Calculation of Two-Phonon Conductivity in Semiconductors*

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(Received 1 May 1969)

Photoconductivity produced in semiconductors by the two-photon excitation process is analyzed theoretically by considering surface and volume recombinations. The photoconductivity is shown to depend, in general, on the intensity of the excitation light, the thickness L of the crystal, the diffusion length $1/\lambda$ of the free carriers, and the surface-recombination velocity. When the excitation intensity is such that λ is much greater than the absorption constant of the two-photon process and $\lambda L \gg 1$, the expression for the photoconductivity reduces to a very simple form and is found to be relatively independent of the surface-recombination velocity. Calculations using various values of surface-recombination velocity and diffusion length are presented.

INTRODUCTION

WITH the advent of the laser, investigation of photoconductivity is no longer limited to low-intensity light excitation. A number of workers have investigated photoconductivity by using highly intense laser beams and have shown that one can thereby obtain valuable information about the impurity centers of crystals.[1–3] In these experiments the electrons and holes created by the excitation were mainly due to the one-photon absorption process, since the energy of the photons in the laser beam was greater than the band-gap energy of the crystal. Electrons and holes can also be created by photons of energy less than the band-gap energy—by two-photon or multiphoton excitation processes, as we know from perturbation theory of quantum mechanics—when the excitation is very intense. Recently, the two-photon excitation process has been observed in a number of experiments.[4,5] In fact, a few workers have even observed the production of stimulated emission (laser action) in a crystal by the two-photon excitation process when the crystal was exposed to an intense laser beam.[4,6]

This paper generalizes on a previous result reported recently by the author in a paper on the calculation of the two-photon conductivity in semiconductors.[7] Whereas the calculation described in the previous paper was subject to two restrictions—namely, $\lambda L \gg 1$ and $1 > \beta L$, where λ is the reciprocal of the diffusion length, β is the absorption constant for the two-photon excitation process, and L is the crystal thickness—the present paper puts no such restrictions on the calculation. Consequently, the results are more general; and more interesting conclusions can be drawn from them.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] M. Gershenzon and R. M. Mikulyak, Appl. Phys. Letters **8**, 245 (1966).
[2] K. Maeda and S. Iida, Appl. Phys. Letters **9**, 92 (1966).
[3] R. H. Bube and C.-T. Ho, J. Appl. Phys. **37**, 4132 (1966).
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[5] R. Braunstein and N. Ockman, Phys. Rev. **134**, A499 (1964).
[6] S. Wang and C. C. Chang, Appl. Phys. Letters **12**, 193 (1968).
[7] J. H. Yee, Appl. Phys. Letters **14**, 232 (1969).

For simplicity, we consider a photoconductor as shown in Fig. 1. When it is illuminated by a laser beam of intensity $\bar{I}_0$, electrons and holes are created in the conductor. Some of the free carriers will recombine inside the crystal with the volume lifetime τ, and some will diffuse to the surface and recombine at the surface with surface-recombination velocity V_s'.

In a one-photon absorption process, the rate of generation of free carriers by the incident light per unit volume and per unit time can be written as $\alpha I_0 e^{-\alpha x}$, where the absorption coefficient α is independent of intensity.[8] In the case where the free carriers are generated by the two-photon absorption process, the absorption coefficient is linearly dependent on light intensity[4,5]: $\alpha = 2\hbar\omega A_0 I$. A_0 is a constant dependent on the band structure of the crystal. For cubic crystals whose conduction and valence bands have the form shown in Fig. 2, A_0 can be written as follows[4]:

$$A_0 = \frac{2^{17/2}\pi q^4}{\epsilon c^2(\hbar\omega)^6}(2\hbar\omega - E_g)^{3/2} \times \left(\frac{|\langle\alpha P\rangle|^2}{m^2} m_{CV_1} + \frac{|\langle\alpha P\rangle|^2}{m^2} m_{CV_2}\right), \quad (1)$$

where $(m_{CV_i})^{-1} = m_C^{-1} + m_{V_i}^{-1}$, $i = 1, 2$; ϵ is the dielectric constant; $\hbar\omega$ is the energy of the laser light; and I is the intensity of the light in the crystal. According to Basov *et al.*,[4] I takes the following form:

$$I(x) = I_0/(1 + 2\hbar\omega A I_0 x). \quad (2)$$

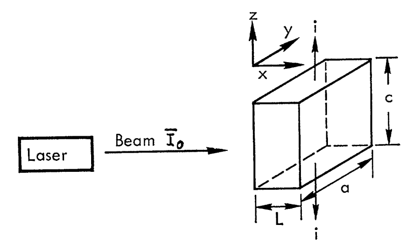

FIG. 1. Schematic of photoconductor excited by laser beam.

