$1-n_\uparrow$, and for the improved calculation, for which

$$Z=1/(1-d\Sigma/d\omega). \tag{53}$$

We see that the two results are almost identical.

DISCUSSION

Perhaps the most interesting result of the present work is that for U_{eff} in Fig. 5. If U_{eff} is to be equated with the exchange splitting in the Stoner model, then we find that the Kanamori result for this quantity departs more rapidly from the low-density result than does the present calculation, as the number of carriers increases. If we believe that an improved calculation gives more correlation reduction of this quantity, then we must say that our result is an improvement and that the Kanamori result overestimates the increase of U_{eff}. The above belief would be a fact if the claim made in I for a minimum principle were verified. At any rate, these considerations may have serious consequences as for example in the work of Calloway's[8] recent article in which he finds that the increase in U_{eff} is very important in stabilizing the ferromagnetic state for small but finite n. This effect seems also to be important in Ni.[7]

There are several mitigating factors here. One is that we are considering the worst possible case, namely, $U\rightarrow\infty$. Secondly, we are using the scnn model, and the work of Calloway in particular is based on a second-neighbor face-centered cubic model which has, like nickel, a peak in the density of states near the band edge. The argument here is that the Kanamori result should be good in this case, as the Fermi level is very close to the band edge. We should examine U_{eff} in this case and hope to do so in the future. Calloway has also applied his result to Ni[13] and the same remarks apply, with the additional factor that he considers, and we neglect, degeneracy. Finally, to compare our work with that of Lang and Ehrenreich,[7] for example, we need to find the corresponding quantity for the paramagnetic state. U_{eff} may not be the same for the ferromagnetic and paramagnetic states, as pointed out recently by Penn.[14]

Nevertheless, it does appear that the Kanamori result can be improved upon and extended to higher densities by methods based on the present work, and also that the two pole approximation gives a reasonable value for U_{eff} at finite carrier densities.

[13] J. Calloway and H. M. Zhaver, Phys. Letters 28A, 292 (1969).
[14] D. R. Penn, Phys. Rev. 177, 839 (1969).

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Friedel Density Oscillations about a Coulombic Impurity in a High Magnetic Field

Norman J. Horing
Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030
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The density perturbation $\delta\rho(\mathbf{r})$ about a Coulombic impurity in strong magnetic fields is analyzed in the degenerate case. The results show that (a) anisotropic Friedel oscillatory behavior is discernible along the direction parallel to the magnetic field, and (b) exponential decay along the direction perpendicular to the magnetic field deprives the result of long-range character. These results are in accord with those of Rensink, save for the fact that the δ-function impurity used by Rensink results in a Gaussian decay in the perpendicular direction, in contrast to the exponential decay in the perpendicular direction seen to result from a Coulombic impurity here. This indicates that the qualitative nature of the spatial decay in the perpendicular direction is dependent upon the character of the impurity potential. The results obtained here are valid at large distances from the impurity for a wide range of field strengths. A qualitative discussion of low-field behavior in the approach to the zero-field limit is presented, and the role of self-consistent aspects of response is briefly indicated.

I. INTRODUCTION AND FORMULATION

THE study[1] of the dynamical properties of a quantum plasma in a high magnetic field has revealed the existence of interesting physical effects due to Landau quantization and nonlocality. Moreover, one may expect static properties to be correspondingly

[1] N. J. Horing, Ann. Phys. (N. Y.) 31, (1965). The notation of this reference will be maintained here, and a brief review follows: $\omega_p=(4\pi e^2\rho/m)^{1/2}$; ρ is the density; m is the effective mass; ω_c is the cyclotron frequency $(=eH/mc)$; $\mathbf{p}$ is the wave vector $[=(p_z,\bar{p})]$; $\mathbf{r}$ is the position vector from impurity $[=(r_z,\bar{r})]$; $\mathbf{H}$ is the magnetic field along the z axis; $\beta=(KT)^{-1}$; K is Boltzmann's constant; T is the absolute temperature; ζ is the chemical potential, equal to the Fermi energy E_F in the degenerate limit; $f_0(\omega)$ is the Fermi-Dirac distribution function $[=(1+e^{(\omega-\zeta)\beta})^{-1}]$; p_F is the Fermi wave number $[=(2m\zeta)^{1/2}/\hbar]$; and p_D is the Debye-Thomas-Fermi (DTF) wave number $[=(4\pi e^2\partial\rho/\partial\zeta)^{1/2}]$. (Also, r_0 is the interparticle spacing, a_0 is the Bohr radius, and $r_s=r_0/a_0$.) Note that two-vectors referring to the plane perpendicular to the magnetic field are represented by lightface barred letters.

influenced.[2] In particular, the Friedel density oscillation which occurs about an impurity is a relatively high wave number phenomenon ($p\sim 2p_F$) of the quantum plasma, and as such, one may expect it to be sensitive to the effects of Landau quantization. Alternatively, one may say that the disruption of the Fermi surface by Landau quantization disturbs the theoretical basis of the Friedel density oscillation, and different behavior is to be expected. In a recent publication,[3] Rensink has investigated the effects of Landau quantization on the Friedel density oscillation about a δ-function impurity. His results indicate that behavior in the nature of the Friedel density oscillation is still discernible in an anisotropic fashion along the direction parallel to the magnetic field, whereas the density perturbation due to the δ-function impurity suffers a Gaussian type of spatial decay along the direction perpendicular to to the magnetic field. The first part of these results is in good qualitative accord with our earlier work[2] as far as behavior along the magnetic field direction is concerned. However, the qualitative nature of the spatial decay in the transverse direction is, in fact, dependent upon the character of the impurity potential. We shall investigate here the density perturbation about a Coulombic impurity in high field, and show directly that the Coulomb impurity potential is characterized by a simple exponential type of spatial decay in the transverse direction, rather than by the Gaussian which arises in conjunction with the δ-function impurity potential.

The density perturbation about a Coulomb impurity may be expressed in terms of thermodynamic Green's functions[1] by virtue of the fact that $\rho=-i\bar{G}_1(1;1^+)$. Moreover, if we neglect interparticle interactions and take $U(1)$ to be the Coulombic impurity potential, then the equation for $\bar{G}_1$, is simply given by

$$[(\bar{G}_1{}^0)^{-1}(1)-U(1)]\bar{G}_1(1;1')=\delta(1,1'), \qquad (1)$$

where $\bar{G}_1{}^0$ is the one-electron thermodynamic Green's function for a Pauli electron in an arbitrarily strong constant uniform magnetic field. This Green's function has been discussed in detail in Ref. 1. It should be remarked that the Pauli-spin term must be included in studies of high-field phenomena, since spin splitting of levels is comparable with Landau-level separation. The $\bar{G}_1$ equation (1) may be converted to an integral equation and iterated for solution. The leading term, which is linear in the Coulombic impurity potential, is given by

$$\bar{G}_1(1;1')=\bar{G}_1{}^0(1;1')+\int d(2)\bar{G}_1{}^0(1;2)U(2)\bar{G}_1{}^0(2;1'), \quad (2)$$

which corresponds to saying that the response to the impurity is governed by

$$\delta\bar{G}_1(1;1')/\delta U(2)=\bar{G}_1{}^0(1;2)\bar{G}_1{}^0(2;1'). \qquad (3)$$

