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Relativistic Thermodynamics*

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It is shown that a generalized formulation of statistical mechanics provides a unified logical basis for the construction of a manifestly covariant theory of relativistic thermodynamics in contrast to heuristic approaches, such as the original theory by Planck and Einstein, and more recent ones by Ott and Møller. The generalized formalism is applied to discuss the relativistic thermodynamics of blackbody radiation, including such processes as the absorption and reflection of a light beam as well as the adiabatic cooling of cosmic blackbody radiation in Milne's model of the universe. The measurement of the temperature of a light beam is also discussed.

I. INTRODUCTION

RELATIVISTIC thermodynamics has been characterized by a certain degree of arbitrariness and confusion. Most of the existing formulations of the theory are based on discussions of particular examples, which are used heuristically to generalize the mathematical forms of the First and Second Laws of thermodynamics in order to take relativity into account. These formulations depend both on the particular examples chosen for constructing the theory, and on the authors' personal preferences in defining certain fundamental quantities.

We shall construct relativistic thermodynamics on the basis of a generalized formulation of statistical mechanics.[1,2] The resulting formalism will be automatically covariant and, in principle, of general applicability. Our main interest lies in analyzing the physical significance of fundamental quantities, as well as in determining their transformation properties under the Lorentz group. The generalized statistical mechanical approach to relativistic thermodynamics is not new and is mentioned in Ref. 2.

In Sec. II we will present two of the existing formulations of relativistic thermodynamics. We will sketch a generalized theory of statistical mechanics for quantized systems and state the generalized forms of the First and Second Laws in Sec. III. A discussion of the relativistic thermodynamics of blackbody radiation will then be given in full detail, including such processes as the absorption and reflection of a light beam as well as the adiabatic cooling of cosmic blackbody radiation in Milne's model of the universe (Sec. IV). Finally, we will summarize results important either from a historical or observational standpoint (Sec. V).

II. PREVIOUS FORMULATIONS OF RELATIVISTIC THERMODYNAMICS

The original approach of Planck and Einstein to relativistic thermodynamics is an example of a heuristic construction of a physical theory. In their treatment,[3] they consider a system composed of radiation, or of an ideal gas, in equilibrium inside a box. Drawing upon the relativistic mechanics of continuous media, they compute the total energy and linear momentum of the system. Then, imagining that the system undergoes an infinitesimal transition from one state of equilibrium to another, they calculate the change in the total energy of the system δE and express it as the sum of the transfer

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[1] D. van Dantzig, Physica 6, 673 (1939); Proc. Koninkl. Ned. Akad. Wetenschap. 42, 601 (1939); 42, 608 (1939).
[2] P. G. Bergmann, Phys. Rev. 84, 1026 (1951).
[3] Although the method described in the first part of Sec. II is not the approach originally adopted by Planck [M. Planck, Sitzber. Preuss. Akad. Wiss. Berlin 542 (1907)], it is the one followed by Einstein [A. Einstein, Jahrb. Rad. E. 4, 411 (1907)] and the way in which the theory is presented in most textbooks [see, e.g., W. Pauli, *Theory of Relativity* (Pergamon Press Ltd., London, 1958), p. 134]. For a historical sketch of the development of relativistic thermodynamics, see L. A. Schmid, Goddard Space Flight Center, Maryland, Report, 1969, p. 59 (unpublished).

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of heat to the system ΔQ and the performance of work on the system ΔA. Such a decomposition maintains the form of the First Law in all inertial frames. Planck adopts as a definition of ΔA the expression

$$\Delta A = \mathbf{u}\cdot\delta\mathbf{P} - p\delta V\,, \tag{1}$$

where $\mathbf{u}$ is the velocity of the system, $\mathbf{P}$ is its total linear momentum, p is the isotropic pressure, and V is the volume of the box. He justifies the first term, $\mathbf{u}\cdot\delta\mathbf{P}$, as representing that part of the work done by external forces in changing the total linear momentum by $\delta\mathbf{P}$. The second term is the usual expression for the "internal work." With this choice for ΔA, the transfer of heat is necessarily given by[4]

$$\Delta Q = \Delta\hat{Q}(1-u^2)^{1/2}\,, \tag{2}$$

where $\Delta\hat{Q}$ is the transfer of heat to the system as measured in the rest frame ($\mathbf{u}=0$). In this formulation, expression (2) represents the transformation law for the transfer of heat under the Lorentz group. Under the assumptions that the entropy S is a scalar under the Lorentz group, that the form of the Second Law is the same in all inertial frames, and that the process considered is reversible, the transformation law for the temperature T,

$$T = \hat{T}(1-u^2)^{1/2}\,, \tag{3}$$

readily follows ($\hat{T}$ is the rest temperature of the system).