[8] H. B. DeVore, Phys. Rev. **102**, 86 (1956).

The generation rate in this type of crystal for the two-photon excitation process can then be written[4]

$$F(x)=\frac{2^{17/2}\pi e^4}{\epsilon c^2(\hbar\omega)^6}(2\hbar\omega-E_g)^{3/2}$$
$$\times\left(\frac{|\langle\alpha P\rangle|^2}{m}m_{CV_1}+\frac{|\langle\alpha P\rangle|^2}{m}m_{CV_2}\right)[I(x)]^2$$
$$=A_0I^2. \qquad (3)$$

In steady state, the concentration of the generated carriers can be obtained from the following differential equation[7,8]:

$$D_P\frac{\partial^2 P}{\partial x^2}-\frac{P}{\tau}=-\frac{A_0I_0^2}{(1+2\hbar\omega A_0I_0x)^2}=-F. \qquad (4)$$

The general solution $P(x)$ of this differential equation can be obtained by using the method of variation of parameters:

$$P(x)=Ae^{-\lambda x}+Be^{\lambda x}+\frac{1}{2D_P\lambda}\left(e^{-\lambda x}\int_0^x e^{\lambda x'}F(x')dx'\right.$$
$$\left.-e^{\lambda x}\int_0^x e^{-\lambda x'}F(x')dx'\right), \qquad (5)$$

where

$$\lambda=(D_P\tau)^{-1/2}.$$

Boundary conditions are determined by assuming that recombination occurs at each surface at a rate which may be represented by a recombination current $I_s=\pm V_s'P_s$, where P_s is the density of the carriers at the surface. Thus, at the first surface $I_s=V_s'P_s|_{x=0}$ and at the second surface $I_s=-V_s'P_s|_{x=L}$. Using these boundary conditions, we obtain the two unknowns in Eq. (5) for any crystal thickness and for any light intensity as follows[7]:

$$A=\frac{(\lambda-V_s)e^{-\lambda L}}{(\lambda+V_s)^2-(\lambda-V_s)^2e^{-2\lambda L}}$$
$$\times[-V_sf_1(L)-V_sf_2(L)-f_1'(L)-f_2'(L)], \qquad (6)$$

$$B=\frac{(\lambda+V_s)e^{-\lambda L}}{(\lambda+V_s)^2-(\lambda-V_s)^2e^{-2\lambda L}}$$
$$\times[-V_sf_1(L)-V_sf_2(L)-f_1'(L)-f'(L)], \qquad (7)$$

where

$$f_1=\frac{1}{2D\lambda}e^{-\lambda x}\int_0^x F(x')e^{\lambda x'}dx', \qquad (8)$$

$$f_2=\frac{-1}{2D\lambda}e^{\lambda x}\int_0^x F(x')e^{-\lambda x'}dx', \qquad (9)$$

$$V_s=V_s'/D_P.$$

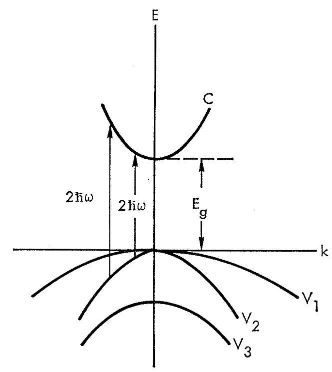

FIG. 2. Band structure of GaAs. C is conduction band; V_1, V_2, V_3 are valence bands; E_g is width of forbidden band.