Now $\bar{G}_1{}^0(1;1')$ depends on the gauge ϕ and on $\mathbf{r}_1+\mathbf{r}_1'$ as well as $\mathbf{r}_1-\mathbf{r}_1'$, in the following manner (see Ref. 1):

$$\bar{G}_1{}^0(1;1')=C(\mathbf{r}_1,\mathbf{r}_1')\bar{G}_1{}^{0'}(\mathbf{r}_1-\mathbf{r}_1',t_1-t_1'), \qquad (4a)$$

where

$$C(\mathbf{r}_1,\mathbf{r}_1')=\exp[i(\tfrac{1}{2}e)\mathbf{r}_1\cdot\mathbf{H}\times\mathbf{r}_1'-i\phi(\mathbf{r}_1)+i\phi(\mathbf{r}_1')], \quad (4b)$$
$$C(\mathbf{r}_1,\mathbf{r}_1)=1, \quad \text{and} \quad C(\mathbf{r}_1,\mathbf{r}_2)C(\mathbf{r}_2,\mathbf{r}_1)=1.$$

These properties clearly allow the replacement of $\bar{G}_1{}^0$ by $\bar{G}_1{}^{0'}$ when Eq. (2) is used to calculate the density $\rho=-i\bar{G}_1(1;1^+)$:

$$\bar{G}_1(1;1^+)=\bar{G}_1{}^{0'}(1;1^+)+\int d(2)\bar{G}_1{}^{0'}(1-2)U(2)\bar{G}_1{}^{0'}(2-1^+). \quad (5a)$$

Introducing spatial Fourier transforms, one finds

$$\bar{G}_1(1;1^+)=\bar{G}_1{}^{0'}(1;1^+)+\int\frac{d\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{r}_1}U(\mathbf{p})\times\int dt\int\frac{d\mathbf{q}}{(2\pi)^3}\bar{G}_1{}^{0'}(\mathbf{q},-t)\bar{G}_1{}^{0'}(\mathbf{q}-\mathbf{p},t-0^+). \quad (5b)$$

A thorough discussion of the integral

$$I=\int dt\int\frac{d\mathbf{q}}{(2\pi)^3}\bar{G}_1{}^{0'}(\mathbf{q},-t)\bar{G}_1{}^{0'}(\mathbf{q}-\mathbf{p},t-0^+) \quad (6a)$$

has been carried out in Ref. 1, employing the thermodynamic Green's function appropriate to arbitrary magnetic field strength. Actually, it should be observed that Eq. (5b) is subject to two equivalent interpretations, depending on whether $\delta\bar{G}_1(1;1')/\delta U(2)$ is taken to represent the "periodic" response of the system or, alternatively, the physical (retarded) response. For the former, one may make the specific identification [Eq. (I.33) of Ref. 1]

$$I\rightarrow I(\mathbf{p},\nu\rightarrow 0), \qquad (6b)$$

whereas for the latter, one must make the specific identification (footnote 6 of Ref. 1)

$$I\rightarrow I(\mathbf{p},\omega=0+i\epsilon)\rightarrow \mathrm{Im}I(\mathbf{p},\omega=0+i\epsilon). \quad (6c)$$

Since we are directly interested in the physical response, we shall adhere to the latter interpretation, given by Eq. (6c) [where we have already noted that $\mathrm{Re}I(\mathbf{p},\omega)$ does not contribute for $\omega=0$, since it is an odd function of ω].[1] The fact that zero frequency is involved here reflects the fact that we are calculating the density perturbation a very long (infinite) time after the Coulomb impurity is "turned on," so that all transients have died out and the density perturbation is independent of time (corresponding to $\omega\rightarrow 0$). Noting that $U(\mathbf{p})=4\pi Ze^2/p^2$ for a Coulomb impurity (having Z units of charge), one then obtains

$$\delta\rho(\mathbf{r})=\rho(\mathbf{r})-\rho^0(\mathbf{r})=\int\frac{d\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{r}}\frac{4\pi Ze^2}{p^2}\times\mathrm{Im}I(\mathbf{p},\omega=0+i\epsilon). \quad (7)$$

[2] N. J. Horing, J. Phys. Soc. Japan Suppl., 21, 704 (1966).
[3] M. E. Rensink, Phys. Rev. 174, 744 (1968).

This result may also be understood in terms of usual linear response theory, employing a longitudinal dielectric function in the ordinary Hartree-Fock approximation. The important integral $I(\mathbf{p}, \omega=0+i\epsilon)$ is applied to the degenerate case of interest in the Appendix of this paper, utilizing its development in Ref. 1.

II. QUANTUM STRONG-FIELD LIMIT

The evaluation of $\delta\rho(\mathbf{r})$ in the quantum strong-field limit may be undertaken using the result for $\mathrm{Im}I(\mathbf{p}, \omega=0+i\epsilon)$ as given in the Appendix in the case $\hbar\omega_c>\zeta$:

$$\mathrm{Im}I(\mathbf{p}, \omega=0+i\epsilon)=-\frac{\rho_0}{\hbar}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2}\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n \times\ln\left|\frac{n\omega_c+\hbar p_z^2/2m+(2p_z^2\zeta/m)^{1/2}}{n\omega_c+\hbar p_z^2/2m-(2p_z^2\zeta/m)^{1/2}}\right|, \quad (8)$$

where $\rho_0=m^{3/2}\omega_c\zeta^{1/2}/2^{1/2}\pi^2\hbar^2$. It should be noted that the argument of the logarithm may be factored as follows:

$$\ln|\cdots|\to\ln\left|\frac{[p_z+p_F(1+i\gamma_n)][p_z+p_F(1-i\gamma_n)]}{[p_z-p_F(1-i\gamma_n)][p_z-p_F(1+i\gamma_n)]}\right|, \quad (9)$$

where $i\gamma_n=(1-n\hbar\omega_c/\zeta)^{1/2}$ so that γ_n is real for $n\geqslant 1$, ($\hbar\omega_c>\zeta$), but $i\gamma_0=1$. Thus the branch point of the $n=0$ term is real, whereas the branch points of $n\geqslant 1$ terms are complex:

$$\mathrm{Im}I(\mathbf{p}, \omega=0+i\epsilon)=-\frac{\rho_0}{\hbar}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2}\exp\left(-\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\times\left[\ln\left|\frac{p_z+2p_F}{p_z-2p_F}\right|+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\times\ln\left|\frac{[p_z+p_F(1+i\gamma_n)][p_z+p_F(1-i\gamma_n)]}{[p_z-p_F(1-i\gamma_n)][p_z-p_F(1+i\gamma_n)]}\right|\right]. \quad (10)$$

