Three-Alpha Model of C^{12} Nucleus†

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(Received 3 June 1965)

The C^{12} nucleus is studied from the three-α-particle point of view based on a wave-function method in which the Hermiticity of the Fredholm kernel is preserved. The interaction between two α particles, obtained from the resonating-group calculation of Okai and Park, is recast in a separable form with the aid of Hille's formula. For the numerical calculation, we have employed three simpler types of the potential: Gaussian, Tabakin, and Yamaguchi. It is shown that Harrington's theory, based on the T-matrix method, reduces to a result of the present study, but not to those of other wave-function methods. Contrary to a conclusion of the previous work, we have succeeded in finding an excited state with the same (J, π) as the ground state. The present method also supplies the mean height and side of the equilateral triangles formed by three α particles. These values are found to imply a root-mean-square radius of the C^{12} nucleus which agrees well with the experimental value.

I. INTRODUCTION

AN integral equation with a separable (or degenerate) kernel can be solved exactly by an algebraic method.[1,2] The two-nucleon problem was first treated in this way by Yamaguchi,[3] who employed a nonlocal but separable (NLS) potential. This potential has since been widely used in the three-body problem. In fact, most of the recent formulations of the three-body problem presuppose the use of NLS potential.[4,5] While some fundamental question regarding the nonlocality of the basic nucleon-nucleon interaction may exist, such a question does not exist for the nucleon-nucleus interaction. Any theory in which antisymmetrization is involved will lead to an integrodifferential equation, well-known examples being the atomic Hartree-Fock equation and the method of resonating group structure in nuclear physics.[6] In the latter case the nonlocality of the interaction is reinforced by the exchange nature of the nuclear forces. Thus as far as nuclear physics is concerned the nonlocality of the interaction is a general feature rather than an exception. It remains to be seen how well the ansatz of separability is justified.

In the present paper we first investigate the separability of the α-α interaction, which was treated earlier by Okai and Park[7] by use of the method of resonating group structure. We shall recast the kernel in the momentum space and write down the separable form of the kernel with the aid of Hille's formula. This is done in the first part of Sec. II.

The second objective is to calculate the binding energy and other observables of C^{12} nucleus by use of NLS two-body potentials between alpha particles[8] This problem has already been treated by Harrington. and Wong,[9] whose methods were based on the Faddeev-Lovelace formulation of the three-body problem, in which the T matrix plays a central role. For the bound-state problem, however, the Schrödinger wave function has greater physical ramification. The method of wave function employed here is similar to methods used by Mitra[4] and Eyges[10] but differs from them in details and in that we preserve the Hermiticity of the integral kernel. Furthermore, it turns out that Harrington's final result can be shown to lead to Eq. (20) below, thus implying that for the present simplest three-body problem the T-matrix method is equivalent to the wave-function method adopted here. We have employed three types of NLS potentials: a Yamaguchi potential, Tabakin-type potentials,[11] and Gaussian potentials. These are discussed in detail in the second half of Sec. II. The Yamaguchi potential is considered here as a reference and the values of the potential parameters are equivalent to those suggested by Harrington.[8] We find that the potential does give rise to the first excited state of the same (J, π) as the ground state of C^{12} nucleus, contrary to Harrington's finding. The details of the wave-function method are presented in Sec. III, where we also briefly touch upon the Eyges method. On the whole, Tabakin-type potentials reproduce well both the two-body and three-body experimental data, including the prediction of the first excited state of the same (J, π) as that of the ground state. In the present method of wave function it is very natural to talk of the relative motions of the three particles as manifested through the wave functions. We have thus calculated the root-mean-square distances involved in the three-body system. The result predicts, if inter-

† Work in its early stage supported in part by the National Science Foundation.

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preted geometrically, a reasonable value of the root-mean-square radius of the C^{12} nucleus. These are discussed in detail in Sec. IV. Finally, in Sec. V we summarize the physics of the C^{12} nucleus obtained in this study.

II. SEPARABILITY OF α-α INTERACTION

A simple example of a nonlocal interaction between complex nuclei is afforded by the α-α interaction as calculated by the method of resonating group structure.[6] According to this method, the α-α interaction is described by an integrodifferential equation of the form

$$(\nabla^2+k^2)\psi(\mathbf{r})=\int K(\mathbf{r},\mathbf{r}_0)\psi(\mathbf{r}_0)\,d\mathbf{r}_0, \tag{1}$$

where the kernel $K(\mathbf{r},\mathbf{r}_0)$ arises from the antisymmetrization and the exchange nature of the two-nucleon forces. There is, in addition, the direct potential, which will be regarded as having been absorbed in the kernel. In the momentum space, Eq. (1) reads