In order to evaluate the two integrals in (8) and (9), we can expand either the numerator or the denominator of the integrands. But to expand the denominator, one has to restrict the value of βx to less than unity. Therefore, we expand $e^{\pm\lambda x}$. The result of the integration can be written as

$$f_1(x)=\frac{A_0I_0^2}{2D\lambda}e^{-\lambda x}\left\{\frac{e^{-\lambda/\beta}}{\beta}\left(\frac{\beta x}{(1+\beta x)}+\frac{\lambda}{\beta}\ln(1+\beta x)\right)\right.$$
$$+\frac{e^{-\lambda/\beta}}{\beta}\left[\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\beta}\right)^n\frac{(1+\beta x)^{n-1}}{(n-1)}\right.$$
$$\left.\left.-\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n-1}\right]\right\}, \qquad (10)$$

$$f_2=-\frac{A_0I_0^2}{2D\lambda}e^{\lambda x}\left\{\frac{e^{\lambda/\beta}}{\beta}\left(\frac{\beta x}{(1+\beta x)}-\frac{\lambda}{\beta}\ln(1+\beta x)\right)\right.$$
$$+\frac{e^{\lambda/\beta}}{\beta}\left[\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{\beta}\right)^n\frac{(1+\beta x)^{n-1}}{(n-1)}\right.$$
$$\left.\left.-\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{\beta}\right)^n\frac{1}{n-1}\right]\right\}, \qquad (11)$$

where $\beta=2\hbar\omega A_0I_0$. Using the definition for the conductivity,[9]

$$\Delta G=\int_0^L\int_0^C\frac{q(\mu_pP+\mu_eN)dy\,dx}{a}, \qquad (12)$$

and the neutrality condition for the carriers $N=P$, we obtain ΔG:

$$\Delta G=\frac{-\alpha(1-e^{-\lambda L})}{\lambda[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}$$
$$\times\{V_s[f_1(L)+f_2(L)]+f_1'(L)+f_2'(L)\}$$
$$+\alpha\int_0^L[f_1(x)+f_2(x)]dx, \qquad (13)$$

[9] Shyh Wang, *Solid-State Electronics* (McGraw-Hill Book Co., New York, 1966).

where

$$f_1'(L)+f_2'(L)=A_0I_0^2\left\{\frac{\lambda}{D\beta^2}\ln(1+\beta L)\sinh\left(\frac{\lambda}{\beta}(1+\beta L)\right)-\frac{1}{D\beta}\frac{\beta L}{(1+\beta L)}\cosh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\right.$$
$$+\frac{1}{D\beta}\sinh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\left[\sum_{\substack{n=3,5,\cdots\\ \text{(all odd)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{(1+\beta L)^{n-1}}{(n-1)}-\sum_{\substack{n=3,5,\cdots\\ \text{(all odd)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{(n-1)}\right]$$
$$\left.-\frac{1}{D}\frac{1}{\beta}\cosh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\left[\sum_{\substack{n=2,4,\cdots\\ \text{(all even)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{(1+\beta L)^{n-1}}{(n-1)}-\sum_{\substack{n=2,4,\cdots\\ \text{(all even)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{(n-1)}\right]\right\}, \quad (14)$$

$$\int_0^L[f_1(x)+f_2(x)]dx=\frac{A_0I_0^2}{D\lambda\beta^2}\ln(1+\beta L)\sinh\left(\frac{\lambda}{\beta}(1+\beta L)\right)+\frac{A_0I_0^2}{D\lambda^2}\frac{1}{\beta}\frac{\beta L}{(1+\beta L)}\left[1-\cosh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\right]$$
$$+\frac{A_0I_0^2}{D\lambda^2}\frac{1}{\beta}\sinh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\left[\sum_{\substack{n=3,5,\cdots\\ \text{(all odd)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{(1+\beta L)^{n-1}}{n!}\frac{1}{(n-1)}-\sum_{\substack{n=3,5,\cdots\\ \text{(all odd)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{(n-1)}\right]$$
$$-\frac{A_0I_0^2}{D\lambda^2}\frac{1}{\beta}\cosh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\left[\sum_{\substack{n=2,4,\cdots\\ \text{(all even)}}}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{(1+\beta L)^{n-1}}{(n-1)}-\sum_{\substack{n=2,4,\cdots\\ \text{(all even)}}}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{(n-1)}\right], \quad (15)$$