The original formulation of relativistic thermodynamics was generally accepted until Ott's paper[5] set off a controversy involving all aspects of the subject.[6] Ott claimed that formulas (2) and (3) of Planck's theory for the transformation laws of the heat transfer and temperature, respectively, under the Lorentz group were in error and that they should be replaced by the equations

$$\Delta Q = \Delta\hat{Q}(1-u^2)^{-1/2}\,, \tag{4}$$

$$T = \hat{T}(1-u^2)^{-1/2}. \tag{5}$$

Although Ott was the first to attack Planck's theory directly, in presenting the main features of his proposal we shall follow the work of Møller.[7,8] Like Ott, Møller objects to Planck's choice of expression (1) as the total work done by the external forces on the gas, because it is not clear whether or not the total change in the linear momentum $\delta\mathbf{P}$ in that expression has a contribution from "the transfer of heat." If the transfer of heat does contribute, the former interpretation of the product $\mathbf{u}\cdot\delta\mathbf{P}$ is incorrect. To illuminate these points, Møller chooses a system of the same type as the one considered in the first part of this section, designed to lend itself to a careful analysis of the external forces and to a computation of the changes in the linear momentum $\Delta\mathbf{J}$ and the energy ΔA of the system caused by their action. He finds that the total change in linear momentum is different from $\Delta\mathbf{J}$ and that the change in energy ΔA (the work done by the external forces) is given by

$$\Delta A = \mathbf{u}\cdot\Delta\mathbf{J} - p dV\,, \tag{6}$$

an expression of the same form as (1) but with $\Delta\mathbf{J}$ replacing $\delta\mathbf{P}$. Having recognized that the total change in linear momentum is not due only to the action of external mechanical forces, he generalizes the form of the First Law to

$$\delta P^\mu = \Delta Q^\mu + \Delta W^\mu\,, \tag{7}$$

where

$$P^\mu = (E,\mathbf{P})\,,\quad \Delta Q^\mu = (\Delta Q,\Delta\mathbf{G})\,,\quad \Delta W^\mu = (\Delta A,\Delta\mathbf{J})\,,$$

and $\Delta\mathbf{G}$ is the transfer of linear momentum associated with the transfer of heat ΔQ. He discovers that ΔQ^μ transforms as a four-vector under the Lorentz group. In the particular case in which $\Delta\mathbf{G}=0$ in the rest frame, he can write

$$\Delta Q^\mu = \Delta\hat{Q}v^\mu\,,\quad v^\mu = (v^0, v^0\mathbf{u})\,,\quad v^\mu v_\mu = 1\,, \tag{8}$$

and, consequently,

$$\Delta Q = \Delta\hat{Q}(1-u^2)^{-1/2}\,, \tag{9}$$

which is Ott's formula for the transformation of the transfer of heat. Equation (8) [and hence (9)] is valid for a reversible process. Therefore, considering S a scalar, i.e., making in this respect the same assumption that Planck did in his theory, he obtains for the transformation formula of the temperature the expression (5).

Møller also considers the possibility of having irreversible processes. To include such cases in the present formalism, he simply writes the most general form of the Second Law of thermodynamics for systems at rest in terms of moving system quantities[9]:

$$dS \geqslant \Theta^\mu \Delta Q_\mu\,,\quad \Theta^\mu = v^\mu/\hat{T}. \tag{10}$$

This expression is supposed to represent the most general statement of the Second Law in relativistic thermodynamics. We recall that v^μ in (10) represents the four-velocity associated with the particular system of interest.

[4] We take the velocity of light equal to unity throughout this paper and the metric tensor $\eta^{\mu\nu}=0$ for $\mu\neq\nu$; $\eta^{00}=1$, $\eta^{11}=\eta^{22}=\eta^{33}=-1$.

[5] H. Ott, Z. Physik **175**, 70 (1963).

[6] H. Arzelies, Nuovo Cimento **35**, 792 (1965); **41B**, 81 (1966); R. Penney, *ibid.* **43A**, 911 (1966); T. W. B. Kibble, *ibid.* **41B**, 72 (1966); **41B**, 83 (1966); **41B**, 84 (1966); A. Gamba, *ibid.* **37**, 1792 (1965); **41B**, 72 (1966); F. Rohrlich, *ibid.* **45B**, 76 (1966); A. Staruszkiewicz, *ibid.* **45**, 684 (1966); P. T. Landsberg and K. Jones, *ibid.* **52B**, 28 (1967); K. Kuchar, Acta Phys. Polon. **35**, 331 (1969).

[7] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **36**, No. 1 (1967).

[8] C. Møller, in *Old and New Problems in Elementary Particles* (Academic Press Inc., New York, 1968). Møller has recently given a statistical mechanical account of the results presented in this paper and in Ref. 7 (see Ref. 9).

[9] Møller has introduced this form of the Second Law only recently [C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **36**, No. 16 (1968)]. The inverse temperature vector Θ^μ was introduced by van Dantzig (see Ref. 1).

So far, we have presented the main features of two different formulations of relativistic thermodynamics. Neither theory is completely satisfactory from a physical standpoint. In constructing a physical theory it is desirable to have a unified picture, free of the constraints usually imposed by a heuristic approach. In the two formulations of relativistic thermodynamics that we have presented, the consideration of a particular system such as an ideal gas enclosed in a box led the authors to include the velocity of the system in the expressions of the First and Second Laws. Hence, neither theory can be applied to situations in which there is no unique rest frame or no rest frame at all. This is the case, for instance, when we consider the thermodynamics of a beam of photons or of massive particles. These two examples are of primary interest since they provide possible mechanisms for interchanging energy and momentum between components of a general system. It is also important to notice that a heuristic generalization of an existing theory to enlarge its scope of applicability or to agree with underlying physical principles will usually require the identification of certain expressions with generalized quantities—a procedure that in most cases is not unique. Even the explicit computation of these generalized quantities in some special cases is subject to personal preferences and is tied to the characteristic features of the example used in performing the calculations.

