any restrictions on the boundary conditions, this integral formulation would presumably resemble a tensorial analog of Kirchhoff's formula for the scalar inhomogeneous wave equation; that is, it would represent the potential $g^{\mu\nu}$ at any point as the sum of a volume integral over sources plus a surface integral depending on the values of the potential over the bounding surface. The surface integral would represent the contribution from sources outside the volume, and also the contribution from any source-free part of the potential.

It is the source-free contribution to the potential which should vanish according to Mach's principle, and it might be hoped to ensure this by requiring that the surface integral in the theory should tend to a suitable limit when the surface tends to infinity.[2] However, a global condition of this type is difficult to discuss in pseudo-Riemannian space-time, so in this paper we shall retain the surface integral, and confine our attention to a normal neighborhood of the point at which the potential is to be evaluated. We believe that our integral formula (14) for such a neighborhood may be of interest in general relativity independently of the problem of Mach's principle.

There remains our second problem: that of expressing Einstein's equations in integral form. Interesting attempts to solve this problem have already been made by Altshuler[3] and Lynden-Bell.[4] We shall take over many of their ideas and differ from them principally in our choice of kernel for the integral form and in our retention of the surface integral.

The most important idea which we borrow from these authors[5] is that, despite the nonlinearity of Einstein's equations, we may regard each element of the source $T^{\mu\nu}$ as contributing linearly to the potential $g^{\mu\nu}$ in the sense that the influence of each element may be regarded as propagated linearly over the *actual* space-time, whose structure does, of course, depend on all the other sources in the universe. Symbolically we may

[1] D. W. Sciama, Mon. Not. Roy. Astr. Soc. **113**, 34 (1953).

[2] D. W. Sciama, Proc. Roy. Soc. (London) **A273**, 484 (1963); Rev. Mod. Phys. **36**, 463 (1964).

[3] B. L. Altshuler, Zh. Eksperim. i Teor. Fiz. **51**, 1143 (1966); **55**, 1311 (1968) [English transls.: Soviet Phys.—JETP **24**, 766 (1967); **28**, 687 (1969)].

[4] D. Lynden-Bell, Mon. Not. Roy. Astr. Soc. **135**, 413 (1967).

[5] Lynden-Bell attributes this idea to F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. (London) **A282**, 191 (1964).

write

$$g^{\mu\nu}(x)=\int G_{\alpha'\beta'}{}^{\mu\nu}(x,x')T^{\alpha'\beta'}d^4x'+\text{surface term}\,, \quad (1)$$

where the kernel $G_{\alpha'\beta'}{}^{\mu\nu}(x,x')$ is suitably defined for a space-time with metric $g^{\mu\nu}$. In other words, the kernel or Green's function is a functional of the potential instead of being fixed *a priori* as it is in linear theories.

Once we permit the Green's function G to depend explicitly on the potential it becomes possible to represent solutions of Einstein's equations in the integral form (1) with many different definitions of G. These definitions involve linear differential operators $D_{\mu\nu\sigma\tau}$, of which G is the Green's function in the usual sense, that is, D operating on G gives a δ function. We would like to set up conditions which lead to a unique choice of operator D. This we do by demanding that:

(i) A first-order variation of source and metric should lead to an approximate (first-order) integral representation in which the Green's functional remains unchanged.[6] This is the nearest we can get to regarding Einstein's theory as a linear theory.

(ii) $D_{\mu\nu\sigma\tau}=D_{\sigma\tau\mu\nu}$. This condition is required in any case if we wish the volume integral alone to be a solution of Einstein's inhomogeneous field equations and the surface integral to be a solution of the homogeneous equations. This splitting of the potential is customary in the Kirchhoff representation, and has an obvious physical significance.

(iii) Only $g_{\mu\nu}$ or $g^{\mu\nu}$ should be regarded as suitable representations of the potential, rather than $(\sqrt{-g})^n g_{\mu\nu}$ or $(\sqrt{-g})^n g^{\mu\nu}$.