(Appendix A shows the evaluation of this integral), and

$$\alpha=(c/a)q(\mu_e+\mu_p).$$

After substituting into Eq. (13) the functions from (10), (11), and (14) and the integral from (15), we obtain the photoconductivity as follows:

$$\Delta G=\frac{\alpha A_0\tau I_0^2L}{(+\beta L)}-\frac{V_s\alpha A_0\tau I_0^2e^{-(\lambda/\beta)(1+\beta L)}}{\beta[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}\left[\frac{\beta L}{(1+\beta L)}+\frac{\lambda}{\beta}\ln(1+\beta L)\right.$$
$$\left.+\sum_{n=2}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{(1+\beta L)^{n-1}}{(n-1)}-\sum_{n=2}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{(n-1)}\right]-\frac{V_s\alpha A_0I_0^2e^{\lambda/\beta}}{D\lambda^2\beta[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}\left[\frac{\beta L}{(1+\beta L)}-\frac{\lambda}{\beta}\ln(1+\beta L)\right.$$
$$\left.+\sum_{n=2}^{\infty}\left(\frac{\lambda}{\beta}\right)^n(-1)^n\frac{1}{n!}\frac{(1+\beta L)^{n-1}}{(n-1)}\sum_{n=2}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{(-1)^n}{n!}\frac{1}{(n-1)}\right]. \quad (16)$$

It can be shown that after some simple algebraic manipulation, Eq. (16) can be written in the following compact form:

$$\Delta G=\frac{\alpha A_0\tau I_0^2L}{(1+\beta L)}-\frac{\alpha A_0I_0^2e^{-\lambda L}\tau V_s}{[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}\int_0^L\frac{e^{\lambda x}dx}{(1+\beta x)^2}-\frac{\alpha A_0I_0^2\tau V_s}{[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}\int_0^L\frac{e^{-\lambda x}dx}{(1+\beta x)^2}$$
$$=\frac{\alpha A_0I_0^2\tau L}{(1+\beta L)}-\frac{2\alpha A_0I_0^2\tau V_se^{-\lambda L/2}}{[(\lambda+V_s)-(\lambda-V_s)e^{-\lambda L}]}\int_0^L\frac{\cosh\lambda(x-\frac{1}{2}L)dx}{(1+\beta x)^2}. \quad (17)$$

Plots of the quantity $(\lambda^2/\alpha)[4D_pA_0(\hbar\omega)^2]\Delta G$ as a function of β are shown in Figs. 3 and 4, with crystal thickness, wavelength, and surface-recombination velocity as parameters. These graphs show that the photoconductivity induced by two-photon excitation is strongly dependent on not only the surface-recombination velocity and the diffusion length but also on the crystal thickness.

It is of interest to examine several limit cases.

(1) $\lambda L\gg 1$. This implies $(\lambda+V_s)\gg(\lambda-V_s)e^{-\lambda L}$. So we have

$$\Delta G\approx\frac{\alpha A_0I_0^2L}{(1+\beta L)}-\frac{2\alpha A_0\tau V_sI_0^2}{(\lambda+V_s)}e^{-\lambda L/2}\times\int_0^L\frac{\cosh\lambda(x-\frac{1}{2}L)dx}{(1+\beta x)^2}. \quad (18)$$

(2) $\lambda L \gg 1$ and $1 \gg \beta/\lambda$. For this case it is shown in Appendix B that ΔG reduces to the simple form

$$\Delta G \approx (c/a)q(\mu_e+\mu_p)\tau A_0 I_0^2 L/(1+\beta L). \qquad (19)$$

[This case also includes the one we considered in the previous paper[7]; Eq. (8) in the previous paper corresponds to Eq. (19) here.]