III. GENERALIZED STATISTICAL MECHANICS

A. Density Operator

Let ρ represent the density operator characterizing a given state of a quantized system. We define its entropy function S by the expression

$$S=-k\,\mathrm{Tr}(\rho\ln\rho), \tag{11}$$

where k is the Boltzmann constant and ρ is a positive definite operator subject to the normalization condition that its trace is unity. We shall say that a system with given mean values $\langle A^j\rangle$ for a set of observables A^j is in thermodynamic equilibrium if its density operator maximizes the value of the entropy function under the subsidiary conditions

$$\mathrm{Tr}(\rho A^j)=\langle A^j\rangle, \tag{12}$$

i.e., if

$$\delta S=0,\quad \delta^2 S<0 \tag{13}$$

for all variations of ρ that fulfill the requirements

$$\delta[\mathrm{Tr}(\rho A^j)]=0,\quad \delta[\mathrm{Tr}(\rho)]=0. \tag{14}$$

We can use the conditions (13) and (14) to determine the form of the operator ρ. In the ρ representation, these formulas take the form

$$\delta S=-k\sum_a \delta\rho_a(\ln\rho_a+1)=0,$$
$$\sum_a A_a{}^j\delta\rho_a=0,$$
$$\sum_a \delta\rho_a=0.$$

Therefore, by using the method of Lagrange multipliers, we find that ρ is an extremal for the function S consistent with conditions (14) if, and only if,

$$\ln\rho_a+\sum_j \beta_j A_a{}^j+\lambda+1=0, \tag{15}$$

where λ and β_j are Lagrange multipliers. Equation (15) gives

$$\rho_a=Z^{-1}e^{-\Sigma_j\beta_j A_a{}^j}. \tag{16}$$

The function Z (the partition function) is defined by the expression

$$Z(\beta_j;\gamma)=\mathrm{Tr}(e^{-\Sigma_j\beta_j A^j}), \tag{17}$$

with γ representing a possible set of external parameters entering the definition of the operators A^j. The density operator as given by (16) corresponds to the canonical distribution of classical statistical mechanics. To satisfy the conditions (12), we must formally compute the values of β_j from the expression

$$\langle A^j\rangle=-\partial\ln Z/\partial\beta_j. \tag{18}$$

The value of the entropy function for a system in thermal equilibrium is

$$S=k(\sum_j \beta_j\langle A^j\rangle+\ln Z). \tag{19}$$

The proof that (16) represents a true maximum is straightforward and will be omitted here. We notice that the invariance of ρ, with respect to a transformation that leads from the set A^j to a set $A^{j'}$ consisting of linear combinations of the A^j, implies that β_j transforms contragrediently to A^j. We shall refer to the parameter β_j as the generalized temperature of the system.

B. First and Second Laws of Thermodynamics

The concept of performance of work on the system is related to the change in the mean values of the observables A^j when the parameters γ change. We obtain in this way what is called the adiabatic change[2] of $\langle A^j\rangle$,

$$\delta_{\mathrm{ad}}\langle A^j\rangle=\langle\partial A^j/\partial\gamma\rangle\delta\gamma. \tag{20}$$

We will interpret the expression (20) as the "generalized performance of work on the system." The "generalized transfer of heat" is then introduced through the difference between the total change of the mean values of A^j and the expression (20):

$$\delta_Q\langle A^j\rangle=\delta\langle A^j\rangle-\delta_{\mathrm{ad}}\langle A^j\rangle. \tag{21}$$

From expression (19) and relation (18), we immediately find that

$$\delta S = k \sum_j \beta_j \delta_Q \langle A^j \rangle. \tag{22}$$

In thermodynamics, it is customary to designate the change of the measurable quantities $\langle A^j \rangle$ by dU^j, where U^j is a thermodynamic function or variable of state, the generalized work performed on the system by ΔW^j, and the generalized transfer of heat by ΔQ^j, where Δ indicates that the quantity to which it is attached is not a thermodynamic function. Consequently, according to expressions (21) and (22), for those processes during which the system is and remains in thermal equilibrium, the First Law of thermodynamics is

$$dU^j = \Delta Q^j + \Delta W^j. \tag{23}$$

(The total change in the observables $U^j \equiv \langle A^j \rangle$ can be decomposed into the sum of the transfer of heat and the work performed on the system.) The Second Law takes the form

$$dS = k \sum_j \beta_j \Delta Q^j. \tag{24}$$

(For reversible processes the total change in the entropy is obtained through the projection of the transfer of heat along the generalized temperature β_j.)

In order to obtain the generalized mathematical form for the strong formulation of the Second Law, namely, that all thermodynamic processes involving a system that is isolated thermally from its surroundings will increase its entropy, we must supplement our discussion with considerations of nonequilibrium systems and their approach to equilibrium. The line of reasoning is parallel to the one followed in ordinary statistical mechanics[10] and leads to the expression

$$dS \geqslant k \sum_j \beta_j \Delta Q^j, \tag{25}$$

where dS is given by (22) and ΔQ^j enters the First Law as stated in (23). The inequality (25) represents the most general formulation of the Second Law in a generalized thermodynamics. We obtain a relativistic thermodynamics in the particular case in which the symmetry group of our theory is the Lorentz group.