As we shall see, we then arrive at a unique covariant linear second-order differential operator $D_{\mu\nu\sigma\tau}$ given by

$$D_{\mu\nu\sigma\tau}=\tfrac{1}{4}(g_{\mu\sigma}g_{\nu\tau}+g_{\mu\tau}g_{\nu\sigma})\nabla^\rho\nabla_\rho-\tfrac{1}{2}(R_{\mu\sigma\nu\tau}+R_{\mu\tau\nu\sigma}) \quad (2)$$

for defining the Green's function of (1).

2. NOTATION

The metric tensor $g_{\mu\nu}$ of space-time has signature $+---$, and its determinant is denoted by g. Partial derivatives taken with respect to x^μ are denoted by $,\mu$ and covariant derivatives by $;\mu$ or ∇_μ. The block operator (d'Alembertian) is defined by $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$.

The Riemann-Christoffel tensor is defined in terms of the Christoffel connection $\Gamma^\alpha{}_{\mu\nu}$ by

$$R^\alpha{}_{\lambda\mu\nu}=\Gamma^\alpha{}_{\lambda\mu,\nu}-\Gamma^\alpha{}_{\lambda\nu,\mu}+\Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\lambda\mu}-\Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\lambda\nu}.$$

The Ricci tensor is $R_{\lambda\nu}=R^\alpha{}_{\lambda\alpha\nu}$.

Small parentheses indicate symmetrization, e.g.,

$$2R^{(\mu}{}_{(\rho}{}^{\nu)}{}_{\sigma)}=R^\mu{}_\rho{}^\nu{}_\sigma+R^\nu{}_\rho{}^\mu{}_\sigma=R^\mu{}_\rho{}^\nu{}_\sigma+R^\mu{}_\sigma{}^\nu{}_\rho.$$

The speed of light is taken to be unity. The symbol $\approx$ means equality to the first order in a small variation parameter.

3. PERTURBATIONS OF EINSTEIN'S FIELD EQUATIONS

Our aim in this section is to find a way of perturbing Einstein's field equations which is compatible with condition (i) of Sec. 1. Einstein's equations are

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=-\kappa T_{\mu\nu}.$$

Any perturbation of these equations, when linearized, will be of the form

$$D_{\mu\nu\sigma\tau}\delta g^{\sigma\tau}\approx-\kappa\delta T_{\mu\nu}$$

for a suitable differential operator $D_{\mu\nu\sigma\tau}$ defined in terms of the unperturbed metric $g^{\sigma\tau}$. Condition (i) would be satisfied if also

$$D_{\mu\nu\sigma\tau}g^{\sigma\tau}=-\kappa T_{\mu\nu}\,,$$

for then

$$D_{\mu\nu\sigma\tau}(g^{\sigma\tau}+\delta g^{\sigma\tau})\approx-\kappa(T_{\mu\nu}+\delta T_{\mu\nu}).$$

The perturbed metric would then be governed to first order by a linear *homogeneous* differential operator defined in the unperturbed space-time. Accordingly, we could regard $g^{\sigma\tau}+\delta g^{\sigma\tau}$ as a gravitational potential acting to first order in the unperturbed space-time. The equation for this potential would by virtue of the homogeneity of D admit a first-order Kirchhoff representation with a Green's function defined in the unperturbed space-time. In the limit as $\delta g^{\sigma\tau}\to 0$, the potential would coincide with the metric and the integral representation would become exact, and would thus be the representation we are seeking.

We now consider this procedure in more detail. Let us introduce a small variation in the energy-momentum tensor $T^\mu{}_\lambda$ [we choose the mixed form of this tensor in anticipation of our later result that this is the only form for which conditions (i) and (ii) of Sec. 1 are satisfied]:

$$T^\mu{}_\lambda\to\hat{T}^\mu{}_\lambda\equiv T^\mu{}_\lambda+\delta T^\mu{}_\lambda.$$

This will induce a corresponding small variation[7]

$$g^{\rho\sigma}\to\hat{g}^{\rho\sigma}\equiv g^{\rho\sigma}+\delta g^{\rho\sigma}\,,$$

where

$$\begin{aligned}\hat{g}^{\mu\nu}(R_{\nu\lambda}+\delta R_{\nu\lambda})&=-\kappa(\hat{T}^\mu{}_\lambda-\tfrac{1}{2}\hat{T}^\nu{}_\nu\delta^\mu{}_\lambda)\\&=-\kappa{}^*\hat{T}^\mu{}_\lambda\,,\ \text{say}.\end{aligned} \quad (3)$$