$$(-p^2+k^2)\psi(\mathbf{p})=\int\langle\mathbf{p}\mid K\mid\mathbf{q}\rangle\psi(\mathbf{q})\,d\mathbf{q}, \tag{2}$$

where $\psi(\mathbf{p})$ is the Fourier transform of $\psi(\mathbf{r})$ and

$$\langle\mathbf{p}\mid K\mid\mathbf{q}\rangle=\int\int[d\mathbf{r}/(2\pi)^{3/2}]\exp(-i\mathbf{p}\cdot\mathbf{r})\langle\mathbf{r}\mid K\mid\mathbf{r}_0\rangle \times\exp(i\mathbf{q}\cdot\mathbf{r}_0)[d\mathbf{r}_0/(2\pi)^{3/2}]. \tag{3}$$

We see that the separability of $K(\mathbf{r},\mathbf{r}_0)$ always implies that of $\langle\mathbf{p}\mid K\mid\mathbf{q}\rangle$. The kernel $K(\mathbf{r},\mathbf{r}_0)$ derived and used by Okai and Park[7] consists of many terms of the form

$$K(\mathbf{r},\mathbf{r}_0)=F(r)F(r_0)\exp[-\alpha(\mathbf{r}-\mathbf{r}_0)^2], \tag{4}$$

which is symmetric (or Hermitian) in $\mathbf{r}$ and $\mathbf{r}_0$. We see immediately that it is the angular factor $\exp[-\alpha(\mathbf{r}-\mathbf{r}_0)^2]$ which prevents K from being separable. We resolve K into the radial and angular parts:

$$K(\mathbf{r},\mathbf{r}_0)=\sum_{L,M}[4\pi/(2L+1)]K_L(r,r_0)Y_{LM}(\hat{r})Y_{LM}(\hat{r}_0), \tag{5a}$$

where

$$K_L(r,r_0)=\tfrac{1}{2}(2L+1)\int_{-1}^{1}K(r,r_0)P_L(\mu)\,d\mu. \tag{5b}$$

For the specific form of (4) the radial part of K becomes

$$K_L(r,r_0)=(-)^L(2L+1)F(r)F(r_0) \times\exp[-\alpha(r^2+r_0^2)](\tfrac{1}{2}\pi)^{1/2}[I_{L+1/2}(2\alpha rr_0)/(2\alpha rr_0)^{1/2}]. \tag{5c}$$

We now employ Hille's formula[12]

$$\frac{I_\nu(2^{3/2}xy)}{(xy)^\nu}=\frac{\exp(x^2+y^2)}{2^{\nu/2+1}} \times\sum_{n=0}^{\infty}\frac{n!}{2^n\Gamma(n+\nu+1)}L_n^\nu(x^2)L_n^\nu(y^2) \tag{6}$$

to make $K_L(r,r_0)$ formally separable. By substituting (6), (5c), and (5a) into (3), we find

$$\langle\mathbf{p}\mid K\mid\mathbf{q}\rangle=2(\tfrac{1}{2}\pi)^{3/2}\sum_{l,m,n}\frac{(-\frac{1}{2}\alpha)^l n!}{2^n\Gamma(n+1+\frac{3}{2})} \times f_n^{*(l)}(p)f_n^{(l)}(q)Y_{lm}^*(\hat{p})Y_{lm}(\hat{q}), \tag{7a}$$

where

$$f_n^{(l)}(p)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dr\,r^2j_l(pr) \times\left[F(r)\exp\left(-\frac{\sqrt{2}-1}{\sqrt{2}}\alpha r^2\right)r^l L_n^{l+1/2}\left(\frac{\alpha r^2}{\sqrt{2}}\right)\right], \tag{7b}$$

which is the Fourier-Bessel transform of the quantity in square brackets. We note that Eq. (7a), up to a multiplicative factor, is precisely the angular-momentum-dependent separable interaction anticipated by Yamaguchi. If we apply the above prescription to the entire kernel in (1), we would be deriving the separable kernel by starting from the local two-nucleon interaction. The kernel $K(\mathbf{r},\mathbf{r}_0)$ of Ref. 7, which contains the effect of Coulomb interaction as well, is very much complicated, and hence it is not really illuminating to pursue this program.[13] Instead we shall postulate the S-wave NLS interaction of the form

$$f_G(p)=\alpha(p_c^2-p^2)(a+bp^2)\exp(-dp^2). \tag{8}$$

This form of the interaction is implied by (7b) as well as the shell-model type of calculation involving the Laguerre polynomials. Since the S-wave α-α scattering phase shift is known to change sign as a function of the bombarding energy, the value of P_c^2 can be readily determined. We shall also use a Tabakin potential