IV. BLACKBODY RADIATION

A. Thermodynamics of a Pencil of Electromagnetic Radiation

We shall now apply the ideas developed in Sec. III to the relativistic thermodynamics of a pencil of electromagnetic radiation (also referred to as a light beam). With such a pencil, we can associate a single normal mode of vibration that is characterized by a set of operators $(A^\mu \equiv p^\mu)$ corresponding to the four-momentum operator

$$p^\mu = \hbar k^\mu n, \tag{26}$$

where k^μ is a positive-frequency null vector defining the energy and direction of propagation of the pencil of radiation. We identify this four-vector with the external parameters γ; n is the number operator whose eigenvalues are non-negative integers, and $\hbar$ is Planck's constant divided by 2π.

Since the density operator ρ should be a scalar under the Lorentz group, it follows from (16) that β_μ should also transform as a four-vector. The application of the general expressions (17)–(22) to our case yields

$$Z(\beta_\mu; k_\rho) = \langle n \rangle + 1, \tag{27}$$

$$\langle p^\mu \rangle = \hbar \langle n \rangle k^\mu, \tag{28}$$

$$\delta \langle p^\mu \rangle = \hbar (k^\mu \delta \langle n \rangle + \langle n \rangle \delta k^\mu), \tag{29}$$

$$S = k[\hbar k^\mu \beta_\mu \langle n \rangle + \ln(\langle n \rangle + 1)], \tag{30}$$

$$\delta S = k\hbar \delta \langle n \rangle \beta_\mu k^\mu, \tag{31}$$

$$\langle n \rangle = (e^{\hbar \beta_\mu k^\mu} - 1)^{-1}. \tag{32}$$

In expression (29), the second term is associated with the generalized performance of work, while the first is related to the generalized transfer of heat. It is immediately seen from (32) that knowledge of $\langle n \rangle$ and k^μ is not enough to determine a unique value of β_μ. We have only one equation for four unknowns. We also notice that β_μ could not be parallel to k_μ, since this would require the mean value of n to be infinite. We may conclude from the requirement of having $\langle n \rangle$ finite and positive definite for all values of $\mathbf{k}$ that β_μ (assumed independent of $\mathbf{k}$) must be timelike and pointing towards the future; i.e., there exists a frame of reference in which β_μ takes the simple form $\beta_\mu = (\beta, 0, 0, 0)$. In this frame, expression (32) will only depend on k^0 and assume the usual form of the Planck distribution for isotropic radiation in thermal equilibrium. From this analogy, we can identify the parameter β with $1/kT$, where T is the absolute temperature assigned to the radiation beam.[11]

To relate a single normal mode to a pencil of electromagnetic radiation, we recall that the number of normal modes characterized by wave vectors $\mathbf{k}$ with components in the intervals Δk_1, Δk_2, and Δk_3 of a field enclosed in a volume $\Delta x_1 \Delta x_2 \Delta x_3$ is given by[12]

$$2(2\pi)^{-3} \Delta x_1 \Delta x_2 \Delta x_3 \Delta k_1 \Delta k_2 \Delta k_3. \tag{33}$$

The expression (33) is Lorentz-invariant, i.e., the value and form of this formula are independent of the frame of reference employed in the description of the physical system.

[10] See, e.g., P. G. Bergmann, *Heat and Quanta* (Dover Publications, Inc., New York, 1962), §5-2 and §5-3.

[11] M. Planck, *The Theory of Heat Radiation* (Dover Publications, Inc., New York, 1959), §166.

[12] See, e.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1951), Chap. 6, p. 130.

Then, with the help of expressions (28), (30), and (33), we can define

$$I^{\mu}=[2\hbar/(2\pi)^3]\langle n\rangle k^{\mu}|\mathbf{k}|^2 d|\mathbf{k}|d\Omega, \tag{34}$$

$$I_s=[2/(2\pi)^3]S|\mathbf{k}|^2 d|\mathbf{k}|d\Omega \tag{35}$$

as the energy-momentum and entropy, respectively, emitted by (or incident on) a unit of normal area per unit time in a solid angle $d\Omega$, with direction and frequency given by $\mathbf{k}$. We shall refer to I^{μ} and I_s as the specific intensities of energy-momentum and entropy.

Let us now suppose that we are in a region of space that is filled with radiation and that we have a speck of dust acting as a matter catalyst at a given space point. Then, according to the Second Law of thermodynamics, the change in the entropy density σ at that point must satisfy

$$\delta\sigma\geqslant 0, \tag{36}$$

while the condition of conservation of energy-momentum requires

$$\delta u^{\mu}=0, \tag{37}$$

where u^{μ} is the energy-momentum density. With the help of expressions (34) and (35), u^{μ} and σ can be written as

$$u^{\mu}=\frac{2\hbar}{(2\pi)^3}\int k^{\mu}\langle n\rangle|\mathbf{k}|^2 d|\mathbf{k}|d\Omega, \tag{38}$$

$$\sigma=\frac{2}{(2\pi)^3}\int S|\mathbf{k}|^2 d|\mathbf{k}|d\Omega. \tag{39}$$

We can now state the necessary and sufficient condition for having radiation in thermal equilibrium ($\delta\sigma=0$) at a point. Using the expression (31), we see that we can satisfy both $\delta\sigma=0$ and (37) for an arbitrary change in $\langle n\rangle$ if there exists a vector α_{μ} independent of $\mathbf{k}$, so that for all values of $\mathbf{k}$,

$$\alpha_{\mu}k^{\mu}=\beta_{\mu}k^{\mu}. \tag{40}$$

Hence, we see that for a system in thermal equilibrium we can choose the different world vectors β_{μ} of the various pencils of radiation passing through the point to have the same value. That is the case, for instance, for radiation in thermal equilibrium within a cavity of finite volume.