As shown in (11), β cm^{-1} depends on the excitation intensity, the frequency of the laser light, and the band structure of the crystal. In order to know when the inequalities given in (19) are satisfied, one must know something about these parameters. For a crystal whose diffusion length is very short ($\sim 10^{-4}$), like that of GaAs, these inequalities can easily be met when the level of the excitation is not too high. This can be shown by an order-of-magnitude calculation. Using the band parameters[10–13] for GaAs, the light intensity of 25 MW/cm^2, and a laser photon energy of 1.17 eV, we find that β is approximately 90. So β/λ is approximately 10^{-2}.

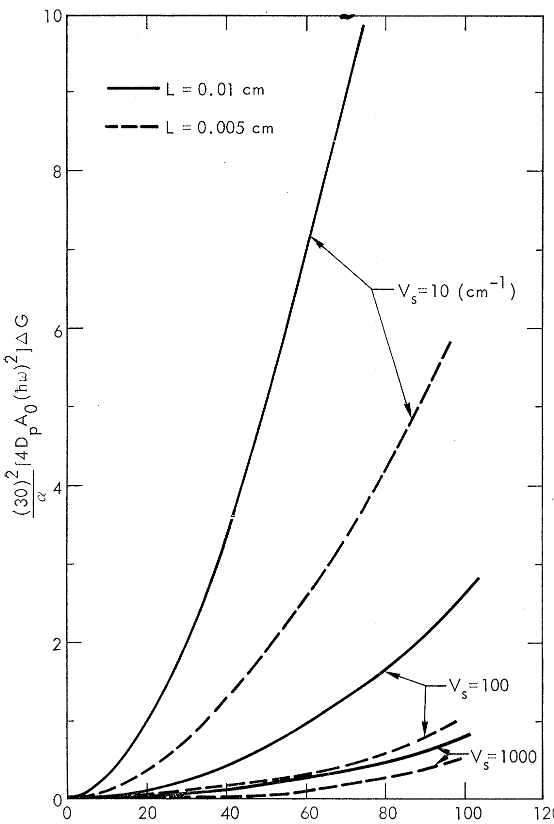

FIG. 3. Plot of the quantity $(\lambda^2/\alpha)[4D_pA_0(\hbar\omega)^2]\Delta G$ as a function of β, with crystal thickness and surface-recombination velocity as parameters, for $\lambda = 30$/cm.

[10] R. Philipp and H. Ehrenreich, Phys. Rev. **129**, 1550 (1963).
[11] J. Vilms, Stanford University, Technical Report No. 5107-1, 1964 (unpublished).
[12] R. J. Archer and D. Kerps, in *Gallium Arsenide* (Institute of Physics and the Physical Society, London, 1967), p. 103.
[13] L. V. Keldysh, Zh. Eksperim. i Teor. Fiz. **45**, 364 (1963) [English transl.: Soviet Phys.—JETP **18**, 253 (1964)].

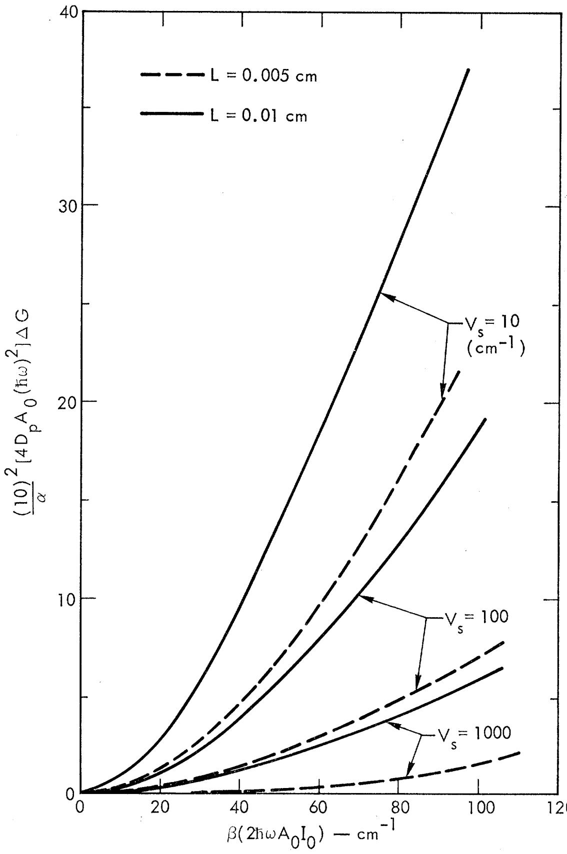

FIG. 4. Plot of the quantity $(\lambda^2/\alpha)[4D_pA_0(\hbar\omega)^2]\Delta G$ as a function of β, with crystal thickness and surface-recombination velocity as parameters, for $\lambda = 100$/cm.