$$f_T(p)=\alpha(p_c^2-p^2)[(p^2+d^2)/(p^2+b^2)](p^4+a^4)^{-1}. \tag{9}$$

This is a variant of the Yamaguchi potential, implying the Yukawa-type spatial dependence, and as such probably has a better theoretical foundation. For the sake of a reference we shall also employ a Yamaguchi potential

$$f_Y(p)=\alpha/(p^2+b^2). \tag{10}$$

The separable two-body potential is then given by

$$V(p,p')=-(\hbar^2/m)(1/2\pi^2)f(p)f(p'), \tag{11}$$

where m is the mass of an α particle. The two-body binding energy $\hbar^2\kappa^2/m$ and the S-wave phase shift are then obtained from the well-known relations

$$1=\frac{2}{\pi}\int_0^\infty dp\,\frac{p^2f^2(p)}{p^2+\kappa^2}, \tag{12}$$

$$\tan\delta(k)=kf^2(k)\Big/\left(1-\frac{2}{\pi}P\int_0^\infty dp\,\frac{p^2f^2(p)}{p^2-k^2}\right). \tag{13}$$

[12] See *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Co., New York, 1953) Vol. 2, p. 189.

[13] S. C. Park, C. C. Leung, and J. P. Rickett, Bull. Am. Phys. Soc. 12, 188 (1967). In this report Hille's formula (6) was applied to the kernel of Ref. 7.

TABLE I. Values of the potential parameters and results of calculation. Energies listed here are the nuclear-binding energies only. In order to obtain the actual binding energies, Coulomb energies must be added. According to Ref. 8, the approximate values of the Coulomb energies are 3.5 and 5.44 MeV for Be^8 and C^{12} nuclei, respectively.

Potential (method used)	a	b	d	α^2	Ground-state energy of Be^8 (MeV)	Ground-state energy of C^{12} (MeV)	Excited-state energy of C^{12} (MeV)	Root-mean-square radius of C^{12} (F)
Yamaguchi potential		0.736		2.36				
(Harrington's result)[a]		(F^{-1})		(F^{-6})	−2.9	−12.8		
(WFM)[a]						−12.8	−3.0	1.3
(Eyges' method)						−13.3	−1.6	
(Mitra's method)						−11.2		
Tabakin potential	1.9	1.3	5.0	3.186	−3.6	−12.2	−6.6	2.0
(WFM)	(F^{-1})	(F^{-1})	(F^{-1})	(F^{-6})				
Gaussian potential	1.0	0.05	0.5	4.432	−3.4	−9.0	−2.2	2.1
(WFM)	0.6	0.1	0.6	14.48	−4.1	−10.0		2.1
	(F^{-2})		(F^2)	(F^2)				

[a] These two works employ the same binding-energy equation (20).

The values of the potential parameters are listed in Table I; those for the Yamaguchi potential are the ones suggested by Harrington. The corresponding shapes of the potential factors are shown in Fig. 1. The S-wave α-α scattering phase shifts predicted by these potentials are shown in Fig. 2, which also contains the result S_{O-P} of the resonating-group calculation. The two versions of Gaussian potential predict almost equal phase shifts, and they are not distinguished from each other in Fig. 2. This means that the determination of the potential parameters is by no means unique, and there exists a certain cancellation effect, the greater attraction of f_{G1} annulling to some extent its greater repulsion, thus producing the same phase shift as f_{G2}. On the other hand, some consequences of these potentials are quite unexpected. For instance, the introduction of the repulsive part as in f_T does not necessarily mean that the corresponding two-body binding energy would decrease, although it yields negative phase shifts. On the contrary, the Tabakin potential in Fig. 1 gives rise to a greater binding energy than the Yamaguchi potential shown there, as attested by the values in Table I. This and other paradoxical features of the Tabakin potentials and the separable potentials in general are discussed in detail elsewhere.[14]

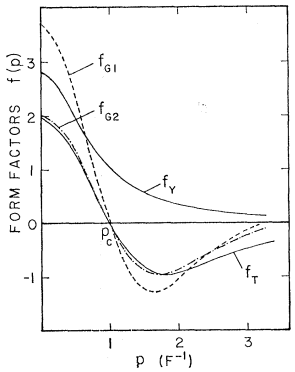

FIG. 1. Potential form factors. The subscripts G, Y, and T correspond to Gaussian, Yamaguchi, and Tabakin potentials, respectively.