Let us now study some typical processes in which a pencil of electromagnetic radiation is involved.

1. *Absorption by a Cold Slab of Matter. Temperature of a Light Beam*

The energy-momentum intensity of a light beam can be altered independently by a transfer of heat and by a performance of work [see expression (29)]. The partial absorption of a light beam by a cold slab of matter illustrates a process in which a pure transfer of heat takes place. Since neither the frequency nor the direction of propagation of the beam changes in a process of absorption, the variations in the specific intensities of energy-momentum and entropy assume the form

$$\delta I^{\mu}=[2/(2\pi)^3]\hbar k^{\mu}\delta\langle n\rangle|\mathbf{k}|^2 d|\mathbf{k}|d\Omega, \tag{41}$$

$$\begin{aligned}\delta I_s&=[2/(2\pi)^3]\delta S|\mathbf{k}|^2 d|\mathbf{k}|d\Omega\\&=k\beta_{\mu}\delta I^{\mu}.\end{aligned} \tag{42}$$

Supposing that the piece of absorbing material is originally at 0°K, we have, according to the Second Law of thermodynamics,

$$\delta I_s+\delta\tilde{I}_s\geqslant 0, \tag{43}$$

where $\delta\tilde{I}s$ represents the change in entropy of the slab per unit of normal area and unit time. The expression (43) can also be written in the form

$$\delta\tilde{I}_s\geqslant -k\beta_{\mu}\delta I^{\mu}. \tag{44}$$

Since energy-momentum is conserved at any instant of time, we also have

$$\delta I^{\mu}+\delta\tilde{I}^{\mu}=0,$$

where $\delta\tilde{I}^{\mu}$ is the change in energy-momentum of the slab per unit of normal area and unit time. Therefore (44) gives

$$\delta\tilde{I}_s\geqslant k\beta_{\mu}\delta\tilde{I}^{\mu}, \tag{45}$$

a more familiar expression of the application of the Second Law to the piece of absorbing material. We notice from (41) and (42) that δI^{μ} and δI_s depend on the projection of β_{μ} along k^{μ}. We cannot determine a unique value for β_{μ} from these expressions. If we choose to fix a convenient value for β_{μ}, we could begin in that frame of reference in which the slab is at rest and assign a temperature T to the beam through the relation[11]

$$\langle n\rangle=(e^{\hbar k^0/kT}-1)^{-1}. \tag{46}$$

This is equivalent to choosing $\beta_{\mu}=\delta_{\mu}{}^0/kT$. Then, in any other frame, we can express β_{μ} as

$$\beta_{\mu}=v_{\mu}/kT, \tag{47}$$

where v_{μ} is the four-velocity of the slab. This is just a particular choice, however, and does not need to be adopted in general.

Because of the invariance of $\langle n\rangle$ under a Lorentz transformation, it is important to notice that according to (46) two different inertial observers making intensity measurements[11] of a pencil of radiation will assign different temperatures T and T' to the beam, which are related by

$$\nu/T=\nu'/T', \tag{48}$$

where ν and ν' are the respective frequencies as measured by the two different observers ($k^0=2\pi\nu$, $k^{0\prime}=2\pi\nu'$).

2. *Reflection by a Moving Mirror*

The following example provides an illustration of a reversible process in which the pencil of radiation is allowed to change its specific intensity of energy-momentum by performing work.

Let us denote the mean occupation number and null vector characterizing the incoming beam by $\langle n\rangle_1$ and k_1^μ, and the corresponding quantities for the reflected beam by $\langle n\rangle_2$ and k_2^μ. It is convenient to choose the generalized temperature β_μ of the incoming beam to be $\beta_\mu=\delta_\mu{}^0/kT$ in that frame of reference in which the mirror is at rest. T is determined from $\langle n\rangle_1$ by using expression (46). In this same frame of reference, we also have

$$\langle n\rangle_1=\langle n\rangle_2\,,\quad k_1{}^0=k_2{}^0. \tag{49}$$

Consequently, if we move to a frame in which the four-velocity of the mirror is v^μ, we shall have

$$\beta_\mu k_1{}^\mu=\beta_\mu k_2{}^\mu\,,\quad \beta_\mu=v_\mu/kT. \tag{50}$$

To be definite, we assume that the mirror is plane and has a four-velocity $v^\mu=(v^0,v^0\mathbf{u})$, where $\mathbf{u}$ is along the direction of the normal to the mirror. Then, it is possible to prove that (see Fig. 1)

$$(\cos\alpha_1-u)d^3k_1=(-\cos\alpha_2+u)d^3k_2\,,\quad u=|\mathbf{u}| \tag{51}$$

