

operations: (1) If a vector is off the surface and off the contour, we must replace it by one which is closer to the surface and contour; and (2) having a vector that is on the surface and on the contour, we must generate a new vector which is also on the surface and contour but which had advanced by a known angle.

We define the following four unit vectors:

$$\begin{aligned}\hat{n}_1 &= \nabla E / |\nabla E|, & \hat{n}_2 &= \mathbf{k} \times \hat{n}_1 / |\mathbf{k} \times \hat{n}_1|, \\ \hat{n}_3 &= \hat{n}_1 \times \hat{n}_2, & \hat{n}_4 &= \mathbf{k} / |\mathbf{k}|.\end{aligned}$$

Consider first process (1). If we have a vector  $\mathbf{k}$  which is off the surface and contour, we want to replace it by a vector  $\mathbf{k}' = \mathbf{k} + \delta\mathbf{k}$  which is closer to the surface and contour. We require  $\delta\mathbf{k}$  to lie in the plane formed by the vectors  $\hat{n}_1$  and  $\hat{n}_3$ , i.e.,

$$\delta\mathbf{k} = a\hat{n}_1 + b\hat{n}_3.$$

The values of  $a$  and  $b$  are found using Eq. (6) together with the condition  $k_0^2 = |\mathbf{k} + \delta\mathbf{k}|^2 \cong k^2 + 2\mathbf{k} \cdot \delta\mathbf{k}$ , where  $k_0$  is the radius of the contour under consideration. The

result is

$$a = \frac{E_F - E(\mathbf{k})}{|\nabla E|}$$

and

$$b = \frac{1}{\hat{n}_3 \cdot \mathbf{k}} \left( \frac{k_0^2 - k^2}{2} - \frac{E_F - E(\mathbf{k})}{|\nabla E|} \hat{n}_1 \cdot \mathbf{k} \right).$$

We repeat this process until  $\delta k/k$  is smaller than some specified number (0.001, in our case). Now consider process (2). Having a vector  $\mathbf{k}$  which is on the surface and contour, we wish to replace it with a vector  $\mathbf{k}' = \mathbf{k} + \delta\mathbf{k}$  which is quite close to the surface and contour but which makes an angle  $\theta$  with  $\mathbf{k}$ . We require  $\delta\mathbf{k}$  to lie in the plane formed by the vectors  $\hat{n}_2$  and  $\hat{n}_4$ . A short calculation shows that

$$\delta\mathbf{k} = 2k \sin \frac{1}{2}\theta (\hat{n}_2 \cos \frac{1}{2}\theta - \hat{n}_4 \sin \frac{1}{2}\theta).$$

By alternatively applying operations (1) and (2), we may advance around a contour of constant radius. Figure 10 shows the contours of constant radius for the  $\Gamma$ -centered surface of platinum.

## Point-Defect Interactions in Elastic Materials of Grade 2

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The main result of this paper is that two centers of dilatation in an infinite, isotropic domain do possess an interaction energy according to the linearized version of Toupin's strain-gradient theory of elasticity—in contrast to the classical theory of elasticity, where the interaction energy was shown to be zero. The center of dilatation has been adopted by many investigators as a representation of point defects in crystalline solids, so the above result may have significance in evaluating the long-range interactions which may occur between point defects.

### I. INTRODUCTION

THE classical theory of elasticity has been used by a number of authors to study the behavior of point defects and dislocations in crystalline solids. While this approach admittedly may not be appropriate in a number of cases to which it has been applied, it has yielded several useful results. In order to simplify the analysis the crystalline solid is assumed to be isotropic and infinite in extent. Within this general framework the present paper applies a recently developed, non-classical theory of elasticity to the problem of the interaction of two centers of dilatation in an infinite, isotropic, elastic medium. In contrast to the statement which is frequently made, based on the work of Bitter,<sup>1</sup> that the interaction energy of two centers of dilatation in an infinite, isotropic, elastic medium is zero, an

explicit expression for the interaction energy will be given. This seems to be more in line with the results obtained from digital-computer computations based on lattice models. The classical result has been especially noted and criticized by Hardy and Bullough.<sup>2</sup>

A number of investigators have considered generalizations of the fundamental model of a continuum which forms the basis of the classical theory of elasticity. The generalized continuum models have the common feature that they take into account more of the details around a material point. There is evidence that the generalized continuum models may incorporate some of the lattice characteristics of crystalline solids in the continuum approach. However, most of the nonclassical theories are in their early stages of development, so it is still somewhat premature to judge the potential value of these new models.

<sup>1</sup> F. Bitter, *Phys. Rev.* **35**, 1527 (1931).

<sup>2</sup> J. R. Hardy and R. Bullough, *Phil. Mag.* **15**, 237 (1967).

The nonclassical theory used here will be the linearized form of Toupin's strain-gradient theory.<sup>3</sup> This theory is based on a strain-energy density which includes the components of the first gradient of the strain as well as the strain components. Toupin<sup>4</sup> indicates that the configuration of the points in the neighborhood of a particular material point is more accurately described as higher-order gradients of the strain are included in the strain-energy density. He refers to a material governed by a strain-energy density which is a function of the  $(N-1)$ -order strain gradients as an elastic material of grade  $N$ . Thus, the strain-gradient theory used here is the theory corresponding to elastic materials of grade 2.

The interaction problem for two centers of dilatation (which can be thought of as equivalent to two spherical cavities subject to an internal pressure  $p$ ) can be approached from the viewpoint of applied mechanics in terms of stress concentration factors or from the viewpoint of chemistry and physics in terms of the interaction energy. From the standpoint of applied mechanics, it is necessary to provide an explicit representation of the stress field. This problem has been investigated by Sternberg and Sadowsky<sup>5</sup> using the classical theory of elasticity. They showed that the tangential stress components at the surface of one sphere were changed by as much as 14% due to the presence of the second sphere when the spheres were separated by a distance of four radii. At a separation distance of eight radii, the two spheres appear to have negligible influence on each other. The computations were based on only one value of Poisson's ratio:  $\nu = 0.25$ .

The physicist, in dealing with imperfections in solids, is primarily interested in various energies (the binding energy, migration energy, interaction energy, etc.) which may be associated with a particular defect or distribution of defects. The energy approach is based on the assumption that the stress and strain fields arising from any number of individual defects can be superimposed (which is true for the linear theory of elasticity); however, while this method provides the correct value for the strain energy, the boundary conditions are not satisfied when more than one defect is considered. Thus, the precise stress distribution for several defects cannot be readily determined by this method. The solution given by Sternberg and Sadowsky indicates the modification that occurs in the stresses at the boundary of one pressurized sphere due to the presence of a second pressurized sphere. If the total strain energy for this exact solution could be evaluated, it should be equal to the sum of the self-energy for each sphere.