III. THREE-BODY EQUATION

We shall now present the formulation of the three-body bound state problem based on the wave-function method (abbreviated as WFM), which is similar to methods used by Eyges and Mitra. We introduce three vectors

$$\mathbf{P}=\tfrac{1}{3}(\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_3),$$
$$\mathbf{p}_1=\tfrac{1}{2}(\mathbf{K}_2-\mathbf{K}_3), \qquad (14)$$
$$\mathbf{q}_1=\tfrac{1}{3}(\mathbf{K}_2+\mathbf{K}_3-2\mathbf{K}_1),$$

where $\hbar\mathbf{K}_i$ is the momentum of the ith α particle

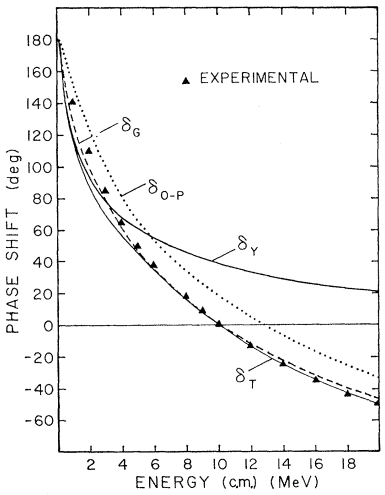

FIG. 2. S-wave α-α scattering phase shifts. The subscripts G, O-P, Y, and T correspond to Gaussian, Okai-Park, Yamaguchi, and Tabakin potentials.

[14] C. C. Leung and S. C. Park (to be published in Phys. Rev.).

referred to a fixed coordinate system. We shall work in the barycentric coordinate system, $\mathbf{P}=\mathbf{O}$, throughout the work. We can similarly introduce other pairs of momenta, which can be obtained from the pair $(\mathbf{p}_1, \mathbf{q}_1)$ by the transformation

$$\begin{pmatrix}\mathbf{p}_2\\ \mathbf{q}_2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & \frac{3}{4}\\ -1 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\mathbf{p}_1\\ \mathbf{q}_1\end{pmatrix},$$
$$\begin{pmatrix}\mathbf{p}_2\\ \mathbf{q}_3\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{3}{4}\\ -1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\mathbf{p}_1\\ \mathbf{q}_1\end{pmatrix}. \tag{15}$$

These momenta are invariant in the sense that

$$p_i^2+\tfrac{3}{4}q_i^2=p_1^2+\tfrac{3}{4}q_1^2 \qquad (i=1, 2, 3).$$

The three-body nuclear binding energy will be denoted by

$$B=\hbar^2K^2/m,$$

where m is the mass of an α particle.

Our starting point is the homogeneous integral equation for the three-body bound-state function $\Psi(\mathbf{p}, \mathbf{q})$ in any pair of the variables $(\mathbf{p}_i, \mathbf{q}_i)\equiv(\mathbf{p}, \mathbf{q})$:

$$(\hbar^2/m)(p^2+\tfrac{3}{4}q^2+K^2)\Psi(\mathbf{p}, \mathbf{q})$$
$$=-\int\int d\mathbf{p}'d\mathbf{q}'\langle \mathbf{p}, \mathbf{q} \mid V_1+V_2+V_3 \mid \mathbf{p}', \mathbf{q}'\rangle\Psi(\mathbf{p}', \mathbf{q}'), \tag{16}$$

where V_i denotes the interaction between the pair of α particles j and k $(i, j, k=1, 2, 3)$. We shall restrict ourselves to only the S-wave interaction between two α particles; we shall not consider any three-body forces. The separable interaction V_i is then assumed to be of the form

$$\langle \mathbf{p}_i, \mathbf{q}_i \mid V_i \mid \mathbf{p}_i', \mathbf{q}_i'\rangle$$
$$=-(\hbar^2/m)(1/2\pi^2)f(p_i)f(p_i')\delta(\mathbf{q}_i-\mathbf{q}_i'). \tag{17}$$

The basic equation (16) then suggests the introduction of the following three functions:

$$\psi_i(\mathbf{p}_i, \mathbf{q}_i)=\frac{-1}{(\hbar^2/m)(p_i^2+\frac{3}{4}q_i^2+K^2)}\times\int\int dp_i'dq_i'\langle \mathbf{p}_i, \mathbf{q}_i \mid V_i \mid \mathbf{p}_i', \mathbf{q}_i'\rangle\Psi(\mathbf{p}_i', \mathbf{q}_i')$$
$$=(1/2\pi^2)\left[f(p_i)/(p_i^2+\tfrac{3}{4}q_i^2+K^2)\right]\times\int dp_i'f(p_i')\Psi(\mathbf{p}_i', \mathbf{q}_i) \tag{18}$$
$$\equiv\left[f(p_i)/(p_i^2+\tfrac{3}{4}q_i^2+K^2)\right]\eta(\mathbf{q}_i) \qquad (i=1, 2, 3),$$

where $\eta(\mathbf{q}_i)$ is an unknown function yet to be determined. The three-body wave function $\Psi(\mathbf{p}, \mathbf{q})$ is then the sum of the following three functions:

$$\Psi(\mathbf{p}, \mathbf{q})=\psi_1+\psi_2+\psi_3. \tag{19a}$$

By introducing the change of variables as in (15), we have

$$\Psi(\mathbf{p}, \mathbf{q})=1/(p^2+\tfrac{3}{4}q^2+K^2)\times\left[f(p)\eta(q)+f(-\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q})\eta(-\mathbf{p}-\tfrac{1}{2}\mathbf{q})+f(-\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}\mathbf{q})\eta(-\mathbf{p}+\tfrac{1}{2}\mathbf{q})\right]. \tag{19b}$$