Now, to first order, we have

$$\delta R_{\mu\nu}\approx(\delta\Gamma^\lambda{}_{\mu\lambda})_{;\nu}-(\delta\Gamma^\lambda{}_{\mu\nu})_{;\lambda}\,,$$

$$\delta\Gamma^\lambda{}_{\mu\nu}\approx\tfrac{1}{2}g^{\lambda\alpha}(\delta g_{\alpha\mu;\nu}+\delta g_{\alpha\nu;\mu}-\delta g_{\mu\nu;\alpha})\,,$$

$$\delta g_{\alpha\mu;\nu;\lambda}\approx\delta g_{\alpha\mu;\lambda;\nu}-(R^\rho{}_{\alpha\nu\lambda}\delta g_{\rho\mu}+R^\rho{}_{\mu\nu\lambda}\delta g_{\alpha\rho})\,,$$

$$\delta g_{\mu\nu}\approx g_{\mu\nu}-g_{\mu\alpha}g_{\nu\beta}\hat{g}^{\alpha\beta}\,,$$

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[6] The importance of this condition was stressed to us by D. Lynden-Bell.

[7] J. N. Goldberg and E. T. Newman [J. Math. Phys. 10, 369 (1969)] have independently used this device of varying $g^{\mu\nu}$ to obtain a Green's theorem for general relativity.

where the covariant derivatives are taken in the unperturbed space-time. These relations enable us to write (3) in the first-order form

$$g_{\lambda\alpha}\Box\hat{g}^{\mu\alpha}+(g_{\lambda\rho}R^{\mu}{}_{\sigma}-\delta^{\mu}{}_{\rho}R_{\sigma\lambda}-2R^{\mu}{}_{\rho\lambda\sigma})\hat{g}^{\rho\sigma} +g^{\mu\nu}(\hat{\psi}_{\nu;\lambda}+\hat{\psi}_{\lambda;\nu})\approx 2\kappa{}^{*}\hat{T}^{\mu}{}_{\lambda}, \quad (4)$$

where

$$\hat{\psi}_{\mu}=\tfrac{1}{2}g_{\rho\sigma}\hat{g}^{\rho\sigma}{}_{;\mu}-g_{\mu\rho}\hat{g}^{\rho\sigma}{}_{;\sigma}.$$

It is convenient to write (4) in the symmetrized form

$$\Box\hat{g}^{\mu\nu}-2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\hat{g}^{\rho\sigma}+g^{\mu\alpha}g^{\nu\beta}(\hat{\psi}_{\alpha;\beta}+\hat{\psi}_{\beta;\alpha})\approx 2\kappa\hat{K}^{\mu\nu}, \quad (5)$$

where

$$\hat{K}^{\mu\nu}=\tfrac{1}{2}(g^{\mu\alpha}{}^{*}\hat{T}^{\nu}{}_{\alpha}+g^{\nu\alpha}{}^{*}\hat{T}^{\mu}{}_{\alpha}).$$

(Owing to the ambiguity involved, we do not raise or lower indices of varied quantities.)

Equation (5) has the required form, that is, $\hat{g}^{\mu\nu}$ is acted on by a linear homogeneous differential operator defined entirely in the background space-time. The linearity follows from the fact that the operator is independent of $\delta g^{\mu\nu}$, which is a trivial consequence of our rejection of all terms of order higher than the first. The homogeneity, that is, the absence of a term on the left-hand side which is independent of $\hat{g}^{\mu\nu}$, is nontrivial. Had we varied $T_{\mu\nu}$, $T^{\mu\nu}$, $\mathfrak{T}_{\mu\nu}$, $\mathfrak{T}^{\mu}{}_{\nu}$, or $\mathfrak{T}^{\mu\nu}$ instead of $T^{\mu}{}_{\nu}$, there would have been an additional term, namely, $G_{\mu\nu}$, $-G^{\mu\nu}$, $3(\sqrt{-g})G_{\mu\nu}$, $2(\sqrt{-g})G^{\mu}{}_{\nu}$, or $(\sqrt{-g})G^{\mu\nu}$ (where $G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}$), which would have spoiled the homogeneity (except in empty space-times). This in turn would have prevented us from superposing solutions to first order and so would have made a Kirchhoff representation impossible. In passing we note that varying $\mathfrak{T}^{\mu}{}_{\nu}$ does lead to a linear homogeneous differential operator acting on the covariant tensor $\hat{g}_{\mu\nu}$, but this operator does not satisfy our symmetry condition (ii) and so would not be a physically satisfactory starting point for a Kirchhoff representation.