Plots of the quantity $(\lambda^2/\alpha)[4D_pA_0(\hbar\omega)^2]\Delta G$ as a function of β cm^{-1} are shown in Fig. 5, with crystal thickness as a parameter. As shown in Appendix B, change in photoconductivity for this case is independent of the surface-recombination velocity.

To get an idea how ΔG changes with excitation intensity, we applied Eq. (19), using the value of A_0 obtained from Eq. (1), to the case of a GaAs crystal excited by a neodymium-glass laser with photon energy 1.17 eV. Figure 6 shows the resultant plots for three crystal thicknesses. The values of m_c, m_{V_1}, m_{V_2}, ϵ, E_g, τ, V_s, λ, and $|\langle\alpha p\rangle|^2/m^2$ used in obtaining Fig. 6 are those of other workers.[10–13]

(3) $\lambda L \gg 1$, $1 \gg \beta/\lambda$, and $\beta L \gg 1$. Then

$$\Delta G = (c/a)q[(\mu_p+\mu_e)\tau I_0/2\hbar\omega].$$

In this case the photoconductivity is linearly dependent on the excitation intensity.

CONCLUSION

The result of this derivation shows that in general the photoconductivity induced in a crystal by two-photon excitation is strongly dependent on the crystal's surface-recombination velocity and on its thickness (see Figs. 3

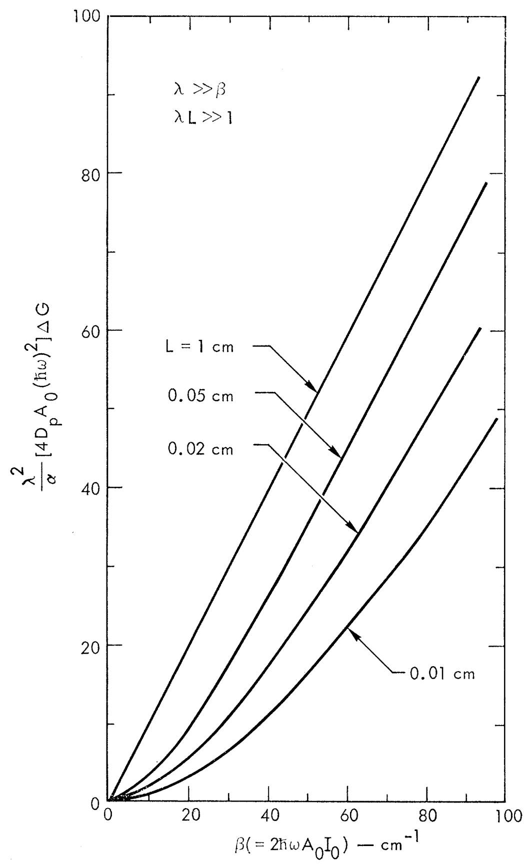

FIG. 5. Plot of the quantity $(\lambda^2/\alpha)[4D_pA_0(\hbar\omega)^2]\Delta G$ as a function of β, with crystal thickness as the parameter [case (2)].

and 4). But for the case of a crystal having a very short disffuion length for free carriers and excited by a laser beam whose intensity is not too high, the derivation shows that, to a good approximation, the photoconductivity does not depend on the surface-recombination velocity (see Fig. 5).

This conclusion seems reasonable because at such levels of light intensity the absorption coefficient for two-photon excitation is not nearly as high as that for ordinary one-photon excitation, and therefore the light can create electrons and holes throughout a large volume of the crystal. If the diffusion length is short, not many of these electron-hole pairs will be lost as a result of surface recombination. Accordingly, the conductivity depends strongly only on the intrinsic properties of the crystal and the light excitation intensity.

