Relaxation and Electromagnetic Response of a Superconductor in a State of Supercurrent Flow

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The situation is investigated in which a superconductor in a state of supercurrent flow and near the transition temperature is exposed to an alternating electric field. The induced alternating current is shown to exhibit a frequency dependence $(1-i\Omega\tau_R)^{-1}$, which is due to a relaxation process in the Cooper-pair density. The relaxation time τ_R is determined by electron-phonon collisions (collision time τ_c), in which case $\tau_R \simeq \tau_c[T_c/(T_c-T)]^{1/2}$; impurity scattering is important only under certain conditions, in which case $\tau_R \simeq 10^{-11}[T_c/(T_c-T)]^{-5}$ sec.

I. INTRODUCTION

TIME-DEPENDENT processes in superconductors which affect the density of Copper pairs may occur under two quite different conditions. In one case, the superconductor is gapless. This is realized, for instance, in the Abrikosov mixed state[1]; where vortex motion is the prominent time-dependent process.[2] In the other case, the superconductor has a finite energy gap; and as a consequence, energy conservation severely restricts changes in the density of Cooper pairs.

A time-dependent process of the last kind has been realized by Gittleman and co-workers,[3] in their experimental work on the hf conductivity of a superconducting filament as a function of a large dc through the filament. Since the density of Cooper pairs decreases with increasing momentum of the pairs (depairing effect), the hf conductivity due to accelerated supercurrents has to decrease with increasing dc. However, the exact relationship depends on the frequency Ω. If Ω is sufficiently small, the density of Cooper pairs will change in time corresponding to the hf part of the current, and it will not change if Ω is very large. In a formal way, these two cases are distinguished by $\Omega\tau_R \ll 1$ and $\Omega\tau_R \gg 1$, respectively, where τ_R is the relaxation time of the considered process.

In the experiment mentioned above, the frequency (23 kMc/sec) could not be varied. In analyzing their data, the authors chose the case $\Omega\tau_R \gg 1$, i.e., $\tau_R \ll 10^{-11}$ sec, and obtained consistent results (the criterion was the magnitude of the penetration depth).

Theoretical calculations,[4–6] however, result in much larger values of τ_R. Though there are discrepancies among these calculations,[7] the general trend is clear and results from the common assumption that inelastic electron-phonon collisions furnish the energy required to break up Cooper pairs. The corresponding collision time τ_c is, even at higher temperatures ($T \lesssim T_c$), seldom much smaller than 10^{-9} sec.

This discrepancy has led the author to investigate more closely the case in which superconducting films in a current-carrying state are exposed to a small homogeneous hf electric field. Only the vicinity of the transition temperature ($T_c - T \ll T_c$) has been studied. It is found that for clean samples (no residual resistance), the relaxation time τ_R is essentially the same as given in a previous paper,[6]

$$\tau_R \simeq \tau_c[T_c/(T_c-T)]^{1/2}.$$

Impurity scattering does not change this result if the dc is zero (or sufficiently small). This reflects the fact that impurity scattering is elastic. However, a peculiar situation develops with larger dc. One obtains here a finite relaxation time even without electron-phonon scattering, under favorable circumstances (largest dc), as small as

$$\tau_R \simeq 10^{-11}[T_c/(T_c-T)]^5 \text{ sec.}$$

In order to indicate how this particular result arises, let us consider a clean metal in a current-carrying state. Here, the quasiparticle energy depends on the direction, such that the kinetic energy and the binding energy of the electronic system is no longer constant on a surface of constant energy of a quasiparticle. If a quasiparticle scatters at an impurity, a transition between states of different binding energy takes place. Although the difference in binding energy is quite small, a series

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[1] Another example is provided by a superconductor with paramagnetic impurities of sufficient concentration. This situation has recently been investigated by L. P. Gorkov and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **54**, 612 (1968) [English transl.: Soviet Phys.—JETP **27**, 328 (1968)].

[2] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **139**, A1163 (1965); J. Bardeen and M. J. Stephen, *ibid.* **140**, A1197 (1965); M. Tinkham, Phys. Rev. Letters **13**, 804 (1964); A. Schmid, Physik Kondensierten Materie **5**, 302 (1966); C. Caroli and K. Maki, Phys. Rev. **164**, 591 (1967).

[3] J. Gittleman, B. Rosenblum, T. E. Seidel, and A. W. Wicklund, Phys. Rev. **137**, A527 (1965).