Before we can contemplate defining a Green's function associated with our differential operator we must allow for the well-known fact that (5) admits an invariance group of gauge transformations in addition to its coordinate invariance. This guage group arises from our ability to make a first-order coordinate transformation in the perturbed space-time without making a corresponding transformation in the unperturbed space-time. It is convenient to work in a particular gauge framc, which must of course be chosen in such a way that it implies no restriction when we eventually pass to the limit $\hat{g}^{\mu\nu}\longrightarrow g^{\mu\nu}$. With this in mind we make the first-order coordinate transformation

$$x'^{\mu}=x^{\mu}+\xi^{\mu}$$

in the perturbed space-time, choosing ξ^{μ} so that[8]

$$\hat{\psi}'_{\mu}=0 \quad (6)$$

in a finite region. As is well known, this requires that ξ^{μ} be a solution of the inhomogeneous wave equation

$$\Box\xi_{\mu}-R^{\lambda}{}_{\mu}\xi_{\lambda}\approx\hat{\psi}_{\mu}. \quad (7)$$

Henceforth, we shall assume that this choice has been made, and so suppress the dashes.

It is known that Cauchy's problem for (7) has a unique local solution. This uniqueness ensures that our supplementary conditions eliminate the degenerate case in which the perturbation leading to the varied form of Einstein's equations is entirely due to a localized infinitesimal coordinate transformation of the original space-time.

We thus arrive at the following equation for $\hat{g}^{\mu\nu}$:

$$\Box\hat{g}^{\mu\nu}-2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\hat{g}^{\rho\sigma}\approx 2\kappa\hat{K}^{\mu\nu}. \quad (8)$$

In operator form, this equation is

$$D^{\mu\nu}{}_{\sigma\tau}\hat{g}^{\sigma\tau}=\kappa\hat{K}^{\mu\nu},$$

where

$$D^{\mu\nu}{}_{\sigma\tau}=\tfrac{1}{2}\delta^{(\mu}{}_{(\sigma}\delta^{\nu)}{}_{\tau)}\Box-R^{(\mu}{}_{(\sigma}{}^{\nu)}{}_{\tau)},$$

which clearly satisfies the symmetry relation

$$D_{\mu\nu\sigma\tau}=D_{\sigma\tau\mu\nu}.$$

4. KIRCHHOFF REPRESENTATION OF PERTURBED EINSTEIN EQUATIONS

The starting point for the Kirchhoff representation of any differential equation is the existence of a Green's function or elementary solution. The modern theory of such solutions for second-order hyperbolic differential equations was developed to avoid the divergent integrals used by Hadamard.[9] A number of people[10] have extended this development to include tensorial wave equations. We record here some conventions and formulas relevant to the construction of the elementary solution which is required in the present work.

The two-point scalar $\Gamma(x',x)$ denotes the square of the proper distance along the geodesic interval between x' and x, and is positive, zero, or negative according as the interval is timelike, null, or spacelike. This scalar has the property that

$$g^{\mu\nu}\Gamma_{,\mu}\Gamma_{,\nu}=4\Gamma.$$

To ensure that $\Gamma(x',x)$ is single-valued in a compact four-dimensional domain Ω containing x' in its interior,

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[8] D. Hilbert, Nachr. Akad. Wiss. Goettingen, Math.-Physik. Kl. **53**, 127 (1917); K. Lanczos, Z. Physik **31**, 112 (1925); M. Mathisson, *ibid.* **67**, 270 (1931); B. S. de Witt, in *Recent Developments in General Relativity* (Pergamon Press, Inc., New York, 1962), p. 175; *Gravitation, An Introduction to Current Research*, edited by L. Witten (Wiley-Interscience, Inc., New York, 1962), p. 266; also Ref. 4.