The center of dilatation has been used by several authors in a variety of contexts. It has provided a

representation of a substitutional atom, an interstitial atom, and a vacancy in an otherwise perfect lattice. The interaction energy for two centers of dilatation is defined as the difference between the total strain energy and the sum of the self-energies for the isolated dilatation centers. The usual criterion for the existence of an interaction energy is that the cubical dilatation have a nonzero value. Since a center of dilatation according to the classical theory of elasticity produces a pure shear stress, Eshelby<sup>6</sup> showed that an interaction energy can be found only for finite bodies where an interaction occurs via the image displacements which must be applied to the surface of the body in order to satisfy the boundary conditions. The interaction energy for substitutional atoms has been based on the assumption that the elastic constants of the foreign atom (if one considers this to be a valid representation) are different than those of the host material. In an infinite domain the interaction energy (based on the inclusion problem from the classical theory of elasticity) was shown to vary as  $R^{-6}$ , where  $R$  is the separation distance between the two defects.<sup>7</sup> When the elastic constants of the foreign atom and the host material are identical, the interaction energy vanishes in agreement with our earlier remarks.

In Sec. II the stresses and displacements for a spherical cavity subject to a uniform internal pressure  $p$  in an infinite domain are evaluated according to the strain-gradient theory of elasticity. We observe that the cubical dilatation is not zero, so we proceed to define and evaluate the interaction energy for two centers of dilatation. This involves lengthy algebraic manipulations. An outline of the procedure is given in the Appendix. Fortunately, the solution can be expressed in terms of two dimensionless parameters involving the five additional elastic constants introduced in the strain-gradient theory. The experimental determination of the new elastic constants is not yet available for any material, so one can only speculate on the appropriate values to assign to these parameters. Some restrictions on the magnitude of the two parameters are imposed by requiring that the strain-energy density function be positive definite. Numerical values of the displacements and interaction energy are given for selected values of the material constants.

## II. ELASTIC MATERIALS OF GRADE 2

The classical theory of elasticity is based on a strain-energy density function  $W$  of the general form

$$W = f^{(0)}(\xi_{kl}), \quad (1)$$

where  $\xi_{kl} = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ ;  $u_k$  is the displacement field and  $\partial_k(\ )$  denotes differentiation with respect to the spatial variables. Toupin<sup>3</sup> considers a strain-energy

<sup>3</sup> R. A. Toupin, Arch. Ration. Mech. Anal. **11**, 385 (1962).

<sup>4</sup> R. A. Toupin, Arch. Ration. Mech. Anal. **17**, 85 (1964).

<sup>5</sup> E. Sternberg and M. A. Sadowsky, J. Appl. Mech. **19**, 19 (1952).

<sup>6</sup> J. D. Eshelby, J. Appl. Phys. **25**, 255 (1954).

<sup>7</sup> I. M. Lifshits and L. V. Tanatarov, Fiz. Metal. i Metalloved. **12**, 331 (1961).

density function which depends on the first and second gradients of the deformation; Mindlin<sup>8</sup> displayed the linearized form of Toupin's strain-gradient theory which is obtained from a strain-energy density function of the form

$$W = f(\xi_{kl}, \kappa_{klm}), \quad (2)$$

where  $\kappa_{klm} = \partial_k \partial_l u_m$ . Materials governed by a strain-energy density function of the above form will be referred to as elastic materials of grade 2.

### A. Governing Equations

Recently Bleustein<sup>9</sup> rederived the linear form of Toupin's equations defining a different set of forces to be present on the surface  $S$  and within the interior of a body with volume  $V$ . We shall use the equations developed by Bleustein in our subsequent work.

In Cartesian tensor notation

$$\partial_j(\tau_{jk} - \partial_i \mu_{ijk}) + f_k - \partial_j \Phi_{jk} = 0, \quad \text{in } V \quad (3)$$

$$n_j(\tau_{jk} - \partial_i \mu_{ijk}) - D_j(n_i \mu_{ijk}) = t_k + n_j \Phi_{jk} - D_j T_{jk}, \quad \text{on } S \quad (4)$$

$$n_j n_i \mu_{ijk} = n_j T_{jk}, \quad \text{on } S \quad (5)$$

where  $n_i$  is the unit outward normal to the surface  $S$  and  $D_j(\ )$  is the surface-gradient operator  $(\delta_{jk} - n_j n_k) \partial_k(\ )$ , where  $\delta_{jk}$  is the Kronecker delta. The vectors  $f_k$  and  $t_k$  are the body force per unit volume and the surface force per unit area, respectively, as in the classical theory of elasticity, while  $\Phi_{jk}$  represents the body double force per unit volume, and  $T_{jk}$  is the surface double force per unit area. The stresses corresponding to the kinematic variables  $\xi_{kl}$  and  $\kappa_{klm}$  are

$$\tau_{kl} \equiv \frac{\partial W}{\partial \xi_{kl}} = \tau_{lk} \quad (\text{force-stress components}), \quad (6)$$

$$\mu_{klm} \equiv \frac{\partial W}{\partial \kappa_{klm}} = \mu_{lkm} \quad (\text{double-stress components}). \quad (7)$$

We see that third-order stress components arise due to the inclusion of the second gradients of the displacements in the strain-energy density. Physically, the 18 components of  $\mu_{klm}$  represent double forces per unit area. In analogy with the classical theory of elasticity the first subscript indicates the normal to the element of surface across which the component acts; the second subscript gives the direction of the moment arm for the double forces acting in the direction indicated by the third subscript.

If we restrict our attention to linear constitutive relations, we can assume the strain-energy density is a homogeneous, quadratic function of the 24 kinematic variables  $\xi_{kl}$  ( $= \xi_{lk}$ ),  $\kappa_{klm}$  ( $= \kappa_{lkm}$ ). For a centrosymmetric, isotropic material,

$$W = \frac{1}{2} \lambda \xi_{ii} \xi_{jj} + \mu \xi_{ij} \xi_{ij} + a_1 \kappa_{iik} \kappa_{kjj} + a_2 \kappa_{ijj} \kappa_{ikk} + a_3 \kappa_{iik} \kappa_{jjk} + a_4 \kappa_{ijk} \kappa_{ijk} + a_5 \kappa_{ijk} \kappa_{kji}, \quad (8)$$

<sup>8</sup> R. D. Mindlin, Arch. Ration. Mech. Anal. **16**, 51 (1964).

<sup>9</sup> J. L. Bleustein, Int. J. Solids Structures **3**, 1053 (1967).

where  $\lambda$  and  $\mu$  are the Lamé constants and  $a_1, \dots, a_5$  are five new material constants having the dimension of force.