by using formula (50), the law of reflection, and a Lorentz transformation. From expression (34), $I_1{}^\mu(\cos\alpha_1-u)$ represents the energy-momentum incident upon the mirror per unit area and unit time; similarly, $I_2{}^\mu(-\cos\alpha_2+u)$ represents the energy-momentum leaving the mirror per unit area and unit time. Consequently, the difference between these two expressions gives the transfer of energy-momentum $\delta\mathcal{P}^\mu$ to the mirror per unit area and unit time. Taking into account formulas (49) and (51), we obtain

$$\delta\mathcal{P}^\mu=[2h/(2\pi)^3]\langle n\rangle_1(\cos\alpha_1-u)\times(k_1{}^\mu-k_2{}^\mu)|\mathbf{k}_1|^2d|\mathbf{k}_1|d\Omega_1. \tag{52}$$

The zeroth component of this expression is the amount of work done on the mirror per unit area and unit time and equals the product of the speed u and the pressure p exerted on the mirror, i.e.,

$$\delta\mathcal{P}^0=pu. \tag{53}$$

By a straightforward procedure, we can compute p from the expression (53) and obtain the result

$$p=2I_1{}^0[(\cos\alpha_1-u)^2/(1-u^2)], \tag{54}$$

which agrees with the value obtained by other methods.[13]

Considering the relation (51) and that $\delta\langle n\rangle=0$ $(\langle n\rangle_1-\langle n\rangle_2)$, we find that the entropy incident upon the mirror per unit area and unit time equals the entropy leaving the mirror per unit area and unit time. The reflection of the beam is therefore a reversible process. There is no transfer of heat to the beam, and consequently,

$$\delta\mathcal{P}^\mu=\Delta\mathcal{W}^\mu\,, \tag{55}$$

where $\Delta\mathcal{W}^\mu$ represents the generalized work done on the mirror per unit area and unit time. The zeroth component of $\Delta\mathcal{W}^\mu$ is related to the performance of work, while the spatial components represent a transfer of linear momentum.

3. *Cosmic Blackbody Radiation in Special Relativity*

An interesting application of the fact that different radiation temperatures T and T' are measured by different inertial observers (see example 1) is involved in a special relativistic model of the expanding universe.[14]

In a simplified version of the model, it is usually assumed that all stars and galactic objects were created in an explosion occurring at the point 0 in a space-time diagram (see Fig. 2) and propelled in every direction with all possible speeds. The influence of gravity is not considered, and consequently all particles are represented in the diagram by straight lines passing through 0. We shall assume that there is a uniform distribution

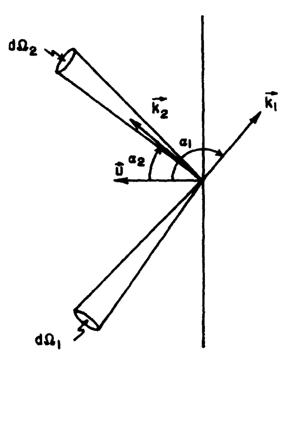

FIG. 1. Diagram of the reflection of a light beam by a moving mirror. The mirror moves toward the left with velocity $\mathbf{u}$ in the figure. The incident light within the elementary cone $d\Omega_1$ is reflected by the mirror into the elementary cone $d\Omega_2$.

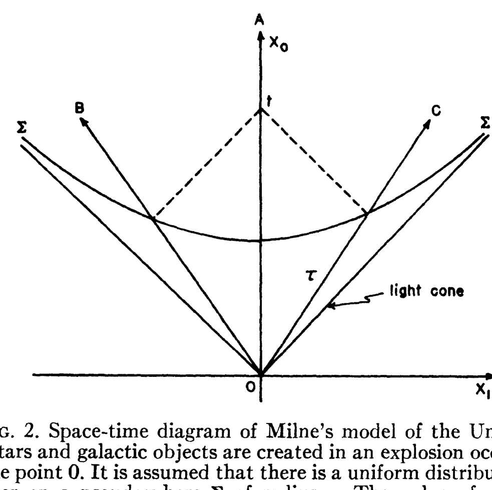

FIG. 2. Space-time diagram of Milne's model of the Universe. All stars and galactic objects are created in an explosion occurring at the point 0. It is assumed that there is a uniform distribution of matter on a pseudosphere Σ of radius τ. The value of τ could correspond to the time in which matter and radiation cease to be in thermal equilibrium. Any observer will detect isotropic blackbody radiation at any point on his world line. The temperature of the radiation decreases as the observer progresses along his trajectory.

[13] A. Einstein, Ann. Physik 17, 891 (1905).