[4] G. Lucas and M. J. Stephen, Phys. Rev. **154**, 349 (1967).

[5] J. W. F. Woo and E. Abrahams, Phys. Rev. **169**, 407 (1968).

[6] A. Schmid, Physik Kondensierten Materie 8, 129 (1968). This paper is the basis of the present investigations.

[7] The treatment of Ref. 4 has been critized in Ref. 5. The calculations of Ref. 5 and Ref. 6 share a common starting point. The author feels that the difference in the results is due, at least partially, to an ansatz in Ref. 5 for the electronic self-energy which is not adequate for temperatures close to T_c. Cf. Ref. 19.

of successive scatterings add up to the energy required to break a Cooper pair.

This discussion shows that, in the vicinity of the transition temperature, there is a considerable disagreement between calculated values of τ_R and those found by experiment.

In the following investigation, we will first, in Sec. II, analyze phenomenologically the experiment of Gittleman and co-workers. The microscopic analysis is based on relations introduced by Eliashberg[8] in the case of electron-phonon interaction and by Abrikosov and Gorkov[9] in the case of impurity scattering; these relations will be presented in Sec. III. In Sec. IV, the equilibrium Green's function of a current-carrying state will be discussed. The relations which determine the self-consistent response to an external electric field will be derived in Sec. V, and approximations will be introduced according to arguments given in the previous paper.[6] Sections VI and VII contain the results and discussion in the clean and the dirty case, respectively.

II. PHENOMENOLOGICAL ANALYSIS

In a state of homogeneous supercurrent flow,[10] the Ginsburg-Landau wave function Ψ varies as $e^{(i/\hbar)\mathbf{q}\cdot\mathbf{r}}$, where $\mathbf{q}$ is the c.m. momentum of the Cooper pairs. Their density $\rho_s=|\Psi|^2$ and their current density $\mathbf{j}_s=(e/m)\rho_s\mathbf{q}$ are found to be equal to

$$\rho_s=(1/\beta)(\alpha-q^2/2m),$$
$$\mathbf{j}_s=(2e/m\beta)(\alpha-q^2/2m)\mathbf{q}, \tag{2.1}$$

where α and β are the Ginsburg-Landau parameters. The current is maximal for $q_m^2=\frac{2}{3}m\alpha$. Note that q_m cannot be surpassed in flow experiments.[11]

Consider the case of an alternating electric field $\mathbf{E}\propto e^{-i\Omega t}$. We will assume that the acceleration law of free particles holds also for the c.m. of the Cooper pairs. Therefore

$$\dot{\mathbf{q}}=2e\mathbf{E} \tag{2.2}$$

and

$$\mathbf{q}=\mathbf{q}_0+(2e/-i\Omega)\mathbf{E}, \tag{2.3}$$

where $\mathbf{q}_0$ is the momentum corresponding to the dc. The change $\delta\mathbf{q}=(2e/-i\Omega)\mathbf{E}$ causes a change in ρ_s, $\delta\rho_s=-(1/\beta Vm)\mathbf{q}_0\cdot\delta\mathbf{q}$, provided the frequency Ω is low enough to allow ρ_s to follow freely. For high frequencies however, $\delta\rho_s=0$. The simplest expression which interpolates between the two limiting cases, and which is a holomorphic function of Ω in the upper complex Ω half-plane (causal response), is given as follows:

$$\delta\rho_s=-i(2e/\beta m\Omega)\mathbf{q}_0\cdot\mathbf{E}(1/1-i\Omega\tau_R). \tag{2.4}$$

Here, τ_R is the relaxation time introduced in Sec. I. In the following, we assume $\mathbf{q}_0\|\mathbf{E}$, since this is the most advantageous situation.

The alternating part of the current is proportional to $\delta\rho_s\mathbf{q}_0+\rho_s\delta\mathbf{q}$. From this, we find the contribution σ_s to the hf conductivity

$$\frac{\sigma_s}{\sigma_{s0}}=1-\frac{q_0^2}{3q_m^2}\left(1+\frac{2}{1-i\Omega\tau_R}\right),$$
$$\sigma_{s0}=i\frac{e^2\rho_{s0}}{m}\frac{1}{\Omega}=i\frac{c^2}{4\pi\lambda^2}\frac{1}{\Omega}, \tag{2.5}$$

where λ is London's penetration depth. In the case $\Omega\tau_R\ll1$, the value of σ_s decreases from σ_{s0} at $q_0=0$ to zero at $q_0=q_m$; whereas, in the case $\Omega\tau_R\gg1$, the value of σ_s decreases from σ_{s0} to $\frac{2}{3}\sigma_{s0}$.