### B. Displacement Field for a Center of Dilatation

The displacement fields for a center of dilatation and a spherical cavity subject to a uniform internal pressure in an infinite domain are of the same general form in the classical theory of elasticity; therefore, the problem described below will be interchangeable with what is usually referred to as a center of dilatation. The strength of the center of dilatation can be represented in terms of the pressure acting on a spherical surface.<sup>10</sup>

Consider a spherical cavity of radius  $R_0$  subject to a uniform internal pressure  $p$  in an infinite domain. Using the center of the spherical cavity as an origin, we introduce the spherical coordinates  $r, \theta$ , and  $\varphi$  with unit vectors  $\mathbf{e}_r, \mathbf{e}_\theta$ , and  $\mathbf{e}_\varphi$ . Since the present problem involves only radial displacements, we can write

$$\mathbf{u} = u(r) \mathbf{e}_r. \quad (9)$$

For this case the nonvanishing components of the strain tensor and the strain-gradient tensor are

$$\xi_{rr} = u', \quad \xi_{\theta\theta} = \xi_{\varphi\varphi} = u/r \quad (10)$$

and

$$\kappa_{rrr} = u''',$$

$$\kappa_{r\theta\theta} = \kappa_{\theta r\theta} = \kappa_{\theta\theta r} = \kappa_{r\varphi\varphi} = \kappa_{\varphi r\varphi} = \kappa_{\varphi\varphi r} = (u/r)',$$

where  $( )'$  denotes differentiation with respect to  $r$ .

From Eq. (8) in conjunction with Eqs. (6) and (7) we find for a centrosymmetric isotropic material

$$\tau_{rr} = (\lambda + 2\mu)u' + (2\lambda u/r), \quad (11)$$

$$\tau_{\theta\theta} = \tau_{\varphi\varphi} = \lambda u' + 2(\lambda + \mu)u/r$$

and

$$\begin{aligned} \mu_{rrr} &= 2(a_1 + a_2 + a_3 + a_4 + a_5)u'' + 4(a_1 + a_2 + a_3)(u/r)', \\ \mu_{r\theta\theta} &= \mu_{\theta r\theta} = \mu_{r\varphi\varphi} = \mu_{\varphi r\varphi} = \frac{1}{2}(a_1 + 2a_2)[u'' + 2(u/r)'] \\ &\quad + 2(a_4 + a_5)(u/r)', \end{aligned} \quad (12)$$

$$\mu_{\theta\theta r} = \mu_{\varphi\varphi r} = (a_1 + 2a_3)[u'' + 2(u/r)'] + 2(a_4 + a_5)(u/r)'. \quad (13)$$

These are the only nonvanishing components of the force-stress and double-stress tensors.

Equations (3)–(5) can be rewritten in dyadic notation and then transformed to spherical coordinates. Using Eqs. (11) and (12), we obtain a single displacement equation of equilibrium and two boundary conditions expressed in terms of the displacements. Setting the body force, body double force, and surface double force equal to zero, we have for  $\mathbf{t} = -p\mathbf{e}_r$ ,

$$\frac{d}{dr} \left[ u''' + \frac{4}{r} u'' - \frac{1}{r^2} \left( u' + \frac{2}{r} u \right) \right] = 0, \quad (13)$$

<sup>10</sup> A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, Inc., New York, 1944), 4th ed.

$$\begin{aligned}
& 2(a_1+a_2+a_3+a_4+a_5) \left[ u''' - \frac{6}{r} \left( \frac{u}{r} \right)' \right] \\
& - 4(a_1+a_2+a_3) \frac{1}{r} \left( \frac{u}{r} \right)' - (\lambda+2\mu)u' - 2\lambda \frac{u}{r} = p \\
& \text{on } r=R_0, \quad (14) \\
& \mu_{rrr} = 2(a_1+a_2+a_3+a_4+a_5)u'' + 4(a_1+a_2+a_3)(u/r)' = 0 \\
& \text{on } r=R_0, \quad (15)
\end{aligned}$$

where

$$l^2 = 2(a_1+a_2+a_3+a_4+a_5)/(\lambda+2\mu).$$

The quantity  $l$  is often referred to as the characteristic length of the material.

$$u = \frac{pR_0^3}{4\mu r^2} \left[ \frac{\gamma^2(1+\gamma) + 4\alpha\gamma^2 + 12\alpha(1+\gamma) - 12\alpha(1+\gamma r/R_0)e^{\gamma(1-r/R_0)}}{\gamma^2(1+\gamma) + 4\alpha\gamma^2 + 12\alpha(1+\gamma) + 6\alpha(1+\gamma)(1+\nu)/(1-2\nu)} \right], \quad (17)$$

where

$$2\alpha = (a_4+a_5)/(a_1+a_2+a_3+a_4+a_5)$$

and  $\gamma = R_0/l$ .

We see that the displacement field found from the classical theory of elasticity

$$u_c = pR_0^3/4\mu r^2 \quad (18)$$

is modified by including strain-gradient effects. Large values of  $\gamma$  indicate that the radius of the spherical cavity is large in relation to the characteristic length  $l$  of the material. As  $\gamma$  approaches infinity, the classical displacement field is obtained from Eq. (17).

The solution for the displacement field in Eq. (17) agrees with that given by Cook and Weitsman<sup>11</sup> for a spherical cavity acted upon by a hydrostatic tension at infinity. The theory they employed, however, is based on a strain-energy density function of the form

$$W = \tilde{f}(\xi_{kl}, \tilde{\kappa}_{kl}, \tilde{\kappa}_{klm}), \quad (19)$$

where  $\tilde{\kappa}_{kl} = \frac{1}{2}\epsilon_{lmn}u_{n,mk}$  are the gradients of the rotation vector, and

$$\tilde{\kappa}_{klm} = \frac{1}{3}(\kappa_{klm} + \kappa_{lmk} + \kappa_{mkl})$$

are the components of the symmetric part of  $\kappa_{klm}$ ;  $\epsilon_{lmn}$  is the permutation symbol. A detailed discussion of the relation between various decompositions of the strain-energy density function can be found in a paper by Mindlin and Eshel.<sup>12</sup>

The couple-stress theory of elasticity is a special case of the strain-gradient theories mentioned above. It is based on a strain-energy density function of the form

$$W = \tilde{f}(\xi_{kl}, \tilde{\kappa}_{kl}). \quad (20)$$

<sup>11</sup> T. S. Cook and Y. Weitsman, Int. J. Solids Structures 2, 333 (1966).

<sup>12</sup> R. D. Mindlin and N. N. Eshel, Int. J. Solids Structures 4, 109 (1968).

The general solution to Eq. (13) is

$$\begin{aligned}
u = & A \frac{R_0^3}{r^2} + B \frac{R_0^3}{r^2} (1+r/l)e^{-r/l} \\
& - C \frac{R_0^3}{r^2} (1-r/l)e^{r/l} + Dr. \quad (16)
\end{aligned}$$

In addition to the boundary conditions on  $r=R_0$  we have the uniformity condition for the infinite domain:

$$u \rightarrow 0 \text{ as } r \rightarrow \infty.$$

This implies that  $C=D=0$ . The integration constants  $A$  and  $B$  can now be found by evaluating the boundary conditions on  $r=R_0$ ; we find

The couple stresses are related to the antisymmetric portion of  $\mu_{klm}$  and thereby are combinations of only double forces with moment. Some of the unusual effects predicted by the couple-stress theory have motivated Toupin to indicate a preference for the full strain-gradient theory.<sup>4</sup>

As indicated in the Introduction, an interaction energy will exist if the cubical dilatation is not zero. The cubical dilatation is zero in both the classical and couple-stress theories of elasticity for a center of dilatation. It is only when the strain-gradient theory is introduced that the cubical dilatation for a center of dilatation will not vanish.