[9] J. Hadamard, in *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923). A modern treatment of the wave equation in a curved space-time is given by F. G. Friedlander in a forthcoming book in the Cambridge Series of Monographs on Relativity.

[10] B. S. de Witt and R. W. Brehme, Ann. Phys. (N. Y.) **9**, 220 (1960); B. S. de Witt, Ref. 8; A. Lichnerowicz, Publ. Math. de l'Inst. des Hautes Études, Paris **10**, 1 (1961); D. Robaschik, Acta Phys. Polon. **24**, 299 (1963); Y. Bruhat, Annali di Matematica, Serie 4, **64**, 191 (1964); H. P. Kunzle, Proc. Cambridge Phil. Soc. **64**, 779 (1968).

it will be assumed throughout that Ω lies within a normal neighborhood of x'.

Let $V_{\alpha'}$ be a local vector specified at x'. (Singly primed indices refer to the point x'.) A vector field can be generated in Ω by parallel transport of $V_{\alpha'}$ along the geodesics emanating from x'. The field vector obtained at x is $g^{\alpha'}{}_{\mu}V_{\alpha'}$, where $g^{\alpha'}{}_{\mu}(x',x)$ denotes the two-point vector of geodesic parallel transport[11] which is covariant at x and contravariant at x', and satisfies

$$\lim_{x'\to x} g^{\alpha'}{}_{\mu}(x',x) = (x',x) = \delta^{\alpha'}{}_{\mu}.$$

We shall be concerned with an elementary solution $E^{\alpha'\beta'}{}_{\mu\nu} = E^{(\alpha'\beta')}{}_{(\mu\nu)}(x',x)$ which is a two-point tensorial distribution satisfying

$$\Box E^{\alpha'\beta'}{}_{\mu\nu} - 2R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}E^{\alpha'\beta'}{}_{\rho\sigma} = g^{\alpha'}{}_{(\mu}g^{\beta'}{}_{\nu)} \times [g(x')g(x)]^{-1/4}\delta(x',x), \quad (9)$$

where $\delta(x',x)$ denotes the four-dimensional δ distribution of the two points x' and x.

In a normal neighborhood, this elementary solution can be taken in the form

$$E^{\alpha'\beta'}{}_{\mu\nu} = (1/4\pi)[\delta(\Gamma)g^{\alpha'}{}_{(\mu}g^{\beta'}{}_{\nu)}\Sigma(x',x) + H(\Gamma)V^{\alpha'\beta'}{}_{\mu\nu}(x',x)], \quad (10)$$

where $\delta(\Gamma)$ is the Dirac δ distribution, $H(\Gamma)$ is the Heaviside step function

$$H(\Gamma)=1 \quad \text{when} \quad \Gamma>0$$
$$=0 \quad \text{when} \quad \Gamma<0,$$

$\Sigma(x',x)$ ($\Sigma=1$ in flat space-time) is a two-point scalar defined by

$$\Sigma = \tfrac{1}{4}[g(x')g(x)]^{-1/4}[-\det(\Gamma_{,\alpha',\mu})]^{1/2},$$

and $V^{\alpha'\beta'}{}_{\mu\nu}(x',x)$ ($V^{\alpha'\beta'}{}_{\mu\nu}=0$ flat space-time) is a regular two-point tensor. The substitution of (10) into (9) shows that $V^{\alpha'\beta'}{}_{\mu\nu}$ is the unique solution of the characteristic boundary value problem for

$$\Box V^{\alpha'\beta'}{}_{\mu\nu} - 2R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}V^{\alpha'\beta'}{}_{\rho\sigma} = 0,$$

which satisfies the following relation along each null geodesic $x^{\mu}=x^{\mu}(\lambda)$ issuing from x':

$$\frac{\delta}{\delta\lambda}\left[\frac{\lambda}{\Sigma}V^{\alpha'\beta'}{}_{\mu\nu}\right] + \frac{1}{4\Sigma}[\Box(g^{\alpha'}{}_{(\mu}g^{\beta'}{}_{\nu)}\Sigma) - 2R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}g^{\alpha'}{}_{(\rho}g^{\beta'}{}_{\sigma)}\Sigma] = 0,$$

where $\delta/\delta\lambda$ denotes an intrinsic derivative taken with respect to the admissible parameter λ ($\lambda=0$ at x').