[14] E. A. Milne, *Relativity, Gravitation and World-Structure* (Clarendon Press, Oxford, 1935). See also J. L. Synge, *Relativity: the Special Theory* (North-Holland Publishing Co., Amsterdam, 1965), p. 149.

of matter on a pseudosphere Σ of radius τ.[15] Consequently, the universe will appear uniform and isotropic to all observers on their respective world lines. We shall also assume that any observer on the surface Σ can detect isotropic blackbody radiation of absolute temperature T. This means that the mean occupation number $\langle n\rangle$ for any observer, say, B, on Σ will be given by

$$\langle n\rangle=(e^{\hbar k^0/kT}-1)^{-1}. \tag{56}$$

In the rest frame of the observer A, the expression (56) will assume the form

$$\langle n\rangle=(e^{\hbar\beta_\mu k^\mu}-1)^{-1}, \tag{57}$$

where $\beta_\mu=u_\mu/kT$; u_μ is the four-velocity of B. It is clear from Fig. 2 that the observer A will also detect isotropic radiation at his proper time t of intensity proportional to $\langle n\rangle$. The mean occupation number $\langle n\rangle$ depends on k^0 according to (57); i.e.,

$$\langle n\rangle=(e^{\eta\hbar k^0/kT}-1)^{-1},\quad k^\mu u_\mu=k^0\eta,\quad \eta=[(1+u)/(1-u)]^{1/2}. \tag{58}$$

The radiation detected by A at his proper time t preserves its thermal character and possesses an effective temperature T' given by

$$T'=T/\eta. \tag{59}$$

It is very easy to show that $\eta=t/\tau$. Therefore, formula (59) becomes

$$T'=T(\tau/t). \tag{60}$$

Since the radius of the universe is given by t in our model, we can conclude that the radiation temperature varies inversely as the radius of the universe.

The same arguments are applied to any other world line; accordingly, two different observers will detect cosmic blackbody radiation of the same temperature for the same value of their proper time. The radiation at any point in our space-time diagram will be represented, in general, by a distribution of the form

$$\langle n\rangle=(e^{\hbar\beta_\mu' k^\mu}-1)^{-1}, \tag{61}$$

where $\beta_\mu'=v_\mu/kT'$; T' is given by the expression (60); t is the proper time of the point considered along the world line of four-velocity v_μ. Since the propagation of a beam of radiation is reversible, the cooling in the course of the expansion is adiabatic. It is interesting to notice that although $\langle n\rangle$ is constant along a given null ray for a particular value of $\mathbf{k}$, the temperature vector β_μ' changes from point to point.

B. Thermal Radiation within a Cavity of Finite Volume

In this section we shall return to the subject of radiation in equilibrium within a box (cf. Sec. II). Through the expressions (38) and (39), the total energy P^0, the total linear momentum $\mathbf{P}$, and the entropy S of such a system are defined in all frames by the expressions

$$P^\mu=\frac{2h}{(2\pi)^3}\int f(k_\rho\beta^\rho)k^\mu d^3x d^3k, \tag{62}$$

$$S=\frac{2}{(2\pi)^3}\int g(k_\rho\beta^\rho)d^3x d^3k, \tag{63}$$

where f and g are obtained through formulas (28) and (30); i.e.,

$$f(k_\rho\beta^\rho)=\langle n\rangle, \tag{64a}$$

$$g(k_\rho\beta^\rho)=k[\hbar\beta_\rho k^\rho\langle n\rangle+\ln(\langle n\rangle+1)]. \tag{64b}$$

The integral over d^3x extends over the entire volume of the cavity, and the integral over d^3k covers all possible values of $\mathbf{k}$ between minus and plus infinity. To simplify the computation of the expressions (62) and (63), it is convenient to introduce the three-dimensional element of hypersurface $d\Sigma_\nu=(d^3x,0,0,0)$ and write

$$P^\mu=\int T^{\mu\nu}d\Sigma_\nu, \tag{65}$$

$$S=\int \sigma^\nu d\Sigma_\nu, \tag{66}$$

where

$$T^{\mu\nu}=\frac{2h}{(2\pi)^3}\int f(k_\rho\beta^\rho)k^\mu k^\nu\frac{d^3k}{k^0}, \tag{67}$$

$$\sigma^\mu=\frac{2}{(2\pi)^3}\int g(k_\rho\beta^\rho)k^\mu\frac{d^3k}{k^0}. \tag{68}$$

Both $T^{\mu\nu}$ and σ^μ have a well-defined tensor character under the Lorentz group. They correspond to the energy-momentum tensor and the entropy current for the radiation. In the case of thermal equilibrium, there exists a frame of reference in which β_μ takes the form $\hat\beta_\mu=\beta\delta_\mu{}^0$, with β independent of $\mathbf{k}$ and $\mathbf{x}$, since we can choose β_μ at each point in the cavity independent of $\mathbf{k}$ (and in our particular case independent of $\mathbf{x}$ as well). In that frame, we shall have

$$\hat T^{\mu\nu}=(\rho+p)\delta_0{}^\mu\delta_0{}^\nu-p\eta^{\mu0}, \tag{69}$$

$$\hat\sigma^\mu=\sigma\delta_0{}^\mu, \tag{70}$$

where

$$\rho(\beta)=\frac{2h}{(2\pi)^3}\int f(k^0\beta)k^0d^3k, \tag{71}$$

$$p(\beta)=\tfrac{1}{3}\rho, \tag{72}$$

$$\sigma(\beta)=\frac{2}{(2\pi)^3}\int g(k^0\beta)d^3k. \tag{73}$$

[15] The value of τ could correspond to the time in which matter and radiation cease to be in thermal equilibrium [see, e.g., R. H. Dicke, P. J. Peebles, P. G. Roll, and D. T. Wilkinson, Astrophys. J. **142**, 414 (1965)].