### C. Interaction Energy for Two Centers of Dilatation

In this section we shall consider the interaction energy for two centers of dilatation in an infinite domain from the viewpoint of the theory of elastic materials of grade 2.

Having the solution for a single center of dilatation, we can use the superposition principle for linear elasticity to evaluate the interaction energy for two centers of dilatation. The strain energy is defined by

$$E = \frac{1}{2} \int_V \tau_{kl} \xi_{kl} dV + \frac{1}{2} \int_V \mu_{klm} \kappa_{klm} dV.$$

We now consider two dilatation centers which we label  $A$  and  $B$ ; then

$$\begin{aligned}
E = & \frac{1}{2} \int_V (\tau_{kl}^A \xi_{kl}^A + \mu_{klm}^A \kappa_{klm}^A) dV \\
& + \frac{1}{2} \int_V (\tau_{kl}^B \xi_{kl}^B + \mu_{klm}^B \kappa_{klm}^B) dV \\
& + \int_V (\tau_{kl}^A \xi_{kl}^B + \mu_{klm}^A \kappa_{klm}^B) dV, \quad (21)
\end{aligned}$$

TABLE I. Nondimensional displacements.

$\gamma$	$r/R_0$	$\alpha=10^0$	$\alpha=10^{-1}$	$\alpha=10^{-2}$	$\alpha=10^{-3}$	$\alpha=10^{-4}$
1	1	0.0769	0.2500	0.7391	0.9653	0.9964
	2	0.2148	0.3621	0.7781	0.9705	0.9970
	4	0.3463	0.4689	0.8153	0.9754	0.9975
	10	0.3844	0.4998	0.8260	0.9769	0.9976
5	1	0.5365	0.8811	0.9859	0.9986	0.99986
	2	0.6891	0.9202	0.9905	0.9990	0.99990
	4	0.6910	0.9207	0.9906	0.9990	0.99990
	10	0.6910	0.9207	0.9906	0.9990	0.99990
10	1	0.7911	0.9664	0.9964	0.9996	0.99996
	2	0.8608	0.9776	0.9976	0.9998	0.99998
	4	0.8608	0.9776	0.9976	0.9998	0.99998
	10	0.8608	0.9776	0.9976	0.9998	0.99998

where the symmetry relations  $\tau_{kl}^A \xi_{kl}^B = \tau_{kl}^B \xi_{kl}^A$  and  $\mu_{klm}^A \kappa_{klm}^B = \mu_{klm}^B \kappa_{klm}^A$  have been used. The interaction energy is defined as the difference between the total energy and the sum of the self-energy for each dilatation center, or

$$E_{\text{int}} = \int_V (\tau_{kl}^A \xi_{kl}^B + \mu_{klm}^A \kappa_{klm}^B) dV. \quad (22)$$

This integration extends over an infinite domain. The interaction energy is more easily evaluated when the volume integral is converted to a surface integral; therefore consider

$$\tau_{kl}^A \xi_{kl}^B + \mu_{klm}^A \kappa_{klm}^B = \partial_l (\tau_{lm}^A u_m^B) + \partial_l \partial_k (\mu_{klm}^A u_m^B) - 2 \partial_l (\partial_k \mu_{klm}^A u_m^B) - (\partial_l \tau_{lm}^A - \partial_l \partial_k \mu_{klm}^A) u_m^B,$$

where the last term on the right-hand side is zero in accordance with the equilibrium equations (3). Hence,

$$\begin{aligned} E_{\text{int}} &= \int_S n_l [\tau_{lm}^A u_m^B + \partial_k (\mu_{klm}^A u_m^B) - 2 (\partial_k \mu_{klm}^A u_m^B)] dS \\ &= \int_S n_l (\tau_{lm}^A u_m^B - \partial_k \mu_{klm}^A u_m^B + \mu_{klm}^A \partial_k u_m^B) dS, \end{aligned} \quad (23)$$

where  $n_l$  is the unit outward normal to the surface  $S$ .

We will divide the interaction energy into two parts,

$$E_{\text{int}} = E_{\text{int}}' + E_{\text{int}}'', \quad (24)$$

where  $E_{\text{int}}'$  is the interaction energy which can be attributed to the force-stress components, while  $E_{\text{int}}''$  is the interaction energy due to the double-stress components.

Bitter<sup>1</sup> outlined a procedure for evaluating the interaction energy according to the classical theory of elasticity; he found that the interaction energy was zero. We shall follow Bitter's approach to evaluate the interaction energy in Eq. (23). Consider two spherical cavities,  $A$  and  $B$ , of radius  $R_0$  separated by a distance  $R$  along the line between their centers. The integration in Eq. (23) can be carried out over the surface of each

sphere; however, we observe that

$$\begin{aligned} E_{\text{int}}' &= \frac{1}{2} \int_S n_l (\tau_{lm}^A u_m^B + \tau_{lm}^B u_m^A) dS \\ &= \frac{1}{2} \int_{S_A} n_l^A (\tau_{lm}^A u_m^B + \tau_{lm}^B u_m^A) dS \\ &\quad + \frac{1}{2} \int_{S_B} n_l^B (\tau_{lm}^A u_m^B + \tau_{lm}^B u_m^A) dS \\ &= \int_{S_A} n_l^A (\tau_{lm}^A u_m^B + \tau_{lm}^B u_m^A) dS, \end{aligned} \quad (25)$$

where  $n_l^A$  is the unit outward normal to the surface  $S_A$ . Similarly,

$$E_{\text{int}}'' = \int_{S_A} n_l^A (-\partial_k \mu_{klm}^A u_m^B - \partial_k \mu_{klm}^B u_m^A + \mu_{klm}^A \partial_k u_m^B + \mu_{klm}^B \partial_k u_m^A) dS. \quad (26)$$