The term containing $\delta(\Gamma)$ in (10) is the parametrix and is nonvanishing only when $\Gamma(x',x)=0$, whereas the term containing $H(\Gamma)$ is nonvanishing for $\Gamma(x',x)>0$. Therefore, the contribution to the elementary solution of the former term is sharp, whereas that of the latter term is diffusive (this diffusive contribution vanishes in flat space-time).

We may regard x' as fixed and allow x to range throughout Ω. It is then evident that the support of the distribution $E^{\alpha'\beta'}{}_{\mu\nu}(x',x)$ is the interior and the surface of the whole (future and past) null cone with vertex x'. Furthermore, by putting

$$\delta(\Gamma) = \delta^{+}(\Gamma) + \delta^{-}(\Gamma),$$
$$H(\Gamma) = H^{+}(\Gamma) + H^{-}(\Gamma),$$

where the distributions with (+) have support in the future of x', and those with (−) in the past, we can decompose the elementary solution (10) into unique advanced and retarded Green's functions (because of the uniqueness of Cauchy's problem in a normal neighborhood) as follows:

$$E^{\alpha'\beta'}{}_{\mu\nu} = \tfrac{1}{2}(G^{+\alpha'\beta'}{}_{\mu\nu} + G^{-\alpha'\beta'}{}_{\mu\nu}),$$

where

$$G^{\pm\alpha'\beta'}{}_{\mu\nu} = (1/2\pi)[\delta^{\pm}(\Gamma)g^{\alpha'}{}_{(\mu}g^{\beta'}{}_{\nu)}\Sigma + H^{\pm}(\Gamma)V^{\alpha'\beta'}{}_{\mu\nu}].$$

Note that both these Green's functions are elementary solutions which satisfy (9).

The existence of these Green's functions enables us to derive a Kirchhoff representation in the usual way. Adapted to our present problem, the procedure is as follows. Let (α) denote the equation resulting from the multiplication of (9) by $\hat{g}^{\mu\nu}[-g(x)]^{1/2}$, and let ($\beta$) denote the equation resulting from the multiplication of (8) by $E^{\alpha'\beta'}{}_{\mu\nu}[-g(x)]^{1/2}$. Then the equation ($\alpha$)−($\beta$) is

$$\hat{g}^{\mu\nu}g^{\alpha'}{}_{\mu}g^{\beta'}{}_{\nu}[g(x)g(x')]^{1/4}\delta(x',x) - 2\kappa E^{\alpha'\beta'}{}_{\mu\nu}\hat{K}^{\mu\nu}[-g(x)]^{1/2} \approx \{\Box E^{\alpha'\beta'}{}_{\mu\nu}\hat{g}^{\mu\nu} - E^{\alpha'\beta'}{}_{\mu\nu}\Box\hat{g}^{\mu\nu}\}[-g(x)]^{1/2}. \quad (11)$$

The right-hand side of this equation can be rewritten in the form

$$\{g^{\rho\sigma}[E^{\alpha'\beta'}{}_{\mu\nu;\rho}\hat{g}^{\mu\nu} - E^{\alpha'\beta'}{}_{\mu\nu}\hat{g}^{\mu\nu}{}_{;\rho}][-g(x)]^{1/2}\}_{,\sigma}.$$

Now let us integrate (11) throughout the domain Ω. After taking into account the properties of $\delta(x',x)$ and the $g^{\alpha'}{}_{\mu}$, this integration yields

$$\hat{g}^{\alpha'\beta'} - 2\kappa\int_{\Omega} E^{\alpha'\beta'}{}_{\mu\nu}\hat{K}^{\mu\nu}[-g(x)]^{1/2}d^4x \approx \int_{\Omega}\{g^{\rho\sigma}[E^{\alpha'\beta'}{}_{\mu\nu;\rho}\hat{g}^{\mu\nu} - E^{\alpha'\beta'}{}_{\mu\nu}\hat{g}^{\mu\nu}{}_{;\rho}] \times [-g(x)]^{1/2}\}_{,\sigma}d^4x, \quad (12)$$

where $d^4x = dx^0dx^1dx^2dx^3$ is the coordinate volume element in Ω.