Because of the symmetry of the system consisting of three identical particles, among which only the S-wave interactions are operative, we must have

$$\psi_1=\psi_2=\psi_3$$

and

$$\eta(\mathbf{q})=\eta(q),$$

where $q=|\mathbf{q}|$.

In view of this, Eq. (18) becomes a single integral equation for η, which now reads

$$\eta(q)=(1/2\pi^2)\int d\mathbf{p}\left[f(p)/(p^2+\tfrac{3}{4}q^2+K^2)\right]\times\left[f(p)\eta(q)+f(-\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q})\eta(-\mathbf{p}-\tfrac{1}{2}\mathbf{q})+f(-\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}\mathbf{q})\eta(-\mathbf{p}+\tfrac{1}{2}\mathbf{q})\right].$$

By suitable changes of the variables in the second and last integrals and by using the evenness of the S-wave potential form factor f, we obtain

$$\left[1-\int\frac{d\mathbf{p}}{2\pi^2}\frac{f^2(p)}{p^2+\frac{3}{4}q^2+K^2}\right]\eta(q)=\frac{1}{\pi^2}\int d\mathbf{p}\frac{f(\mathbf{p}+\frac{1}{2}\mathbf{q})f(\frac{1}{2}\mathbf{p}+\mathbf{q})}{p^2+\mathbf{p}\cdot\mathbf{q}+q^2+K_2}\eta(p). \tag{20}$$

This equation can be solved for the three-body binding energy $\hbar^2K^2/m$ and the function $\eta(q)$ from which the three-body wave function Ψ can be computed. This is true so long as the factor multiplying $\eta(q)$ on the left-hand side of (20) does not vanish. In order to look into this aspect further it is convenient to introduce the factor function

$$H(q)=1-\int(d\mathbf{p}/2\pi^2)\left[f^2(p)/(p^2+\tfrac{3}{4}q^2+K^2)\right]. \tag{21a}$$

If we now compare this with the binding-energy equation (12) of the two-body system, we see that $H(q)=0$ if

$$\tfrac{3}{4}q^2+K^2=\kappa^2, \tag{21b}$$

where $\hbar_2\kappa_2/m$ is the two-body binding energy. This is impossible if $K^2>\kappa^2$ and for real (physical) values of q^2. While the three-body energy $\hbar^2K^2/m$ is certainly greater than the two-body binding energy $\hbar^2\kappa^2/m$, there is a possibility that the binding energy of an excited state of the three-body system is so small that Eq. (21b) can be satisfied for a particular value of q^2. For the present problem this possibility does not exist, but we are pointing out this possibility in other problems. Equation (20) can be put in a more symmetric form if

we introduce an unknown function $F(q)$ through

$$F(q)=\eta(q)H^{1/2}(q),$$

in terms of which Eq. (20) now reads

$$F(q)=\frac{1}{\pi^2}\int d\mathbf{p}\,\frac{f(\mathbf{p}+\frac{1}{2}\mathbf{q})f(\frac{1}{2}\mathbf{p}+\mathbf{q})}{(p^2+\frac{3}{4}q^2+K^2)H^{1/2}(p)H^{1/2}(q)}F(p).$$

It is clear that this equation possesses a Hermitian kernel.

We shall now discuss how the present formulation is related to the other formalisms of the three-body problem. First, we observe that the Eyges method consists in projecting the three-body function onto the two-body space:

$$\psi_i(\mathbf{p}_i,\mathbf{q}_i)=\tilde{\phi}(p_i)\eta(q_i)=[\tilde{f}(p_i)/(p_i^2+K^2)]\eta(q_i), \quad (22)$$

where the tilde denotes that the quantity is of two-body nature. Thus, $\tilde{\phi}(p_i)$ above is the two-body bound-state function corresponding to a two-body binding energy conveniently set equal to $\hbar^2K^2/m$, and $\tilde{f}(p_i)$ is the corresponding two-body potential form factor with the values of the strength parameter $\tilde{\alpha}^2$ adjusted to yield this two-body binding energy. If ψ_i of (22) are used in (19a) and (16), we obtain an integral equation for $\eta(q_i)$, identical to Eq. (23) of Ref. 10, whose kernel is non-Hermitian, however. If we take the Hermiticity of the integral kernel as one of the guiding ideas and symmetrize the kernel, we obtain