The evaluation of the integrals in Eqs. (25) and (26) involves rather lengthy algebraic manipulations. An outline of the procedure used is given in the Appendix; the final result is

$$\begin{aligned} \bar{E}_{\text{int}}' &= \frac{\alpha \gamma^2}{C^2} \left[ (1 + \gamma + 4\alpha) \left( \cosh \gamma - \frac{1}{\gamma} \sinh \gamma \right) - \frac{1 - \nu}{1 - 2\nu} \right. \\ &\quad \times \{ 3\alpha \cosh \gamma + [\frac{1}{4}\gamma^2(1 + \gamma) + \alpha\gamma^2 - 3\alpha] \} \\ &\quad \left. \times \frac{1}{\gamma} \sinh \gamma \right] \frac{e^{\gamma(1-\zeta)}}{\zeta}, \end{aligned} \quad (27)$$

$$\begin{aligned} \bar{E}_{\text{int}}'' &= \frac{\alpha}{C^2} \frac{1 - \nu}{1 - 2\nu} \{ [2\alpha^2\gamma^2 + \alpha(18 - \gamma^2)(1 + \gamma)] \cosh \gamma \\ &\quad - [2\alpha^2\gamma^2 + \alpha(2(9 + \gamma^2)(1 + \gamma) - \gamma^4) - \frac{1}{4}\gamma^4(1 + \gamma)] \\ &\quad \times (1/\gamma) \sinh \gamma \} e^{\gamma(1-\zeta)}/\zeta, \end{aligned} \quad (28)$$

where

$$C = \gamma^2(1 + \gamma) + 4\alpha\gamma^2 + 12\alpha(1 + \gamma) + 6\alpha(1 + \gamma) \frac{1 + \nu}{1 - 2\nu}$$

and  $\zeta = R/R_0$ . The bars over  $E_{\text{int}}'$  and  $E_{\text{int}}''$  indicate that these are nondimensional energies which have been normalized with respect to  $(1/\mu)24\pi p^2 R_0^3$ . The total energy is given by

$$\begin{aligned} \bar{E}_{\text{int}} &= \frac{\alpha}{C^2} \left[ \left( \gamma^2(1 + \gamma + 4\alpha) \right. \right. \\ &\quad \left. \left. + \alpha \frac{1 - \nu}{1 - 2\nu} [18(1 + \gamma) - \gamma^2(4 + \gamma) + 2\alpha\gamma^2] \right) \right. \\ &\quad \times \left( \cosh \gamma - \frac{1}{\gamma} \sinh \gamma \right) \\ &\quad \left. - 3\alpha \frac{1 - \nu}{1 - 2\nu} (1 + \gamma) \sinh \gamma \right] \frac{e^{\gamma(1-\zeta)}}{\zeta}. \end{aligned} \quad (29)$$

TABLE II. Nondimensional interaction energies.

$\gamma$	$\zeta$	$\alpha=10^0$	$\alpha=10^{-1}$	$\alpha=10^{-2}$	$\alpha=10^{-3}$	$\alpha=10^{-4}$
1	2	$3.744 \times 10^{-4}$	$3.529 \times 10^{-4}$	$2.023 \times 10^{-4}$	$3.184 \times 10^{-5}$	$3.363 \times 10^{-6}$
	4	$2.534 \times 10^{-5}$	$2.388 \times 10^{-5}$	$1.369 \times 10^{-5}$	$2.154 \times 10^{-6}$	$2.275 \times 10^{-7}$
	10	$2.512 \times 10^{-8}$	$2.368 \times 10^{-8}$	$1.357 \times 10^{-8}$	$2.136 \times 10^{-9}$	$2.256 \times 10^{-10}$
5	2	$-1.004 \times 10^{-4}$	$6.981 \times 10^{-5}$	$1.249 \times 10^{-5}$	$1.325 \times 10^{-6}$	$1.333 \times 10^{-7}$
	4	$-2.278 \times 10^{-9}$	$1.585 \times 10^{-9}$	$2.834 \times 10^{-10}$	$3.007 \times 10^{-11}$	$3.025 \times 10^{-12}$
	10	$-8.528 \times 10^{-23}$	$5.932 \times 10^{-23}$	$1.061 \times 10^{-23}$	$1.126 \times 10^{-24}$	$1.132 \times 10^{-25}$
10	2	$-7.745 \times 10^{-5}$	$1.343 \times 10^{-5}$	$1.966 \times 10^{-6}$	$2.037 \times 10^{-7}$	$2.045 \times 10^{-8}$
	4	$-7.981 \times 10^{-14}$	$1.384 \times 10^{-14}$	$2.026 \times 10^{-15}$	$2.100 \times 10^{-16}$	$2.107 \times 10^{-17}$
	10	$-2.796 \times 10^{-40}$	$4.846 \times 10^{-41}$	$7.097 \times 10^{-42}$	$7.355 \times 10^{-43}$	$7.381 \times 10^{-44}$

The form of the interaction energy in Eq. (29) is rather fortuitous in that the strain-gradient effects can be represented in terms of only two additional parameters:  $\alpha$  and  $\gamma$ . Although problems have been solved using the strain-gradient theory (and the other newly developed, nonclassical theories of elasticity referred to in the Introduction), we note that only qualitative results are obtained, since, so far, there has been no experimental determination of the five new material constants in the isotropic case and the numerous additional constants which would be present for the various crystal classes. It is therefore only by indirect procedures that some knowledge of the probable values for these constants can be found.

Upon imposing the condition that the strain-energy density be positive definite, we find that, in addition to the usual restrictions on the Lamé constants,<sup>13</sup>

$$a_4 + a_5 > 0 \text{ and } (10/3)(a_1 + a_2 + a_3) + 2(a_4 + a_5) > 0, \quad (30)$$

along with some additional inequalities that are not very pertinent to the present work. The last inequality can be rewritten

$$2(a_1 + a_2 + a_3 + a_4 + a_5) > (8/10)(a_4 + a_5) > 0. \quad (31)$$

These restrictions can also be expressed in terms of  $\alpha$  and  $\gamma$ :

$$0 < \alpha < 5/4 \text{ and } \gamma > 0. \quad (32)$$

Table I gives some idea of the magnitude of the non-dimensional displacements due to a single spherical cavity. The nonclassical displacements can be quite different from the classical case, but a preference was given to those cases which do not deviate too much from the classical values. As  $\gamma$  approaches zero, the maximum displacement at the spherical surface approaches zero for the range of  $\alpha$  given in Eq. (32). As  $\gamma$  approaches infinity, the nonclassical displacements rapidly approach the classical values. It can be seen from Table I that

<sup>13</sup> These inequalities are obtained from those given by Mindlin and Eshel (Ref. 12) for the strain-energy density given in Eq. (19) and then relating the material coefficients for this decomposition of the strain-energy density to the one given in Eq. (2). We have considered Eq. (8) directly, but we were only able to show that

$$a_4 + a_5 > 0 \text{ and } 2(a_1 + a_2 + a_3 + a_4 + a_5) > 0.$$

The remaining inequalities could not be rearranged into a form that would yield the more restrictive conditions given in the second of Eqs. (30).

the difference between the two cases is quite small when  $\gamma \approx 10$ , especially for small values of  $\alpha$ .