By using (32) and (64), we can readily show that σ and ρ are related by

$$\sigma=\tfrac{4}{3}k\beta\rho. \tag{74}$$

In a general frame ($\beta_\mu=v_\mu\beta$, $v_\mu v^\mu=1$), we have

$$T^{\mu\nu}=(\rho+p)v^\mu v^\nu-\eta^{\mu\nu}p, \tag{75}$$

$$\sigma^\mu=\sigma v^\mu. \tag{76}$$

Consequently, after integrating expressions (65) and (66), we obtain

$$P^\mu=(\rho+p)v^\mu v^0 V-p\eta^{\mu 0}V, \tag{77}$$

$$S=\sigma v^0 V, \tag{78}$$

where V is the total volume of the cavity. We see that v^μ defines a linear velocity $\mathbf{u}=(v^0)^{-1}\mathbf{v}$, which may be interpreted as the velocity of the system. We notice that the possibility of assigning a unique velocity to the entire system is a direct consequence of the conditions for thermal equilibrium because we can introduce a vector β_μ, independent of $\mathbf{k}$ and $\mathbf{x}$, as the generalized temperature of the system. The frame of reference in which $\mathbf{u}=0$ ($\mathbf{P}=0$) will be called the rest frame. The particularization of the above expressions to the case $\mathbf{u}=0$ reproduces the formulas of the ordinary theory of blackbody cavity radiation in which we identify β with $1/kT$.[16]

Now let the system make an infinitesimal transition from one state of equilibrium to another. The variations in the values of the total energy, linear momentum, and entropy of the radiation are obtained by taking the variations of the expressions (77) and (78):

$$\delta P^\mu=(\delta\hat{E}+p\delta\hat{V})v^\mu+\hat{H}\delta v^\mu+\delta p\hat{V}v^\mu-\delta(pV)\eta^{\mu 0}, \tag{79}$$

$$\delta S=\delta\hat{S}, \tag{80}$$

where $\hat{H}=\hat{E}+p\hat{V}$, $\hat{E}=\rho\hat{V}$, $\hat{V}=v^0V$, $\hat{S}=\sigma\hat{V}$ are the total rest enthalpy, energy, volume, and entropy, respectively. Projecting (79) along the vector $k\beta_\mu$, we find that

$$\delta S=k\beta_\mu(\delta P^\mu-\Delta W^\mu), \tag{81}$$

where, according to the relations (74) and formula (80), we have identified δS with the expression[17]

$$\delta S=k\beta_\mu[(\delta\hat{E}+p\delta\hat{V})v^\mu+\hat{H}C^\mu], \tag{82}$$

$$\beta_\mu C^\mu=0, \tag{83}$$

and ΔW^μ with

$$\Delta W^\mu=\delta p\hat{V}v^\mu-\delta(pV)\eta^{\mu 0}+\hat{H}D^\mu, \tag{84}$$

where

$$D^\mu=\delta v^\mu-C^\mu.$$

[16] See, e.g., M. Planck, *Theory of Heat* (The Macmillan Co., New York, 1957), Parts 3 and 4, Chap. IV.

[17] The introduction of the vector $\hat{H}C^\mu$ in (82), $\hat{H}D^\mu$ in (84), allows more generality in the coefficient of $k\beta_\mu$.

By (83) and the condition $v^\mu\delta v_\mu\equiv 0$, we have

$$D^\mu v_\mu=0. \tag{85}$$

After a straightforward calculation, we find that

$$\Delta W^0=\Delta\mathbf{W}\cdot\mathbf{u}-p\delta V. \tag{86}$$

The formula (84) is interpreted as the generalized expression for the work performed on the system when the work measured in the rest frame is given by

$$\Delta\hat{W}^0=-p\delta\hat{V}. \tag{87}$$

Whenever (87) is valid, we shall also define the generalized transfer of heat to the system by

$$\Delta Q^\mu=(\delta\hat{E}+p\delta\hat{V})v^\mu+\hat{H}C^\mu. \tag{88}$$

With these definitions, the expressions (79) and (81) can be written in the form

$$\delta P^\mu=\Delta Q^\mu+\Delta W^\mu, \tag{89}$$

$$\delta S=k\beta_\mu\Delta Q^\mu. \tag{90}$$

We see that in the special case of a reversible process, these expressions coincide with those obtained by Møller [see expressions (7) and (10)] if we require that both $\delta\mathbf{u}$ and C^μ (or D^μ) vanish.

It is worth noting that in our discussion we have not considered the radiation container. The problem of including it in an enlarged system will not be treated here.

V. CONCLUSIONS

We have shown that a generalized formulation of statistical mechanics provides us with a unified logical basis for the construction of a manifestly covariant relativistic thermodynamics. Quantities representing temperature, transfer of heat, performance of work, etc., are generalized to tensors whose rank depends on the set of observables chosen to describe the system. Although the generalized temperature is represented by a tensor, it is frequently impossible to determine its value uniquely by performing physical measurements. As we have seen in the case of a pencil of electromagnetic radiation, where the generalized temperature is comprised of four quantities (four-vector), only one parameter is directly measured and properly called the "temperature" of the beam. Only in particular cases (e.g., cavity radiation) can the unit vector v^μ in expressions like $\beta^\mu=v^\mu/kT$ be identified with the four-velocity of the system and a meaning ascribed to T.

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