$$\{\tfrac{3}{4}q^2+[1-(\alpha/\tilde{\alpha})^2](K^2+\int d\mathbf{p}p^2\,|\,\tilde{\phi}(p)\,|^2)\}\eta(q)$$
$$=2(\alpha/\tilde{\alpha})^2\int d\mathbf{p}\{\tfrac{1}{2}[(\mathbf{p}+\tfrac{1}{2}\mathbf{q})^2+(\tfrac{1}{2}\mathbf{p}+\mathbf{q})^2]+K^2\}$$
$$\times\tilde{\phi}(\mathbf{p}+\tfrac{1}{2}\mathbf{q})\tilde{\phi}(\tfrac{1}{2}\mathbf{p}+\mathbf{q})\eta(p). \quad (23)$$

A further and more drastic separation of the variables can be effected by following Mitra's approximations, especially involving the angle integration. The comparison of the numerical results will be discussed later. Both the Eyges and Mitra methods share with the present formalism that they are all wave-function methods. Harrington, on the other hand, bases his formulation on the T-matrix method of Faddeev and Lovelace. It turns out that for the present problem, Harrington's result exactly coincides with a result of the present formalism, but not with Eyges' nor Mitra's. If we use the values

$$\beta=0.736\ \mathrm{F}^{-1},$$
$$C=-2.36\times(4\pi\hbar^2/m)=-\alpha^2(4\pi\hbar^2/m), \quad (24)$$

for the Yamaguchi potential in Eq. (16) of Ref. 8, it reduces to (20) of the present study. It follows, therefore, that we should reproduce Harrington's result. It turns out, however, that this is not the case, and the main reason for the discrepancy seems to lie in the accuracy of the numerical calculation. This aspect is discussed in the next section.

IV. NUMERICAL RESULTS

The three-body binding energy $\hbar^2K^2/m$ as well as the function $\eta(q)$ are obtained from the numerical solution of (20), using a 24-point Gauss-Legendre radial integration and a 24-point Legendre angular integration. The results obtained for different potentials and for different methods of the solution are summarized in Table I. For the Tabakin and Gaussian potentials we employed only WFM, but for the Yamaguchi potential we have also tested the approximate methods suggested by Eyges and Mitra. The values of the ground-state energy of C^{12} predicted by these two approximate methods are not too much different from that of WFM, indicating that these two approximate methods are relatively accurate ones. On the other hand, we believe that these two theoretical approximations do not add any real physical insight and hence are unnecessary, because the three-body binding energy as well as the wave function can be numerically computed in a straightforward manner. For the Gaussian potential we have tried the two versions shown in Fig. 1. They yield almost equal values of the C^{12} ground-state energy, which, however, are smaller than the values resulting from the other two types of the potential. For both Yamaguchi and Tabakin potentials considered here the three-body binding energy is greater than three times the two-body binding energy.

We have also searched, with some success, for the first excited state of the C^{12} nucleus with the same (J,π) as the ground state. For the Yamaguchi potential we found the first-excited-state energy equal to -3.0 MeV, in contrast to the conclusion given by Harrington, who used in effect the same potential and the same Eq. (20). Both the Tabakin and the first version of the Gaussian potentials predict an excited state, as shown in Table I.

In the present WFM, it is very natural to talk of the relative motion of two α particles and the corresponding root-mean-square distance. In order to look into this aspect in detail we rewrite (19b) in the form

$$\Psi(\mathbf{p},\mathbf{q})=[f(p)\eta(q)/(p^2+\tfrac{3}{4}q^2+K^2)][1+A(\mathbf{p},\mathbf{q})],$$

where

$$A(\mathbf{p},\mathbf{q})=f^{-1}(p)\eta^{-1}(q)$$
$$\times[f(-\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q})\eta(-\mathbf{p}-\tfrac{1}{2}\mathbf{q})+f(-\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}q)\eta(\mathbf{p}-\tfrac{1}{2}\mathbf{q})].$$

When the wave function Ψ is squared and integrated over either p or q, we obtain a function of a single variable in the form $f^2(p)F^2(p)$ or $\eta^2(q)G^2(q)$, where

$$F(p)=\left[\int d\mathbf{q}\left(\frac{\eta(q)}{p^2+\frac{3}{4}q^2+K^2}[1+A(\mathbf{p},\mathbf{q})]\right)^2\right]^{1/2},$$

$$G(q)=\left[\int d\mathbf{p}\left(\frac{f(p)}{p^2+\frac{3}{4}q^2+K^2}[1+A(\mathbf{p},\mathbf{q})]\right)^2\right]^{1/2}.$$

For a Yamaguchi potential, $f(p)$ and $\eta(q)$ are smooth and sign-definite functions and so are $F(p)$ and $G(p)$.