Table II lists corresponding values of the nondimensional interaction energy. In most instances the interaction energy is positive, which, by the usual convention, indicates that the two pressurized spheres will repel each other. The interaction energy  $\bar{E}_{\text{int}}'$  is always negative, while  $\bar{E}_{\text{int}}''$  is always positive. Thus, we see that  $\bar{E}_{\text{int}}'$  is generally greater than  $\bar{E}_{\text{int}}'$  in absolute value; however,  $\bar{E}_{\text{int}}'$  is predominant when  $\gamma \geq 0.5$ , and  $\bar{E}_{\text{int}}$  is negative over a narrow range of values of  $\gamma$  when  $\gamma \geq 0.43$ . We note that when  $\gamma = 0.5$ ,  $a_1 + a_2 + a_3 = 0$ , and  $a_1 + a_2 + a_3 < 0$  when  $\gamma > 0.5$ . Some idea of the behavior of the interaction energy in this region can be obtained by referring to Table III. The interaction energy is evaluated for  $\zeta = 2$  only. As  $\alpha$  changes, the corresponding change in the form of the energy curve as a function of  $\gamma$  can be seen by comparing the listed values.

All of the numerical computations have been carried out for  $\nu = \frac{1}{3}$ . It was found that variations in  $\nu$  did not affect the results to any great extent.

### III. POINT DEFECTS

In the macroscopic analysis one crystalline solid is distinguished from another by means of the symmetry group to which it belongs and the magnitude of its elastic constants. In the present analysis the assumed isotropy of the material does not permit us to make a distinction between materials from the various crystal classes, and the lack of information pertaining to the elastic coefficients associated with the strain-gradient theory hinders the presentation of definitive results. If the present analysis is taken to be applicable to problems on the atomic scale (in the same sense that the classical theory has been used in this regard), then we see that the continuum representation of point defects is somewhat limited. From this viewpoint a substitutional (or interstitial atom) with physical properties similar to those of the host atoms but having a radius larger (or smaller) than that of the host atoms will be assumed to cause an outward expansion (or inward contraction) of the surrounding lattice. In reality this is not always the case, as Eshelby<sup>6</sup> points out. Also, a vacancy in an otherwise perfect lattice will produce an inward contraction of the surrounding lattice. It has already been emphasized that the classical theory of elasticity is

TABLE III. Variation of nondimensional interaction energy with respect to  $\gamma$  for  $\zeta=2$ .

$\gamma$	$\alpha=0.001$	$\alpha=0.1$	$\alpha=0.425$	$\alpha=0.43$	$\alpha=0.5$	$\alpha=1.25$
0.001	$2.310 \times 10^{-9}$	$2.310 \times 10^{-9}$	$2.310 \times 10^{-9}$	$2.310 \times 10^{-9}$	$2.310 \times 10^{-9}$	$2.31 \times 10^{-9}$
0.01	$2.269 \times 10^{-7}$	$2.269 \times 10^{-7}$	$2.269 \times 10^{-7}$	$2.269 \times 10^{-7}$	$2.269 \times 10^{-7}$	$2.269 \times 10^{-7}$
0.1	$1.810 \times 10^{-5}$	$1.897 \times 10^{-5}$	$1.899 \times 10^{-5}$	$1.899 \times 10^{-5}$	$1.899 \times 10^{-5}$	$1.902 \times 10^{-5}$
1.0	$3.184 \times 10^{-5}$	$3.529 \times 10^{-4}$	$3.540 \times 10^{-4}$	$3.541 \times 10^{-4}$	$3.560 \times 10^{-4}$	$3.851 \times 10^{-4}$
2.0	$1.077 \times 10^{-5}$	$2.893 \times 10^{-4}$	$2.506 \times 10^{-4}$	$2.502 \times 10^{-4}$	$2.468 \times 10^{-4}$	$2.852 \times 10^{-4}$
4.0	$2.326 \times 10^{-6}$	$1.092 \times 10^{-4}$	$4.376 \times 10^{-5}$	$4.231 \times 10^{-5}$	$2.315 \times 10^{-5}$	$-5.509 \times 10^{-5}$
6.0	$8.221 \times 10^{-7}$	$4.675 \times 10^{-5}$	$5.910 \times 10^{-6}$	$4.660 \times 10^{-6}$	$-1.296 \times 10^{-5}$	$-1.428 \times 10^{-4}$
8.0	$3.781 \times 10^{-7}$	$2.360 \times 10^{-5}$	$8.399 \times 10^{-7}$	$-5.892 \times 10^{-9}$	$-1.238 \times 10^{-5}$	$-1.351 \times 10^{-4}$
10.0	$2.037 \times 10^{-7}$	$1.343 \times 10^{-5}$	$4.106 \times 10^{-7}$	$-1.439 \times 10^{-7}$	$-8.456 \times 10^{-6}$	$-1.087 \times 10^{-4}$
12.0	$1.220 \times 10^{-7}$	$8.327 \times 10^{-6}$	$4.972 \times 10^{-7}$	$1.265 \times 10^{-7}$	$-5.529 \times 10^{-6}$	$-8.399 \times 10^{-5}$
15.0	$6.461 \times 10^{-8}$	$4.564 \times 10^{-6}$	$5.571 \times 10^{-7}$	$3.422 \times 10^{-7}$	$-2.997 \times 10^{-6}$	$-5.671 \times 10^{-5}$
20.0	$2.819 \times 10^{-8}$	$2.059 \times 10^{-6}$	$4.602 \times 10^{-7}$	$3.599 \times 10^{-7}$	$-1.228 \times 10^{-6}$	$-3.107 \times 10^{-5}$
50.0	$1.917 \times 10^{-9}$	$1.483 \times 10^{-7}$	$7.678 \times 10^{-8}$	$6.967 \times 10^{-8}$	$-4.740 \times 10^{-8}$	$-3.011 \times 10^{-6}$

incapable of evaluating the interactions which occur between defects of this type using the energy method. The analysis given in Sec. II remedies this deficiency, but the continuum representation of the physical situation is still somewhat naive. One outstanding difficulty in establishing a correspondence between the continuum and lattice models is the determination of the appropriate characteristic length of the material which arises in the continuum analysis. The characteristic length for the problem considered here is probably a function of the lattice parameter of the crystalline solid we are trying to describe, but we have not found a

suitable means for determining this function. However, in order to obtain some quantitative results, we shall follow a rather direct procedure.

The pressure  $p$  appearing in Eq. (17) is not easily defined when a lattice structure is under consideration; therefore, we will assume the radial displacement is specified instead at some point in the medium. Hence,

$$u|_{r=R_0} = u_0 \text{ (a positive constant).} \quad (33)$$

For a vacancy replace  $u_0$  by  $-u_0$ . It follows from Eq. (17) that

$$u = u_0 \frac{\gamma^2(1+\gamma+4\alpha) + 12\alpha(1+\gamma) - 12\alpha[1+\gamma(r/R_0)]e^{\gamma(1-r/R_0)}}{\gamma^2(1+\gamma+4\alpha)} \frac{1}{(r/R_0)^2}, \quad (34)$$

$$p = 4\mu(u_0/R_0)C/\gamma^2(1+\gamma+4\alpha). \quad (35)$$

Therefore, substituting into Eq. (29), we have

$$E_{\text{int}} = 384\pi\mu\delta_0^2 R_0^3 \{C^2/[\gamma^2(1+\gamma+4\alpha)]^2\} \bar{E}_{\text{int}}, \quad (36)$$

where  $\delta_0 = u_0/R_0$ .