One can readily verify that the contributions to $\delta\rho(\mathbf{r})$ of the terms having $n\geqslant 1$ are smaller than that of the $n=0$ term by factors $\exp(-\gamma_n p_F|r_z|)$ because of the complex nature of their branch points for $n\geqslant 1$. Hence such terms can be ignored in considering the long-distance behavior of $\delta\rho(\mathbf{r})$, and we have ($\hbar\omega_c>\zeta$)

$$\delta\rho(\mathbf{r})=\frac{\rho_0}{\hbar}\left(\frac{m}{2\zeta}\right)^{1/2}\int\frac{d\bar{p}}{(2\pi)^2}e^{i\bar{p}\cdot\bar{r}}\exp\left(-\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\times\int\frac{dp_z}{2\pi}e^{ip_zr_z}\frac{4\pi Ze^2}{\bar{p}^2+p_z^2}\frac{1}{p_z}\ln\left|\frac{p_z-2p_F}{p_z+2p_F}\right|. \quad (11)$$

An asymptotic estimate of the p_z integral is readily obtained for $p_F|r_z|\gg 1$ following the method of Lighthill,[4] with the result

$$\delta\rho(\mathbf{r})=\frac{\rho_0}{\hbar}\left(\frac{m}{2\zeta}\right)^{1/2}\frac{\cos 2p_Fr_z}{2p_F|r_z|}\int\frac{d\bar{p}}{(2\pi)^2}e^{i\bar{p}\cdot\bar{r}}\frac{4\pi Ze^2}{\bar{p}^2+(2p_F)^2}\times\exp\left(-\frac{\hbar\bar{p}^2}{2m\omega_c}\right). \quad (12)$$

The behavior of $\delta\rho(\mathbf{r})$ as a function of $\bar{r}$ is determined by the integral $\mathcal{S}$:

$$\mathcal{S}=\int\frac{d\bar{p}}{(2\pi)^2}e^{i\bar{p}\cdot\bar{r}}\frac{4\pi Ze^2}{\bar{p}^2+(2p_F)^2}\exp(-\bar{p}^2/p_H^2), \quad (13)$$

where we have introduced the notation $p_H=(2m\omega_c/\hbar)^{1/2}=p_F(\hbar\omega_c/\zeta)^{1/2}$. It should be noted that the connection between this calculation and the one of Rensink[3] may be made by replacing the Coulomb pole structure $4\pi Ze^2/[\bar{p}^2+(2p_F)^2]$ by a constant representing the $\delta(\mathbf{r})$ impurity potential in $\mathbf{p}$ space. (Under this replacement, the integral $\mathcal{S}$ would simply become the Fourier transform of a Gaussian in the transverse plane, which yields the Gaussian dependence on $|\bar{r}|$ characterizing Rensink's result.) Actually, the Coulomb pole structure is very important in the asymptotic determination of $\mathcal{S}$, and in fact it yields a simple exponential type of spatial decay as a function of $|\bar{r}|$. In the analysis of $\mathcal{S}$, it is useful to write the Coulomb pole structure in exponential form

$$\alpha^{-1}=\int_0^\infty dx\, e^{-\alpha x}.$$

One may then complete the square in the argument of the exponential integrand of $\mathcal{S}$, translate the integration variable according to $\bar{q}=\bar{p}-i\bar{r}/2(x+p_H^{-2})$, and then evaluate the wave-number integral trivially. The result is given by

$$\mathcal{S}=\frac{4\pi Ze^2}{(2\pi)^2}\int_0^\infty dx\, e^{-\bar{r}^2/4(x+p_H^{-2})}e^{-(2p_F)^2x}\frac{\pi}{x+p_H^{-2}}. \quad (14)$$

Some further manipulation of the integration variable (simple rescaling and translation) then yields

$$\mathcal{S}=Ze^2e^{(2p_F/p_H)^2}\int_{2(p_H\bar{r})^{-2}}^{\infty}dt\frac{1}{t}\times\exp\{-\tfrac{1}{2}(2p_F\bar{r})^2[t+(2p_F\bar{r})^{-2}/t]\}. \quad (15)$$

At large distances the lower limit $\sim 2(p_H\bar{r})^{-2}=(2\zeta/\hbar\omega_c)\times(p_F\bar{r})^{-2}\ll 1$ may be set equal to zero, and then the t integral may be identified in terms of the modified

[4] M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge University Press, New York, 1958).

Hankel function K_0 (BHTF II, p. 82, No. 23),[5]

$$\mathcal{S}=2Ze^2e^{(2p_F/p_H)^2}K_0(2p_F|\bar{r}|). \quad (16)$$

The asymptotic behavior is that of a simple exponential decay given by (BHTF II, p. 86 No. 7)[5]

$$\delta\rho(\mathbf{r})\cong Ze^2\frac{p_0}{\hbar}\left(\frac{m}{2\zeta}\right)^{1/2}e^{(2p_F/p_H)^2}\left(\frac{\pi}{p_F|\bar{r}|}\right)^{1/2}\times e^{-2p_F|\bar{r}|}\frac{\cos 2p_Fr_z}{2p_F|r_z|}. \quad (17)$$

It should be noted that this result is not directly applicable to the special directions strictly parallel and perpendicular to the magnetic field, since we have invoked both the conditions $p_F|\bar{r}|\gg1$ and $p_F|r_z|\gg1$. However, it is applicable for all intermediate directions, thus being of greater interest, and the special directions may be approached arbitrarily closely with this result for sufficiently large distances $p_F|r|\gg1$. Anisotropic Friedel oscillatory behavior is discernible in the r_z direction through the factor $\cos 2p_Fr_z$. However, the exponential decay in the $\bar{r}$ direction $\exp(-2p_F|\bar{r}|)$ deprives the result of long-range character.

III. LOWER MAGNETIC FIELD STRENGTHS

The evaluation of $\delta\rho(\mathbf{r})$ at lower magnetic field strengths $\zeta>\hbar\omega_c\gg kT$ may be undertaken using the corresponding result for $\mathrm{Im}I(\mathbf{p},\omega=0+i\epsilon)$ as given in the Appendix:

$$\mathrm{Im}I(\mathbf{p},\omega=0+i\epsilon)=\frac{m^2\omega_c}{4\pi^2p_z}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\exp\left(-\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\frac{r!}{(n+r)!}\times\left[L_r^n\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\right]^2 2\sum_{\pm,'\pm}\frac{1}{\hbar^3}\eta_+(\zeta-a_{rn}(\pm,\pm'))\ln\left|\frac{\pm'n\omega_c+\hbar p_z^2/2m-(2p_z^2/m)^{1/2}[\zeta-a_{rn}(\pm,\pm')]^{1/2}}{\pm'n\omega_c+\hbar p_z^2/2m+(2p_z^2/m)^{1/2}[\zeta-a_{rn}(\pm,\pm')]^{1/2}}\right|. \quad (18)$$