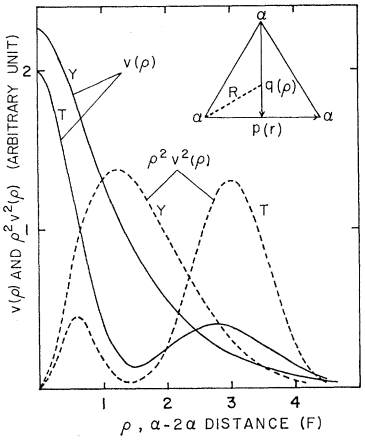

FIG. 3. Radial wave function $v(\rho)$ and radial probability density $\rho^2v^2(\rho)$, obtained from using the Yamaguchi (Y) and Tabakin (T) potentials. $v(\rho)$ is the Fourier transform of the q-dependent part of the three-body wave function $\Psi(p, q)$.

In performing the integration we assign to the value of $\eta(s)$, for any argument s (≥ 0), the value of η at the mesh point closest to s, since only its numerical values at mesh points are available. This is the way the angle integration has been performed. For the radial integration we used 31 mesh points for positive p or q up to 6 F^{-1}. In order to discuss the spatial wave function we now introduce the Fourier transform of $f(p)F(p)$, $u(r)$, and that of $\eta(q)G(q)$, $v(\rho)$. Here $\boldsymbol{\rho}$ and $\mathbf{r}$ represent the position vectors conjugate to $\mathbf{q}$ and $\mathbf{p}$, respectively. The wave function $v(\rho)$ shown in Fig. 3 thus represents the motion of an α particle relative to the center of mass of the remaining two α particles. In the inset of Fig. 3 we show a schematic relation between $\mathbf{p}$ and $\mathbf{q}$. The same diagram can also be used for the pair $(\mathbf{r}, \boldsymbol{\rho})$. The wave function $u(r)$ in Fig. 4, on the other hand, represents the relative motion of two α particles. We observe that $u(r)$ reflects, to some extent, the form of the interparticle potential, i.e., $u(r)$ for the Tabakin potential changes sign, whereas that for the Yamaguchi potential is sign-definite. Having computed $\rho^2v^2(\rho)$ and $r^2u^2(r)$, we can readily calculate the root-mean-square distances of ρ and r, respectively denoted by $\langle\rho\rangle$ and $\langle r\rangle$. These distances correspond to the average values of the height and side length of a triangle. Now this triangle must necessarily be equilateral, because what is true of one height (side) must be true of two remaining heights (sides), thanks to the inherent symmetry of the three-α system. Now suppose we consider two equilateral triangles (or two circles circumscribing the triangles), one with height $\langle\rho\rangle$, the other with side length $\langle r\rangle$. The question then naturally arises whether these two circles imply the same value of the radius R. It has been found in the present numerical calculation that the values of radii corresponding to these triangles do indeed agree with each other within two significant figures. We are inclined to interpret this result as indicating that our numerical wave functions $u(r)$ and $v(\rho)$ are fairly accurate. The values of the root-mean-square radius R, obtained for different potentials, are shown in the last column of Table I. Both Tabakin and Gaussian potentials predict a considerably larger C^{12} nucleus than the Yamaguchi potential. This is easily acounted for from the fact that the former type of the potential contains a repulsion as well as an attraction, in contrast to the Yamaguchi potential which is everywhere attractive. All the values of the root-mean-square radius listed in the table are smaller than the experimental radius, which is about 2.5 F. There are at least two obvious ways of improving the present values. The first is to include the Coulomb interaction, which will certainly increase the size of the C^{12} nucleus. The second way is to explicitly take into account the finite size of an α particle whose experimental radius is about 0.9 F. The value of the radius of C^{12} nucleus obtained from the Yamaguchi potential becomes 2.2 F, which would further increase if the Coulomb potential were explicitly taken into account. The value of the nuclear radius obtained from the Tabakin type of potentials, on the other hand, becomes 2.9 F. However, in this case, the naive model upon which the argument rests also pictures that there exists a *hollow* central region within the circle of radius 1.1 F, which is certainly unrealistic. So we see that the literal inclusion of the effect of the finite size of an α particle probably does injustice to the actual situation. It appears, therefore, that the *explicit* inclusion of the size effect of an α particle is not warranted for the case of the Tabakin and Gaussian potentials, because the fact that these potentials involve a repulsion is itself a reflection of the size effect. In other words, the effect of the finite size of an α particle is already indirectly taken into account when the two-α interaction is determined to fit the *experimental* scattering data. The discussion given above is admittedly