Quantitative results can be obtained from Eq. (36). Let us assume that  $\mu = 10^{12}$  dyn/cm<sup>2</sup>,  $R_0 = 10^{-8}$  cm, and  $\delta_0 = 10^{-2}$ . These values will be used to represent a standard crystalline solid. Equation (36) in conjunction with Table II shows that the interaction energy will generally be quite small. Specific cases can easily be evaluated from the above information.

#### IV. DISCUSSION

The main result of this paper is that two centers of dilatation in an infinite, isotropic domain do possess an interaction energy according to the linearized version of Toupin's strain-gradient theory of elasticity. Since the center of dilatation has been adopted by many investigators as a representation of point defects in crystalline solids, several attempts were made to relate this result to the literature pertaining to point defects. A number of interesting correlations were investigated, but they all seemed to produce undesirable as well as

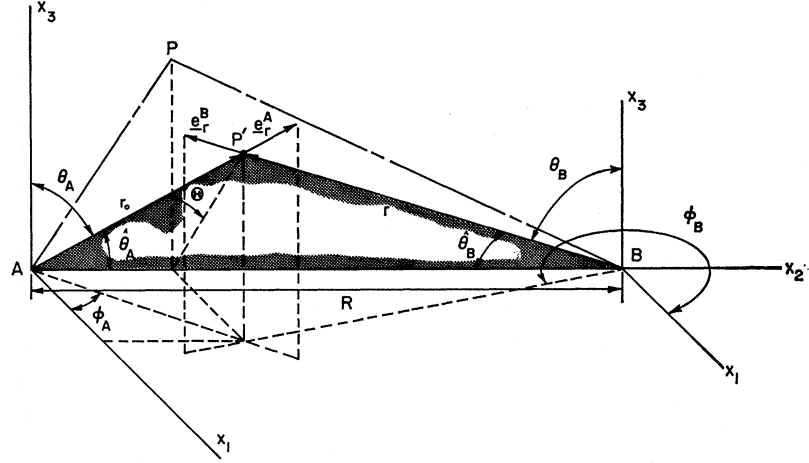
desirable results. Therefore, the straightforward interpretation given in Sec. III was only presented here in order to illustrate the approximate magnitude of the interaction energy.

In evaluating the interaction energy in Sec. II, the simplest boundary condition was assumed. It is feasible analytically to consider substitutional atoms whose mechanical properties are different than those of the host lattice, but this would introduce additional unknown material constants into the final result. It does not seem worthwhile to extend the existing analysis to more complicated problems, until an appropriate physical interpretation of the characteristic length  $l$  has been given for particular materials.

#### APPENDIX

The evaluation of the integrals appearing in Eqs. (25) and (26) is not difficult, but it is rather tedious. We will only outline the procedure used to obtain the final results given in Eqs. (27) and (28). Following Bitter,<sup>1</sup> we see that the integration can be carried out in terms of two variables  $\hat{\theta}_A$  and  $\Theta$ , which are shown in Fig. 1. The point  $P'$  in Fig. 1 identifies an arbitrary point on the

FIG. 1. Coordinate system used for evaluating interaction energy.



surface  $S_A$ . We note, however, that the integrands are independent of  $\Theta$ , which is evident from the symmetry around the line passing through  $A$  and  $B$ . Thus, we will only have to consider the geometric relations between the unit vectors lying in the plane  $APB$ .

The unit base vectors associated with a system of spherical polar coordinates are represented by  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ , and  $\mathbf{e}_\phi$ ; then the unit outward normal to the surface  $S_A$  is  $-\mathbf{e}_r^A$ . Equation (25) may be written

$$E_{\text{int}}' = - \int_0^{2\pi} \int_0^\pi \{ \tau_{rr}^A u^B (\mathbf{e}_r^A \cdot \mathbf{e}_r^B) + [\tau_{rr}^B (\mathbf{e}_r^A \cdot \mathbf{e}_r^B)^2 + \tau_{\theta\theta}^B (\mathbf{e}_r^A \cdot \mathbf{e}_\theta^B)^2] u^A \} \times R_0 \sin \hat{\theta}_A d\Theta R_0 d\hat{\theta}_A, \quad (37)$$

where it is easily shown that

$$\mathbf{e}_r^A \cdot \mathbf{e}_r^B = -\cos(\hat{\theta}_A + \hat{\theta}_B), \quad \mathbf{e}_r^A \cdot \mathbf{e}_\theta^B = \sin(\hat{\theta}_A + \hat{\theta}_B).$$

The displacement field given in Eq. (17) can be substituted into Eq. (11) to obtain explicit expressions for the Cauchy-stress components. Let

$$u = (pR_0/4\mu C)[A - B(1 + \gamma\rho)e^{\gamma(1-\rho)}]1/\rho^2, \quad (38)$$

where  $\rho = r/R_0$  and

$$\begin{aligned} A &= \gamma^2(1 + \gamma + 4\alpha) + 12\alpha(1 + \gamma), \\ B &= 12\alpha, \\ C &= A + 6\alpha(1 + \gamma)(1 + \nu)/(1 - 2\nu). \end{aligned}$$

Then

$$\tau_{rr} = -\frac{p}{C} \left( A - B(1 + \gamma\rho)e^{\gamma(1-\rho)} - \frac{1-\nu}{2(1-2\nu)} B\gamma^2 \rho^2 e^{\gamma(1-\rho)} \right) \frac{1}{\rho^3}, \quad (39)$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \frac{p}{2C} \left( A - B(1 + \gamma\rho)e^{\gamma(1-\rho)} + \frac{\nu}{1-2\nu} B\gamma^2 \rho^2 e^{\gamma(1-\rho)} \right) \frac{1}{\rho^3}.$$

When the expressions for the radial displacement and stresses are substituted into Eq. (37), we find that

$$\begin{aligned} E_{\text{int}}' &= \frac{\pi p^2 R_0^3}{2\mu C^2} B \left[ \left( A - B(1 + \gamma) - \frac{B\gamma^2(1-\nu)}{2(1-2\nu)} \right) \times \int_0^\pi \frac{1}{\rho^2} (1 + \gamma\rho) e^{\gamma(1-\rho)} \cos(\hat{\theta}_A + \hat{\theta}_B) \sin \hat{\theta}_A d\hat{\theta}_A \right. \\ &\quad + \frac{1}{2} [A - B(1 + \gamma)] \int_0^\pi \left( \frac{1}{\rho^3} (1 + \gamma\rho) e^{\gamma(1-\rho)} - \frac{\nu\gamma^2}{1-2\nu} \frac{1}{\rho} e^{\gamma(1-\rho)} - [3(1 + \gamma\rho) + \gamma^2 \rho^2] \frac{1}{\rho^3} e^{\gamma(1-\rho)} \right. \\ &\quad \left. \left. \times \cos^2(\hat{\theta}_A + \hat{\theta}_B) \right) \sin \hat{\theta}_A d\hat{\theta}_A \right]. \quad (40) \end{aligned}$$