It should be noted that the argument of the logarithm may be factored as follows:

$$\ln|\cdots|\to\ln\left|\frac{[p_z/p_F-\beta_{rn}(\pm,\pm')][p_z/p_F-\Delta_{rn}(\pm,\pm')]}{[p_z/p_F+\beta_{rn}(\pm,\pm')][p_z/p_F+\Delta_{rn}(\pm,\pm')]}\right|, \quad (19)$$

where

$$\beta_{rn}(\pm,\pm')=[1-a_{rn}(\pm,\pm')/\zeta]^{1/2}+[1-a_{rn}(\pm,\mp')/\zeta]^{1/2}, \quad (20a)$$

$$\Delta_{rn}(\pm,\pm')=[1-a_{rn}(\pm,\pm')/\zeta]^{1/2}-[1-a_{rn}(\pm,\mp')/\zeta]^{1/2} \quad (20b)$$

and

$$-a_{rn}(\pm,\pm')=[\pm1\pm'n-(n+2r+1)]\tfrac{1}{2}\hbar\omega_c, \quad (21a)$$

$$-a_{rn}(\pm,\mp')=[\pm1\mp'n-(n+2r+1)]\tfrac{1}{2}\hbar\omega_c=-a_{rn}(\pm,\pm')-\pm'n\hbar\omega_c. \quad (21b)$$

Again, the contributions of terms having complex branch points are negligible compared to those of terms having real branch points, so we eliminate the former by inserting a cutoff factor $\eta_+(\zeta-a_{rn}(\pm,\mp'))$. Taken together with the other cutoff function already present, $\eta_+(\zeta-a_{rn}(\pm,\pm'))$, the product of these two, $\eta_+(\zeta-a_{rn}(\pm,\pm'))\eta_+(\zeta-a_{rn}(\pm,\mp'))$, retains all terms in the series having real branch points, and automatically neglects all terms having complex branch points. [Note that these cutoff functions guarantee the reality of $\beta_{rn}(\pm,\pm')$ and $\Delta_{rn}(\pm,\pm')$ for contributing terms which thus have real branch points.] Thus we have the result [$p_H=(2m\omega_c/\hbar)^{1/2}=p_F(\hbar\omega_c/\zeta)^{1/2}$],

$$\delta\rho(\mathbf{r})=\left(\frac{m^2\omega_c}{4\pi^2}\right)\frac{2}{\hbar^3}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm',\pm}\eta_+(\zeta-a_{rn}(\pm,\pm'))\eta_+(\zeta-a_{rn}(\pm,\mp'))\frac{r!}{(n+r)!}\int\frac{d\bar{p}}{(2\pi)^2}e^{i\bar{p}\cdot\bar{r}}(\bar{p}^2/p_H^2)^n\exp(-\bar{p}^2/p_H^2)\times[L_r^n(\bar{p}^2/p_H^2)]^2\int\frac{dp_z}{2\pi}e^{ip_zr_z}\frac{4\pi Ze^2}{\bar{p}^2+p_z^2}\frac{1}{p_z}\ln\left|\frac{[p_z-p_F\beta_{rn}(\pm,\pm')][p_z-p_F\Delta_{rn}(\pm,\pm')]}{[p_z+p_F\beta_{rn}(\pm,\pm')][p_z+p_F\Delta_{rn}(\pm,\pm')]}\right|. \quad (22)$$

Again one can obtain an asymptotic estimate of the p_z integral following the method of Lighthill.[4] The result is

[5] (a) BHTF refers to Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdeli (McGraw-Hill Book Co., New York, 1953); (b) BIT refers to Bateman Manuscript Project, *Tables of Integral Transforms* (McGraw-Hill Book Co., New York, 1954).

[subject to the conditions $|p_F\beta_{rn}(\pm,\pm')r_z|\gg 1$ and $|p_F\Delta_{rn}(\pm,\pm')r_z|\gg 1$]

$$\delta\rho(\mathbf{r})=\frac{m^2\omega_c}{4\pi^2}\frac{2}{\hbar^3}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm',\pm}\eta_+(\zeta-a_{rn}(\pm,\pm'))\eta_+(\zeta-a_{rn}(\pm,\mp'))\frac{r!}{(n+r)!}$$
$$\times\left(\frac{\cos[p_F\beta_{rn}(\pm,\pm')r_z]}{p_F\beta_{rn}(\pm,\pm')|r_z|}\int\frac{d\bar{p}}{(2\pi)^3}e^{i\bar{p}\cdot\bar{r}}(\bar{p}^2/p_H^2)^n\exp(-\bar{p}^2/p_H^2)[L_r^n(\bar{p}^2/p_H^2)]^2\frac{4\pi Ze^2}{\bar{p}^2+p_F^2\beta_{rn}^2(\pm,\pm')}\right.$$
$$\left.+\Phi\frac{\cos[p_F\Delta_{rn}(\pm,\pm')r_z]}{p_F\Delta_{rn}(\pm,\pm')|r_z|}\int\frac{d\bar{p}}{(2\pi)^3}e^{i\bar{p}\cdot\bar{r}}(\bar{p}^2/p_H^2)^n\exp(-\bar{p}^2/p_H^2)[L_r^n(\bar{p}^2/p_H^2)]^2\frac{4\pi Ze^2}{\bar{p}^2+p_F^2\Delta_{rn}^2(\pm,\pm')}\right).\quad(23)$$

[Φ is defined to be zero when $\Delta_{rn}(\pm,\pm')$ vanishes, since inspection of Eq. (22) shows that there can be no such contribution. Otherwise Φ is identically unity, irrespective of whether the nonvanishing value of $\Delta_{rn}(\pm,\pm')$ is positive or negative.]

The behavior of $\delta\rho(\mathbf{r})$ as a function of $\bar{r}$ is determined by integrals of the form [see Eq. (23)]

$$\mathcal{S}(F)=\int\frac{d\bar{p}}{(2\pi)^2}e^{i\bar{p}\cdot\bar{r}}\frac{4\pi Ze^2}{\bar{p}^2+\alpha^2}\exp(-\bar{p}^2/p_H^2)F(\bar{p}^2/p_H^2),\quad(24)$$

where $F(\bar{p}^2/p_H^2)$ is a polynomial function of $\bar{p}^2/p_H^2$. It is immediately clear that

$$\mathcal{S}(F)=F(-(\partial^2/\partial x^2+\partial^2/\partial y^2)/p_H^2)\mathcal{S},\quad(25)$$

where the evaluation of $\mathcal{S}$ has already been carried out in terms of a modified Hankel function [note that α replaces $2p_F$ in Eq. (16)] under the condition that $2(p_H\bar{r})^{-2}=(2\zeta/\hbar\omega_c)(p_F\bar{r})^{-2}\ll 1$. Even at reasonably low fields ($2\zeta/\hbar\omega_c>1$) this condition is still satisfied, since we are interested in large distances, $(p_F\bar{r})^{-2}\ll 1$. Thus the behavior of $\delta\rho(\mathbf{r})$ as a function of $\bar{r}$ for large $\bar{r}$ may be determined from the asymptotic behavior of the modified Hankel function for $\alpha|\bar{r}|\gg 1$ [in the present context this means that $|p_F\beta_{rn}(\pm,\pm')\bar{r}|\gg 1$ and $|p_F\Delta_{rn}(\pm,\pm')\bar{r}|\gg 1$]:

$$\mathcal{S}=2Ze^2e^{(\alpha/p_H)^2}K_0(\alpha|\bar{r}|)\rightarrow Ze^2(2\pi/\alpha|\bar{r}|)^{1/2}e^{(\alpha/p_H)^2}e^{-\alpha|\bar{r}|},\quad(26)$$

where

$$\alpha=p_F\beta_{rn}(\pm,\pm'),\quad p_F|\Delta_{rn}(\pm,\pm')|.$$

Now, in considering the action of spatial derivatives on $\mathcal{S}$ to obtain $\mathcal{S}(F)=F(-(\partial^2/\partial x^2+\partial^2/\partial y^2)/p_H^2)\mathcal{S}$, one should note that the variation of the factor $e^{-\alpha|\bar{r}|}$ dominates such spatial derivatives for $\alpha|\bar{r}|\gg 1$. For example, one can very accurately make the replacement $(\partial^2/\partial x^2+\partial^2/\partial y^2)^n\mathcal{S}\rightarrow\alpha^{2n}\mathcal{S}$ and obtain the result

$$\mathcal{S}(F)=F(-\alpha^2/p_H^2)Ze^2e^{(\alpha/p_H)^2}(2\pi/\alpha|\bar{r}|)^{1/2}e^{-\alpha|\bar{r}|}.\quad(27)$$

Since $F(x)$ represents polynomial functions of the type $F(x)\rightarrow x^n[L_r^n(x)]^2$ [Eq. (23)], the final result is given by

$$\delta\rho(\mathbf{r})=Ze^2\frac{m^2\omega_c}{2\pi^2\hbar^3}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm',\pm}\eta_+(\zeta-a_{rn}(\pm,\pm'))\eta_+(\zeta-a_{rn}(\pm,\mp'))\frac{r!}{(n+r)!}$$
$$\times\left\{e^{[p_F\beta_{rn}(\pm,\pm')/p_H]^2}\left(-\frac{p_F^2\beta_{rn}^2(\pm,\pm')}{p_H^2}\right)^n\left[L_r^n\left(-\frac{p_F^2\beta_{rn}^2(\pm,\pm')}{p_H^2}\right)\right]^2\left(\frac{2\pi}{p_F\beta_{rn}(\pm,\pm')|\bar{r}|}\right)^{1/2}\right.$$
$$\times e^{-p_F\beta_{rn}(\pm,\pm')|\bar{r}|}\frac{\cos[p_F\beta_{rn}(\pm,\pm')r_z]}{p_F\beta_{rn}(\pm,\pm')|r_z|}+\Phi e^{[p_F\Delta_{rn}(\pm,\pm')/p_H]^2}\left(-\frac{p_F^2\Delta_{rn}^2(\pm,\pm')}{p_H^2}\right)^n\left[L_r^n\left(-\frac{p_F^2\Delta_{rn}^2(\pm,\pm')}{p_H^2}\right)\right]^2$$
$$\left.\times\left(\frac{2\pi}{p_F|\Delta_{rn}(\pm,\pm')|\,|\bar{r}|}\right)^{1/2}e^{-p_F|\Delta_{rn}(\pm,\pm')|\,|\bar{r}|}\frac{\cos[p_F\Delta_{rn}(\pm,\pm')r_z]}{p_F\Delta_{rn}(\pm,\pm')|r_z|}\right\}.\quad(28)$$

Again, anisotropic Friedel oscillatory behavior is discernible in the r_z direction, but it is deprived of long-range character by exponential decay in the $\bar{r}$ direction. This multiple series is terminated by the cutoff functions

$$\eta_+(\zeta-a_{rn}(\pm,\pm'))\eta_+(\zeta-a_{rn}(\pm,\mp'))$$

after an appropriate number of terms, depending on the size of $\hbar\omega_c/\zeta$. For example, in the quantum strong-field limit ($\hbar\omega_c/\zeta>1$) only the term characterized by $n=0$, $r=0$, $\pm\rightarrow+$ can contribute, and the double counting of the $n=0$ term (indicated by the prime on the n sum) can be compensated by selecting $\pm'\rightarrow+'$. In this case, $a_{rn}(\pm,\pm')=a_{rn}(\pm,\mp')=0$, so that $\beta_{rn}(\pm,\pm')=2$ and $\Delta_{rn}(\pm,\pm')=0$ (which also means that Φ must be taken to vanish); and noting that $L_0^0\equiv 1$, one recovers the quantum strong-field result of Eq. (17). For lower magnetic fields ($\hbar\omega_c/\zeta<1$), more terms of the multiple series contribute before termination by the cutoff functions. Actually, the result [Eq. (28)] is valid even for rather small magnetic fields as long as we are concerned with large distances, since the only substantive condition on field strength which has been used is that the parameter $(2\zeta/\hbar\omega_c)(p_F\bar{r})^{-2}\ll 1$ should be small; alternatively, the "mixed" parameter $(\hbar\omega_c/2\zeta)(p_F\bar{r})^2\gg 1$ should be large. In terms of this mixed parameter, the magnetic field is effectively large at large distances despite the fact that it may be small in the sense that $\hbar\omega_c/\zeta<1$. In view of the occurrence of the mixed parameter, one can readily see that attempts to analyze $\delta\rho(\mathbf{r})$ for low fields by expanding in powers of $\hbar\omega_c/\zeta$ actually result in expansion in powers of the mixed parameter (in part), which is large at large distances despite the smallness of $\hbar\omega_c/\zeta$; thus such attempts to study the behavior of $\delta\rho(\mathbf{r})$ at large distances are rendered futile, and one must have recourse to Eq. (28).

It should be noted that in taking the largeness of distance to be the dominant consideration in the "mixed" parameter, we are precluding the possibility of approaching the zero-field limit in which the smallness of the field is the dominant consideration (such is the case within the framework of the present level of approximation). Since the "mixed" parameter puts the smallness of the field into competition with the largeness of distance, low-field behavior akin to the zero-field limit can occur only over a limited range of distances corresponding to small values of the mixed parameter. Within this limited range of distances, one may represent the low-field behavior by employing the expansion[6] of $\delta\rho(\mathbf{r})$ in powers of $\hbar\omega_c/\zeta$ mentioned above, since the mixed parameter is small (and its occurrence as an expansion parameter is unobjectionable). Of course, if $\hbar\omega_c/\zeta$ is made sufficiently small, this limited range of distances can be made arbitrarily large—indeed, sufficiently large so that the even larger distances where low-field behavior akin to the zero-field limit does not exist (because of the largeness of the mixed parameter at such large distances) are physically unimportant for the range over which this aspect of response is measured.