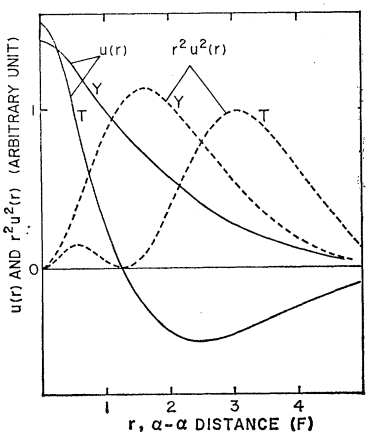

FIG. 4. Radial wavefunction $u(r)$ and radial probability density $r^2u^2(r)$ for the relative motion between two α particles interacting through the Yamaguchi (Y) and Tabakin (T) potentials. $u(r)$ is the Fourier transform of the p-dependent part of the three body wave function $\Psi(p, q)$.

qualitative. It shows nevertheless that the values of the nuclear radius predicted by different types of the potential can be justified qualitatively according to the particular features of the interaction used.

V. CONCLUSION

The three-α model of the C^{12} nucleus is investigated based on a WFM similar to methods used by Eyges and Mitra, but it differs from them in detail and in that the Hermiticity of the Fredholm kernel is preserved through out. We observe that the WFM is appropriate to the bound-state problem treated here, while the T-matrix method is powerful for the scattering problem. However, the two methods yield the same binding-energy equation (20). The fact that Harrington missed the first excited state might be due to the difference in the accuracies of the numerical solutions. We have employed three types of the potential: Gaussian, Tabakin, and Yamaguchi. The Gaussian potential was suggested by the resonating-group calculation for the α-α scattering. The Yamaguchi potential was used as a reference, the values of the parameters being equivalent to those used by Harrington. Throughout the work we are concerned with the nuclear-binding energies only, that is, we have neglected the Coulomb energies, which, according to Ref. 8, are 3.5 and 5.44 MeV for Be^8 and C^{12} nuclei, respectively. Accepting these rough estimates we find that the ground-state and the first-excited-state energies obtained from the Tabakin potential are, respectively, -6.76 and -1.16 MeV, which are not far from the respective experimental values -7.28 and $+0.32$ MeV. Besides the energies of the three-α system, the present study provides the wave functions $v(\rho)$ and $u(r)$ in the configuration space. From these we calculate the mean height $\langle\rho\rangle$ of an equilateral triangle, and the mean side length $\langle r\rangle$ of an equilateral triangle. If we imagine two circles, each circumscribing one of the two triangles, the numerical values of $\langle\rho\rangle$ and $\langle r\rangle$ turn out to predict the equal value of the radii. This close agreement of the two radii is interpreted as indicating a good accuracy of the numerical wave functions $u(r)$ and $v(\rho)$. From these wave functions the root-mean-square radius of the C^{12} nucleus can be calculated. The radius predicted by the Yamaguchi potential is 1.3 F and becomes 2.2 F if the radius 0.9 F of an α particle is added. The inclusion of the Coulomb potential will further increase the radius towards the experimental value 2.5 F. For the case of the Tabakin and Gaussian potentials the explicit inclusion of the size effect of an α particle leads to 2.9–3.0 F for the value of the radius. These potentials, however, do already include the size effect of an α particle in that the repulsive part of the potentials originates from the exclusion principle operating between two *composite* α particles. The literal inclusion of the size effect for these potentials amounts, therefore, to an "over-inclusion" of the effect. For these two potentials, thus, the only effect that should be explicitly included is the Coulomb repulsion.

These considerations, although qualitative, lead us to a consistent picture of the C^{12} nucleus in that its important physical observables, i.e., energies and size, can be deduced from the theory which employs the two-body data only. We have employed three separable approximations to the S-wave α-α interaction. The Tabakin type of potential seems to be preferable in view of the fact that it provides the best account of the two-body scattering as well as three-body observables. Our next objective is to apply the same method to Be^9 and Li^6 in which not only the α-α interaction, but also N-α and N-N interactions enter. Our goal is to correlate these two-body data with the physical observables of these two nuclei as well as C^{12} nucleus based on the method of wave function presented in this paper. The three-body model of Be^9 and Li^6 nuclei will be reported in a forthcoming paper.[15]

ACKNOWLEDGMENTS

We wish to thank Baylor University for the ready availability of Honeywell 1200 computer. One of us (C. C. H. L.) wishes to express sincere thanks to Dr. H. D. Schwetman for his financial support and encouragement.

[15] C. C. H. Leung and S. C. Park (to be published). A summary of the present paper as well as the result on Be^9 and Li^6 nuclei was reported by C. C. H. Leung and S. C. Park, Bull. Am. Phys. Soc. 14, 511 (1969).