We can use the divergence theorem to transform the right-hand side of (12) into an integral taken over the boundary $\partial\Omega$ of Ω. Thus, we obtain the Kirchhoff

[11] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1960).

formula

$$\hat{g}^{\alpha'\beta'}\approx 2\kappa\int_{\Omega} E^{\alpha'\beta'}{}_{\mu\nu}\hat{K}^{\mu\nu}[-g(x)]^{1/2}d^4x + \int_{\partial\Omega} g^{\rho\sigma}[E^{\alpha'\beta'}{}_{\mu\nu;\rho}\hat{g}^{\mu\nu}-E^{\alpha'\beta'}{}_{\mu\nu}\hat{g}^{\mu\nu}{}_{;\rho}] \times[-g(x)]^{1/2}dS_\sigma, \quad (13)$$

where dS_σ is the coordinate surface element on $\partial\Omega$, directed outward from Ω. For consistency, this integral formula must be subjected to the same supplementary conditions (6) as were imposed on the varied field equations.

In practice, it is usually convenient to use the retarded Green's function as the kernel of the Kirchhoff representation. This representation then conforms to the usual notions of causality. However, we emphasize that the advanced Green's function would given an equally valid representation. We believe that the lack of advanced effects in nature is connected with the different limiting behaviors of the advanced and retarded surface integrals as the volume Ω increases until it comprises the whole of space-time.[12]

We now make the usual Kirchhoff requirement, referred to in (ii) of Sec. 1, that the integral $I_\Omega{}^{\alpha'\beta'}$ over Ω in (13) should by itself satisfy the inhomogeneous differential equation, that is

$$D^{\mu'\nu'}{}_{\alpha'\beta'}I_\Omega{}^{\alpha'\beta'}\approx\kappa K^{\mu'\nu'}.$$

This would be true if the action of D on the elementary solution in I_Ω were to produce a δ distribution. However, reference to (8) and (9) shows that we have the wrong pair of indices of D contracted with a pair of indices of I_Ω. Thus, we require that

$$D_{\mu'\nu'\alpha'\beta'}=D_{\alpha'\beta'\mu'\nu'},$$

which is the symmetry condition referred to in (ii). As we have already stated, our operator (2) satisfies this condition.

5. EINSTEIN'S FIELD EQUATIONS IN GENERALLY COVARIANT INTEGRAL FORM

We now proceed to the limit in which the variations of Sec. 4 tend to zero. As $\delta T^\mu{}_\nu\rightarrow 0$, we have

$$\hat{K}^{\mu\nu}\rightarrow g^{\mu\alpha}(T^\nu{}_\alpha-\tfrac{1}{2}T^\lambda{}_\lambda\delta^\nu{}_\alpha),$$
$$\hat{g}^{\mu\nu}\rightarrow g^{\mu\nu},$$
$$\hat{\psi}^\mu\rightarrow 0.$$

Our Kirchhoff representation (13) now tends to an exact generally covariant integral representation of Einstein's field equations:

$$g^{\alpha'\beta'}=2\kappa\int_\Omega G^{-\alpha'\beta'\nu}{}_\mu(T^\mu{}_\nu-\tfrac{1}{2}T^\lambda{}_\lambda\delta^\mu{}_\nu)[-g(x)]^{1/2}d^4x + \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}[-g(x)]^{1/2}dS_\sigma. \quad (14)$$

This result could have been derived directly from Einstein's field equations, without going through the variational procedure. However, such direct considerations would not have revealed the important property of our particular Green's functional, namely, that, subject to the supplementary conditions (6), it may be regarded as remaining unchanged to first order when a small change in the material sources produces a small change in the gravitational potential.[13] We regard this stability property of the Green's functional as of considerable importance both in principle, as representing the nearest approach of Einstein's equations to linearity, and in practice, as suggesting an iterative procedure for solving the field equations which may be convergent in many cases.

Computations based on the integral representation will be presented in later papers.

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[12] For a corresponding discussion of the electromagnetic case see D. W. Sciama, Proc. Roy. Soc. (London) **A273**, 484 (1963).

[13] An integral representation possessing this property is possible with a nonvanishing cosmological constant only if this constant is incorporated into the source term. This would seem to be unacceptable on physical grounds.