The limitations of the approximation under consideration here obscure another important aspect of response which is emphasized by the self-consistency of the random-phase approximation (RPA), namely, the Debye-Thomas-Fermi (DTF) shielding pole[2] at $p_D\cong(4\pi e^2\partial\rho/\partial\zeta)^{1/2}$. That is to say, an improved treatment of the problem using the RPA[2] introduces a distinct contribution to $\delta\rho(\mathbf{r})$ from the vicinity of the DTF pole, and it decays exponentially like $\sim e^{-p_D r}$. If we compare this with the contribution from the vicinity of the logarthmic branch points $\sim p_F=(2\zeta/\hbar\omega_p)p_D\gg p_D$, which we have shown to decay exponentially like $e^{-p_F r}$, then it is clear that the DTF pole contribution is important and likely to be dominant at high density $[\zeta/\hbar\omega_p\sim(a_o/r_o)^{1/2}\sim r_s^{-1/2}\gg 1]$. This is also illuminating in regard to low-field behavior and the zero limit: It indicates that at the "even larger" distances where the contribution from the vicinity of the logarithmic branch points displays no low-field behavior akin to the zero-field limit, this contribution is in fact negligible compared to the DTF pole contribution, which does exhibit appropriate low-field behavior and zero-field limit.

IV. CONCLUSIONS

We have analyzed the density perturbation $\delta\rho(\mathbf{r})$ about a Coulombic impurity in strong magnetic fields in the degenerate case. The results show that anisotropic Friedel oscillatory behavior is discernible in the r_z direction, but that it is deprived of long-range character by exponential decay in the $\bar{r}$ direction. The anisotropic Friedel oscillatory behavior of $\delta\rho(\mathbf{r})$ in the r_z direction is in good accord with the results of Rensink.[3] However, the exponential decay of $\delta\rho(\mathbf{r})$ in the $\bar{r}$ direction for a Coulombic impurity stands in contrast to the Gaussian decay of $\delta\rho(\mathbf{r})$ in the $\bar{r}$ direction for a δ-function impurity found by Rensink, indicating that the qualitative nature of the spatial decay in the transverse direction is dependent upon the character of the impurity potential.

The results obtained here are valid at large distances from the impurity for a wide range of field strengths, including rather low magnetic fields ($\hbar\omega_c<\zeta$), as well as the quantum strong-field limit ($\hbar\omega_c>\zeta$). A qualitative discussion of low-field behavior in the approach to the zero-field limit has been presented, and the role of self-consistent aspects of response to be expected from a RPA analysis has been indicated.

APPENDIX

In this Appendix we shall derive the appropriate expression for $I(\mathbf{p},\Omega=0+i\epsilon)$ in the case of degeneracy at arbitrary magnetic field strength. It is illuminating to calculate $I(\mathbf{p},\Omega+i\epsilon)$ at arbitrary frequency, since its imaginary part yields the plasmon dispersion relation, and its real part yields plasmon damping (see Ref. 1). Of course, the zero-frequency limit must then be taken to give the density perturbation, and static shielding as well. Proceeding from the results obtained in Ref. 1, one has

$$\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)=\int_0^\infty d\omega\frac{f_0(\omega)}{\hbar^3}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{s\omega}\frac{\pi^{3/2}}{(2\pi)^3}\times\left(\frac{2m}{s}\right)^{1/2}\frac{m\hbar\omega_c}{\tanh(\frac{1}{2}\hbar\omega_c s)}\aleph,\quad\text{(A1)}$$

[6] In this connection it should be noted that an attempt to represent magnetic field corrections in this region by expanding $\delta\rho(\mathbf{r})$ in powers of the expansion parameter involving $\hbar w_c/\zeta$ (and the "mixed" parameter) must include a careful accounting of the Lighthill expansion of the exact zero-field limit to an appropriate order in $(p_F r)^{-1}$, such that the approximation thereby constructed is consistent in regard to the magnitudes of the terms retained from the two distinct expansions.

where

$$\aleph=\frac{2i}{\hbar}\int_0^\infty dT\, e^{-i\Omega T}\exp\left(\frac{-\hbar^2p_z^2s}{8m}\right)\times\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\coth(\tfrac{1}{2}\hbar\omega_cs)\right)\left[\exp\left(\frac{-p^2_z}{8ms}(2T-i\hbar s)^2\right)\times\exp\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\frac{\cos[\frac{1}{2}\omega_c(2T-i\hbar s)]}{\sinh\frac{1}{2}\hbar\omega_cs}\right)-(i\to -i)\right]. \quad \text{(A2)}$$

The corresponding plasmon dispersion relation has simple poles at all integral multiples of the cyclotron frequency in the special case of propagation **p** perpendicular to the magnetic field. A representation for arbitrary **p** direction which explicitly corresponds to this may be obtained by expanding the factor

$$e^{y\cos\phi}=\sum_{n=-\infty}^{\infty}e^{in\phi}I_n(y)$$

in terms of modified Bessel functions I_n (BHTF II, p. 7, No. 27).[5] Thus we have

$$\aleph=\frac{2i}{\hbar}\exp\left(\frac{-\hbar^2p_z^2s}{8m}\right)\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\coth\tfrac{1}{2}\hbar\omega_cs\right)\sum_{n=-\infty}^{\infty}I_n\left(\frac{\hbar\bar{p}^2}{2m\omega_c\sinh\frac{1}{2}\hbar\omega_cs}\right)\times\int_0^\infty dT\,e^{-i\Omega T}\left[\exp\left(\frac{-p_z^2}{8ms}(2T-i\hbar s)^2\right)\exp[in(\omega_cT-\tfrac{1}{2}i\hbar\omega_cs)]-(\hbar\to-\hbar)\right]. \quad \text{(A3)}$$

The T integral is readily recognized as a complementary error function (erfc). Noting that $I_{-n}=I_n$, one then obtains

$$\aleph=\frac{i\pi^{1/2}}{\hbar}\left(\frac{2ms}{p_z^2}\right)^{1/2}\exp\left(\frac{-\hbar^2p_z^2s}{8m}\right)\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\coth\tfrac{1}{2}\hbar\omega_cs\right)\sum_{n=0}^{\infty}{}'\,I_n(\hbar\bar{p}^2/2m\omega_c\sinh(\tfrac{1}{2}\hbar\omega_cs))$$
$$\times\{e^{\hbar\Omega s/2}\exp[-m(\Omega-n\omega_c)^2s/2p_z^2]\operatorname{erfc}[i(m(\Omega-n\omega_c)/p_z^2-\tfrac{1}{2}\hbar)(p_z^2s/2m)^{1/2}]-(\hbar\to-\hbar)$$
$$+e^{\hbar\Omega s/2}\exp[-m(\Omega+n\omega_c)^2s/2p_z^2]\operatorname{erfc}[i(m(\Omega+n\omega_c)/p_z^2-\tfrac{1}{2}\hbar)(p_z^2s/2m)^{1/2}]-(\hbar\to-\hbar)\}. \quad \text{(A4)}$$