For ordinary one-photon excitation, by contrast, the absorption coefficient is very high (on the order of 10^4), so that most of the electrons and holes are created near the surface of the crystal. A significant fraction of these free carriers are lost as a result of surface recombination, and so one would expect the observed dependence of photoconductivity on the surface-recombination velocity.

Although the calculation given here is based on a steady-state excitation, one can still make a comparison between it and the photoconductivity from an experiment in which the crystal is excited by a pulse that is long enough so that at the end of the excitation a steady value of photoconductivity is attained.

Note added in proof. The I_0 in Figs. 3–5 is the light intensity just inside the surface of the crystal. This intensity (I_0) is equal to $\bar{I}_0(1-R)$ when and only when all the light entering the crystal is being absorbed. On the other hand, if only a fraction of the light is absorbed, then, by the conservation of energy, I_0 and the incident light intensity $\bar{I}_0$ are related in the following way:

$$I_0=\tfrac{1}{2}\bar{I}_0(1-R-T)\left[1+\left(1+\frac{4}{2\hbar\omega A_0L(1-R-T)\bar{I}_0}\right)^{1/2}\right].$$

Therefore, to plot ΔG as a function of incident light intensity, one must know the reflection and transmission coefficients R and T, respectively, for each crystal of different thickness. Once R, T, and A_0 are known, one can use the information in Figs. 3–5 to obtain the change

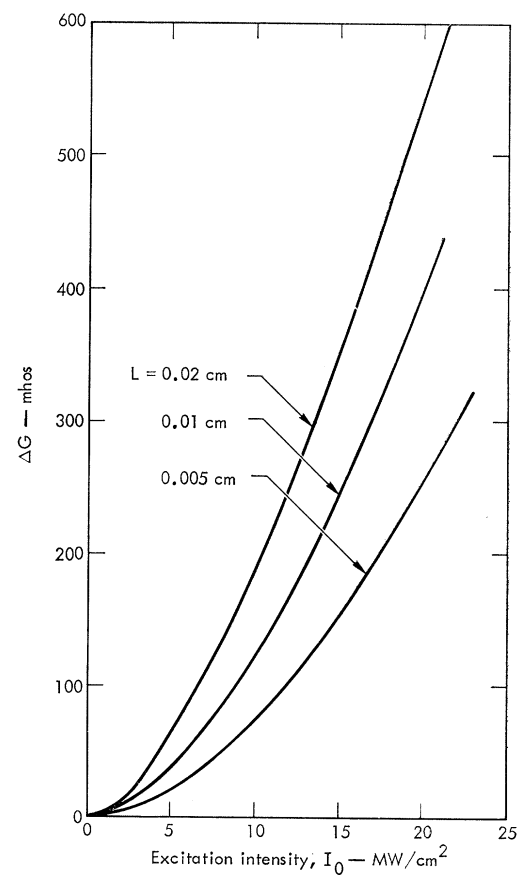

FIG. 6. Change in conductivity of a GaAs crystal due to the two-photon absorption process, as a function of excitation intensity.

of ΔG as a function of $\bar{I}_0$. Due to the lack of information on R and T under the strong laser excitation, the curves in Fig. 6 are obtained by assuming $I_0 \approx \bar{I}_0(1-R)$.

ACKNOWLEDGMENTS

The author is indebted to Dr. R. M. Bevensee and Dr. D. Dawson for their help in computing Eq. (17), and to Dr. E. Fischer-Colbrie for helpful discussions.

APPENDIX A: EVALUATION OF EQ. (15) INTEGRAL

After substituting the functions $f_1(x)$ and $f_2(x)$, given in (10) and (11), into the integral in (15), it can be shown that the result after integration can be written:

$$\int_0^L [f_1(x)+f_2(x)]dx = \frac{A_0 I_0^2}{D\beta\lambda^2}\left[\cosh\frac{\lambda}{\beta} - \cosh\left(\frac{\lambda}{\beta}(1+\beta L)\right)\right] + \frac{A_0 I_0^2}{D\lambda\beta^2}\ln(1+\beta L)\sinh\left(\frac{\lambda}{\beta}(1+\beta L)\right)$$