Working in the  $APB$  plane, we see that

$$\cos \hat{\theta}_B = (1/\rho)(\zeta - \cos \hat{\theta}_A), \quad \sin \hat{\theta}_B = (1/\rho) \sin \hat{\theta}_A. \quad (41)$$

Now consider an integral of the general form

$$\begin{aligned} &\int_0^\pi \left( \frac{1}{\rho} \right)^m e^{-\gamma\rho} \cos^n(\hat{\theta}_A + \hat{\theta}_B) \sin \hat{\theta}_A d\hat{\theta}_A \\ &= \int_0^\pi \left( \frac{1}{\rho} \right)^m e^{-\gamma\rho} \left( \cos \hat{\theta}_A \frac{\zeta - \cos \hat{\theta}_A}{\rho} - \sin \hat{\theta}_A \frac{\sin \hat{\theta}_A}{\rho} \right)^n \\ &\quad \times \sin \hat{\theta}_A d\hat{\theta}_A \\ &= \int_0^\pi \left( \frac{1}{\rho} \right)^{m+n} e^{-\gamma\rho} (\zeta \cos \hat{\theta}_A - 1)^n \sin \hat{\theta}_A d\hat{\theta}_A. \end{aligned}$$

But

$$\rho^2 = \zeta^2 + 1 - 2\zeta \cos \hat{\theta}_A,$$

so

$$\cos \hat{\theta}_A = (1/2\zeta)(\zeta^2 + 1 - \rho^2) \quad \text{and} \quad \sin \hat{\theta}_A d\hat{\theta}_A = \rho d\rho/\zeta.$$



Then

$$\int_0^\pi \left(\frac{1}{\rho}\right)^m e^{-\gamma\rho} \cos^n(\hat{\theta}_A + \hat{\theta}_B) \sin \hat{\theta}_A d\hat{\theta}_A$$

$$= \frac{1}{(2)^n \zeta} \int_{\zeta-1}^{\zeta+1} \left(\frac{1}{\rho}\right)^{m+n-1} e^{-\gamma\rho} [(\zeta^2-1)-\rho^2]^n d\rho. \quad (42)$$

The integrals on the right-hand side of Eq. (42) are in a form which can easily be evaluated. Using the result given in Eq. (42) for the specific values of  $m$  and  $n$  which appear in Eq. (40), the interaction energy  $E_{\text{int}}'$  can be determined. After some algebraic manipulations, we obtain the expression given in Eq. (25).

The evaluation of Eq. (26) is slightly more complex. The explicit form of  $E_{\text{int}}''$  for the problem under consideration is

$$E_{\text{int}}'' = \int_0^\pi \left\{ -\left(\frac{d\mu_{rrr}^A}{dr} - \frac{2}{r}(\mu_{r\theta\theta}^A + \mu_{\theta\theta r}^A)\right) u^B \cos(\hat{\theta}_A + \hat{\theta}_B) - \mu_{r\theta\theta}^A \left[ \left(\frac{du^B}{dr} + \frac{u^B}{r}\right) - \left(\frac{du^B}{dr} - \frac{u^B}{r}\right) \cos^2(\hat{\theta}_A + \hat{\theta}_B) \right] \right.$$

$$+ u^A \left[ \left(\frac{d\mu_{rrr}^B}{dr} - \frac{d\mu_{r\theta\theta}^B}{dr}\right) + \frac{1}{r}(2\mu_{rrr}^B - 5\mu_{r\theta\theta}^B - 3\mu_{\theta\theta r}^B) \right] \cos^2(\hat{\theta}_A + \hat{\theta}_B) + u^A \left(\frac{d\mu_{r\theta\theta}^B}{dr} + \frac{1}{r}(3\mu_{r\theta\theta}^B + \mu_{\theta\theta r}^B)\right)$$

$$+ \left(\frac{du^A}{dr} - \frac{u^A}{r}\right)(\mu_{rrr}^B - 2\mu_{r\theta\theta}^B - \mu_{\theta\theta r}^B) \cos^3(\hat{\theta}_A + \hat{\theta}_B) + \frac{du^A}{dr}(2\mu_{r\theta\theta}^B + \mu_{\theta\theta r}^B) \cos(\hat{\theta}_A + \hat{\theta}_B)$$

$$\left. + \frac{u^A}{r}(\mu_{rrr}^B - \mu_{\theta\theta r}^B) \cos(\hat{\theta}_A + \hat{\theta}_B) \right\} 2\pi R_0^2 \sin \hat{\theta}_A d\hat{\theta}_A, \quad (43)$$

where we have used the boundary condition in Eq. (16) and set  $\mu_{rrr}^A = 0$ . The expressions for the nonvanishing hyperstress components are

$$\mu_{rrr} = \frac{p}{4\mu R_0 C} \left\{ 2(a_4 + a_5) \left[ \frac{6A}{\rho^4} - B \left( \frac{6}{\rho^4} + \frac{6\gamma}{\rho^3} + \frac{3\gamma^2}{\rho^2} + \frac{\gamma^3}{\rho} \right) e^{\gamma(1-\rho)} \right] - 2(a_1 + a_2 + a_3) B \left( \frac{\gamma^2}{\rho^2} + \frac{\gamma^3}{\rho} \right) e^{\gamma(1-\rho)} \right\},$$

$$\mu_{r\theta\theta} = \mu_{\theta r\theta} = \frac{p}{4\mu R_0 C} \left\{ 2(a_4 + a_5) \left[ \frac{-3A}{\rho^4} + B \left( \frac{3}{\rho^4} + \frac{3\gamma}{\rho^3} + \frac{\gamma^2}{\rho^2} \right) e^{\gamma(1-\rho)} \right] - \frac{1}{2}(a_1 + 2a_2) B \left( \frac{\gamma^2}{\rho^2} + \frac{\gamma^3}{\rho} \right) e^{\gamma(1-\rho)} \right\}, \quad (44)$$

$$\mu_{\theta\theta r} = \frac{p}{4\mu R_0 C} \left\{ 2(a_4 + a_5) \left[ \frac{-3A}{\rho^4} + B \left( \frac{3}{\rho^4} + \frac{3\gamma}{\rho^3} + \frac{\gamma^2}{\rho^2} \right) e^{\gamma(1-\rho)} \right] - (a_1 + 2a_3) B \left( \frac{\gamma^2}{\rho^2} + \frac{\gamma^3}{\rho} \right) e^{\gamma(1-\rho)} \right\}.$$

Again the dependence on  $\hat{\theta}_B$  can be eliminated from Eq. (43) by means of the geometric relations given in Eq. (41). Upon substituting the expressions for the radial displacement and the double-stress components into Eq. (43), we find that there are a number of integrals of the general form given in Eq. (42) to evaluate. After carrying out the lengthy algebraic manipulations required, we finally obtain the result given in Eq. (26).