The prime on the summation sign indicates that the $n=0$ term is double-counted and should be divided by 2. For strong magnetic fields ($\hbar\omega_c\sim\zeta$, $\hbar\omega_c\beta\gg1$) it is appropriate to expand the part of $\aleph$ involving hyperbolic functions in a series of powers of $e^{-\hbar\omega_cs/2}$. Such an expansion may be obtained by using the identity[7]

$$\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\coth\tfrac{1}{2}\hbar\omega_cs\right)I_n(\hbar\bar{p}^2/2m\omega_c\sinh\tfrac{1}{2}\hbar\omega_cs)=2\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\times\exp(-\hbar\bar{p}^2/2m\omega_c)\sinh\tfrac{1}{2}\hbar\omega_cs\sum_{r=0}^{\infty}\frac{r!}{(n+r)!}[L_r^n(\hbar\bar{p}^2/2m\omega_c)]^2\exp[-\tfrac{1}{2}(n+2r+1)\hbar\omega_cs]$$

(L_r^n denotes the Laguerre polynomial). The resulting expression for $\aleph$ may be substituted into Eq. (A1), and one then finds that the s integral of (A1) is given by

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{s\omega}\frac{\pi^{3/2}}{(2\pi)^3}\left(\frac{2m}{s}\right)^{1/2}\frac{m\hbar\omega_c}{\tanh\frac{1}{2}\hbar\omega_cs}\aleph=\frac{i}{4\pi}\frac{m^2\omega_c}{p_z}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm'}\sum_{\pm}\frac{r!}{(n+r)!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\exp(-\hbar\bar{p}^2/2m\omega_c)[L_r^n(\hbar\bar{p}^2/2m\omega_c)]^2$$
$$\times\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{\omega s}\exp\{[\pm\tfrac{1}{2}\hbar\omega_c-(n+2r+1)\tfrac{1}{2}\hbar\omega_c-\hbar^2p_z^2/8m-m(\Omega-\pm'n\omega_c)^2/2p_z^2]s\}$$
$$\times\{e^{\hbar\Omega s/2}\operatorname{erfc}[i(m(\Omega-\pm'n\omega_c)/p_z^2-\tfrac{1}{2}\hbar)(p_z^2s/2m)^{1/2}]-(\hbar\to-\hbar)\}. \quad \text{(A5)}$$

(In this equation there are two independent summations over the two possibilities of signature $\pm$: On the one hand, $\sum_\pm$ arises from the two possibilities of signature associated with spin; on the other hand, $\sum_{\pm'}$ arises in connection with folding the summation from $n=-\infty$ to ∞ into one from 0 to ∞. In so doing, the $n=0$ term is double-counted and must be divided by 2; the summation sign carries a prime to indicate this.) The s integral is thus seen to be the inverse Laplace transform of a complementary error function, and this may be evaluated in terms of

[7] Watson, J. London Math. Soc. 8, 189 (1938).

elementary functions (BIT I, p. 266, No. 12).[5] Thus, with a bit of manipulation one obtains the result

$$\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)=-\frac{m^2\omega_c}{4\pi^2p_z}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\right)\frac{r!}{(n+r)!}\left[L_r^n\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\right]^2\sum_{\pm',\pm}\frac{1}{2\hbar^3}\int_0^\infty d\omega'\,f_0(\omega'+a_{rn}(\pm\pm'))$$
$$\times\{[\omega'-(m\omega'/2p_z^2)^{1/2}\Theta_n^{\pm'}]^{-1}-[\omega'+(m\omega'/2p_z^2)^{1/2}\Theta_n^{\pm'}]^{-1}\}+(\Omega\to-\Omega),\quad\text{(A6)}$$

where

$$-a_{rn}(\pm\pm')=[\pm1\pm'n-(n+2r+1)]\tfrac{1}{2}\hbar\omega_c\quad\text{(A7a)}$$

and

$$\Theta_n^{\pm'}=\Omega-\pm'n\omega_c-\hbar p_z^2/2m.\quad\text{(A7b)}$$

This is equivalent to the result obtained by Stephen.[8] It should be noted that in the degenerate case the ω' integral is of the form (η_+ is a cutoff function)

$$\int_0^\infty d\omega'\,f_0(\omega'+a_{rn}(\pm\pm'))\cdots\to\eta_+(\zeta-a_{rn}(\pm\pm'))\int_0^{\zeta-a_{rn}(\pm\pm')}d\omega'\cdots.$$

Thus, in the degenerate case the individual terms of the series (A6) have the role of the chemical potential played by $\zeta-a_{rn}(\pm\pm')$ instead of ζ, and are accompanied by a cutoff factor $\eta_+(\zeta-a_{rn}(\pm\pm'))$ which cuts off the series in the strong-field case. In fact, in the quantum strong-field limit one has $\hbar\omega_c>\zeta$ as well as $\hbar\omega_c\beta\gg1$, and only one term of

$$\sum_{r=0}^{\infty}\sum_{\pm',\pm}$$

survives the cutoff prescription. [This convenience is the reason for introducing the series expansion $\sum_r$ after Eq. (A4).] If the temperature is not exactly zero in the degenerate case, the terms which are "cut off" do not vanish entirely, but are small, being proportional to the small factor $\sim\exp\{[\zeta-a_{rn}(\pm\pm')]\beta\}$, because the ω' integral is of the form

$$\int_0^\infty d\omega'\,f_0(\omega'+a_{rn}(\pm\pm'))\cdots.$$

One can amplify on the remarks above by performing the ω' integration of (A6) in the degenerate case. The integral is elementary,[9] and the result is given by

$$\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)=\frac{m^2\omega_c}{4\pi^2p_z}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\right)\frac{r!}{(n+r)!}\left[L_r^n\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\right]^2$$
$$\times\sum_{\pm',\pm}\frac{1}{\hbar^3}\eta_+(\zeta-a_{rn}(\pm\pm'))\ln\left|\frac{\Theta_n^{\pm'}+(2p_z^2/m)^{1/2}[\zeta-a_{rn}(\pm\pm')]^{1/2}}{\Theta_n^{\pm'}-(2p_z^2/m)^{1/2}[\zeta-a_{rn}(\pm\pm')]^{1/2}}\right|+(\Omega\to-\Omega).\quad\text{(A8)}$$

In the quantum strong-field limit ($\hbar\omega_c>\zeta$) the cutoff function may be taken as $\eta_+(\zeta-a_{rn}(\pm,\pm'))\sim\delta(\pm'=+)\times\delta(\pm=+)\delta(r=0)$ (where the δ functions are Kronecker δ's). Noting that $L_0^n(x)\equiv1$, one finds

$$\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)=\frac{\rho_0}{2\hbar}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2}\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\left(\ln\left|\frac{\Omega-n\omega_c-\hbar p_z^2/2m+(2p_z^2\zeta/m)^{1/2}}{\Omega-n\omega_c-\hbar p_z^2/2m-(2p_z^2\zeta/m)^{1/2}}\right|+(\Omega\to-\Omega)\right).\quad\text{(A9)}$$