$$+\frac{e^{-\lambda/\beta}(e^{-\lambda L}-1)}{2D\lambda^2\beta}\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{(n-1)} + \frac{e^{\lambda/\beta}(e^{\lambda L}-1)}{2D\lambda^2\beta}\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{\beta}\right)^n\frac{1}{(n-1)} + \frac{1}{2D\lambda}\frac{e^{-\lambda/\beta}}{\beta}\sum_{n=2}^{\infty}\left(\frac{1}{n!}\right)\left(\frac{\lambda}{\beta}\right)^n\frac{1}{(n-1)}$$

$$\times\left(\frac{1}{\beta}e^{-\lambda L}\sum_{r=0}^{n-1}\frac{(-1)^r(n-1)!(1+L\beta)^{n-1-r}}{[(n-1)-r]!(-\lambda/\beta)^{r+1}} - \frac{1}{\beta}\sum_{r=0}^{n-1}\frac{(-1)^r(n-1)!}{[(n-1)-r]!(-\lambda/\beta)^{r+1}}\right) - \frac{1}{2D\lambda}\frac{e^{\lambda/\beta}}{\beta}\sum_{n=2}^{\infty}\left(\frac{1}{n!}\right)\left(\frac{-\lambda}{\beta}\right)^n\frac{1}{(n-1)}$$

$$\times\left(\frac{1}{\beta}e^{\lambda L}\sum_{r=0}^{n-1}\frac{(-1)^r(n-1)!(1+\beta L)^{n-1-r}}{[(n-1)-r]!(\lambda/\beta)^{r+1}} - \frac{1}{\beta}\sum_{r=0}^{n-1}\frac{(-1)^r(n-1)!}{[(n-1)-r]!(\lambda/\beta)^{r+1}}\right). \quad (20)$$

Now if for each n, we sum from $r=0$ to $r=n-1$ and add the results of all the n terms, the last two series given in (20)—those led off by the factor $1/2D\lambda$—reduce to

$$-A_0 I_0^2\left\{\frac{e^{-(\lambda/\beta)(1+\beta L)}}{2D\lambda\beta^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^n(1+\beta L)^n\frac{1}{n!}\frac{1}{n}\right] - \frac{e^{-\lambda/\beta}}{2D\lambda\beta^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{n}\right]\right\}$$

$$-A_0 I_0^2\left\{\frac{e^{(\lambda/\beta)(1+\beta L)}}{2D\lambda\beta^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^n(1+\beta L)^n\frac{1}{n!}\frac{(-1)^n}{n}\right] - \frac{e^{\lambda/\beta}}{2D\lambda\beta^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{-\lambda}{\beta}\right)^n\frac{1}{n!}\frac{1}{n}\right]\right\}. \quad (21)$$

In obtaining (21), the following identity was used:

$$\sum_{n=k}^{\infty}\frac{1}{n}\frac{1}{(n-1)} = \frac{1}{k-1} \quad (k=2, 3, 4, \ldots, n).$$

APPENDIX B: SIMPLIFICATION OF EQ. (18) FOR $L\lambda \gg 1$ AND $\beta/\lambda \ll 1$

To show that Eq. (18) is true for $L\lambda \gg 1$ and $\beta/\lambda \ll 1$, we note that

$$\int_0^L e^{\pm\lambda x}dx \geq \int_0^L \frac{e^{\pm\lambda x}dx}{(1+\beta x)^2} \quad \text{for all } \beta.$$

Therefore,

$$e^{-\lambda L}\int_0^L e^{\lambda x}dx + \int_0^L e^{-\lambda x}dx = \frac{2}{\lambda} \geq e^{-\lambda L}\int_0^L \frac{e^{\lambda x}dx}{(1+\beta x)^2} + \int_0^L \frac{e^{-\lambda x}dx}{(1+\beta x)^2}.$$

Now if Eq. (18) is true, we must have $L/(1+\beta L) \gg 2/\lambda$. It follows that

$$\lambda L \gg 2(1+\beta L),$$

which implies that $\lambda L \gg 1$ and $\lambda \gg \beta$. Q.E.D.