[The sum $\sum_{n=0}^{\infty}$ is now unprimed, since the double counting of the $n=0$ term was automatically removed by understanding the $\delta(\pm'=+)$ function to be meaningful for the $n=0$ term as well as for all the nonvanishing n terms.] The appropriate expression for ρ_0 in the quantum strong-field limit is given by

$$\rho_0=m^{3/2}\omega_c\zeta^{1/2}/2^{1/2}\pi^2\hbar^2,\quad\text{(A10a)}$$

and this is a specialization of the more general expression for ρ_0 in the degenerate case:

$$\rho_0=(m^{3/2}\omega_c/2^{1/2}\pi^2\hbar^2)\sum_{\pm}\sum_{r=0}^{\infty}[\zeta\pm\tfrac{1}{2}\hbar\omega_c-(r+\tfrac{1}{2})\hbar\omega_c]^{1/2}\times\eta_+(\zeta\pm\tfrac{1}{2}\hbar\omega_c-(r+\tfrac{1}{2})\hbar\omega_c).\quad\text{(A10b)}$$

Although we are concerned with static properties here, it is of interest to note explicitly the role of $I(\mathbf{p},\Omega+i\epsilon)$ in the determination of the plasmon spectrum.[1] The plasmon dispersion relation is given by

$$1-(4\pi e^2/p^2)\,\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)=0.\quad\text{(A11)}$$

The relative importance of a root $\Omega(\mathbf{p})$ of the plasmon dispersion relation (A11) in response to excitation is measured by the amplitude weight function $Z(\Omega(\mathbf{p}))$, which is given by

$$Z(\Omega(\mathbf{p}))=-\left[\frac{d}{d\Omega}\left(\frac{4\pi e^2}{p^2}\mathrm{Im}I(\mathbf{p},\Omega+i\epsilon)\right)\right]^{-1}_{\Omega=\Omega(\mathbf{p})}.\quad\text{(A12)}$$

[8] M. J. Stephen, Phys. Rev. **129**, 997 (1963).

[9] H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (The Macmillan Co., New York, 1961), 4th ed., No. 188.11.

The damping $\gamma(\Omega(\mathbf{p}))$ of a mode or resonance which arises as a root of the dispersion relation may be expressed as $\gamma=Z\Gamma$, where

$$\Gamma(\mathbf{p},\Omega)=(8\pi e^2/p^2)\,\mathrm{Re}I(\mathbf{p},\,\Omega+i\epsilon)\,. \tag{A13}$$

Proceeding from the results obtained in Ref. 1, one has

$$\mathrm{Re}I(\mathbf{p},\,\Omega+i\epsilon)=\left(\frac{m}{2\pi}\right)^{3/2}\tfrac{1}{2}\omega_c\int_{-\infty}^{\infty}dy\,e^{-i\Omega y/2}\int_0^{\infty}d\omega\frac{f_0(\omega)}{\hbar^3}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{\omega s}$$
$$\times s^{-1/2}\frac{\sinh(\frac{1}{2}\hbar\Omega s)}{\tanh(\frac{1}{2}\hbar\omega_c s)}\exp\left(\frac{-p_z^2}{8ms}(y^2+\hbar^2s^2)\right)\exp\left(\frac{-\hbar\bar{p}^2\,(\cosh\frac{1}{2}\hbar\omega_c s-\cos\frac{1}{2}\omega_c y)}{2m\omega_c\sinh\frac{1}{2}\hbar\omega_c s}\right). \tag{A14}$$

Once again one may introduce an expansion in terms of modified Bessel functions $I_n(\sum_n)$, and then expand the latter in a series of the squares of Laguerre polynomials $L_r^n(\sum_r)$. This results in

$$\mathrm{Re}I(\mathbf{p},\,\Omega+i\epsilon)=(m/2\pi)^{3/2}\omega_c\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm'}\int_0^{\infty}d\omega\frac{f_0(\omega)}{\hbar^3}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{\omega s}$$
$$\times s^{-1/2}\sinh(\tfrac{1}{2}\hbar\Omega s)\cosh(\tfrac{1}{2}\hbar\omega_c s)\exp(-\hbar^2p_z^2s/8m)(\hbar\bar{p}^2/2m\omega_c)^n\exp(-\hbar\bar{p}^2/2m\omega_c)$$
$$\times\frac{r!}{(n+r)!}\left[L_r^n\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\right]^2\exp[-(n+2r+1)\tfrac{1}{2}\hbar\omega_c s]\int_{-\infty}^{\infty}dy\,\exp[-i\tfrac{1}{2}(\Omega-\pm'n\omega_c)y-p_z^2y^2/8ms]. \tag{A15}$$

The y integration is simply evaluated as the Fourier transform of a Gaussian, with the result

$$\int_{-\infty}^{+\infty}dy\cdots=(8\pi ms/p_z^2)^{1/2}\exp[-(\pm'n\omega_c-\Omega)^2ms/2p_z^2].$$

From this it is readily found that the s integration gives rise to $\delta(\omega-a_{rn}(\pm\pm')-m(\Theta_n^{\pm'})^2/2p_z^2)$ functions, so that the ensuing ω integration is trivial. One finally obtains

$$\mathrm{Re}I(\mathbf{p},\,\Omega+i\epsilon)=(m/2\pi)^{3/2}\omega_c(8\pi m/p_z^2)^{1/2}\sum_{n=0}^{\infty}{}'\sum_{r=0}^{\infty}\sum_{\pm}\sum_{\pm'}\frac{r!}{(n+r)!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n$$
$$\times\exp\left(-\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\left[L_r^n\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)\right]^2\left(\frac{1}{4\hbar^3}f_0(a_{rn}(\pm\pm')+m(\Theta_n^{\pm'})^2/2p_z^2)-(\Omega\to-\Omega)\right). \tag{A16}$$

In the degenerate case we have

$$f_0(a_{rn}(\pm\pm')+\cdots)\to\eta_+(\zeta-a_{rn}(\pm\pm')-\cdots)\,,$$

and the same sort of cutoff prescription that follows in the strong-field case above occurs here also. Specifically, the quantum strong-field limit for (A16) is given by the formula

$$\mathrm{Re}I(\mathbf{p},\,\Omega+i\epsilon)=\frac{\pi p_0}{2\hbar}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2}\exp\left(\frac{-\hbar\bar{p}^2}{2m\omega_c}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n\eta_+(\zeta-(\Omega-n\omega_c-\hbar p_z^2/2m)^2m/2p_z^2)-(\Omega\to-\Omega)\,. \tag{A17}$$

As indicated above, the zero-frequency limit $I(\mathbf{p},\,\Omega=0+i\epsilon)$ is necessary for a description of the static density perturbation and/or shielding. In the Introduction we noted that $\mathrm{Re}I(\mathbf{p},\,\Omega=0+i\epsilon)\equiv0$, which onecan see by direct inspection of Eqs. (A14)–(A17). Moreover, $\mathrm{Im}I(\mathbf{p},\,\Omega=0+i\epsilon)$ is readily obtained fromEqs. (A8) and (A9) where

$$\Theta_n^{\pm'}\to-\pm'n\omega_c-\hbar p_z^2/2m\,,$$

since $\Omega\to0$. This yields the results employed in the main body of this paper.