There are, of course, other contributions to the conductivity, e.g., the contribution of normal electrons. Fortunately, their frequency dependence is weak and their relative magnitude is small in most cases.

III. BASIC EQUATIONS

The case of a superconductor with electron-phonon interaction as well as impurity scattering has already been considered by Tsuneto,[12] who combined the results of Eliashberg[8] and Abrikosov and Gorkov.[9] Accordingly, the self-energy $\hat{\Sigma}$ of the electron Green's function $\hat{G}$ (the caret indicates 2×2 matrices as introduced by Nambu[13]) is the sum of a phonon and an impurity contribution,

$$\hat{\Sigma}=\hat{\Sigma}^{\text{ph}}+\hat{\Sigma}^{\text{imp}}. \tag{3.1}$$

The phonon contribution is defined by the following equation:

$$\hat{\Sigma}^{\text{ph}}(1,1')=-D(1,1')\hat{\tau}_3\hat{G}(1,1')\hat{\tau}_3, \tag{3.2}$$

where $D(1,1')$ is the phonon Green's function, and $\hat{\tau}_l$, $l=1, 2, 3$, are the Pauli matrices. Coordinate-temperature representation[14] $(1=\mathbf{r}_1,\tau_1)$ has been used in Eq. (3.2).

According to Eliashberg, $D(1,1')$ is practically unchanged by the transition into the superconducting state. From this we draw the conclusion that it will be affected neither by space- nor time-dependent processes in the superconducting system. This allows us to replace $D(1,1')$ by the equilibrium Green's function $D(1-1')$. Introducing the c.m. coordinate $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_{1'})$, and Fourier transforming with respect to the difference $\mathbf{r}_1-\mathbf{r}_{1'}$, we obtain from Eq. (3.2)

$$\hat{\Sigma}^{\text{ph}}(\mathbf{p};\mathbf{R};\tau,\tau')=-\int\frac{d^3p'}{(2\pi)^3}D(\mathbf{p}-\mathbf{p}';\tau-\tau')\hat{\tau}_3\hat{G}(\mathbf{p}';\mathbf{R};\tau,\tau')\hat{\tau}_3. \tag{3.3}$$

[8] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **38**, 966 (1960) [English transl.: Soviet Phys.—JETP **11**, 696 (1960)].
[9] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **35**, 1558 (1958) [English transl.: Soviet Phys.—JETP **8**, 1090 (1959)].
[10] J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).
[11] A. Schmid, J. Low Temp. Phys. **1**, 11 (1969).
[12] T. Tsuneto, Progr. Theoret. Phys. (Kyoto) **28**, 857 (1962).
[13] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[14] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon Press, Inc., New York, 1965).

On the basis of Migdal's[15] arguments, Eliashberg has shown that the most important region of the integral is where $|\mathbf{p}'|$ is close to the Fermi momentum p_0. Keeping this in mind, and averaging over the directions of $\mathbf{p}'$, we find that $D(\mathbf{p}-\mathbf{p}'; \tau-\tau')$ can be replaced by

$$B(\tau-\tau')=\frac{1}{2p_0^2}\int_0^{2p_0} dp\, pD(p, \tau-\tau'). \quad (3.4)$$

Then,

$$\hat{\Sigma}^{\text{ph}}(p\simeq p_0; \mathbf{R}; \tau,\tau')=-B(\tau-\tau')\hat{\tau}_3\hat{G}(\mathbf{R},\mathbf{R}; \tau,\tau')\hat{\tau}_3. \quad (3.5)$$

It should be noted that in a state of supercurrent flow, the Green's function $G(\mathbf{p}'; \mathbf{R}; \tau,\tau')$ will depend on the direction of $\mathbf{p}'$. In this case, the transition from Eq. (3.3) to Eq. (3.5) involves an additional approximation which corresponds to neglecting the spatial range of the electron-phonon interaction. No inconsistencies have been found using this approximation.[16]

The impurity contribution to the self-energy is of a similar structure to the one discussed above. The difference is connected with the fact that impurity scattering is elastic. Therefore, in the relation corresponding to Eq. (3.5), the temperature-dependent function $-B(\tau-\tau')$ is replaced by a constant:

$$\begin{aligned}\hat{\Sigma}^{\text{imp}}(p\simeq p_0; \mathbf{R}; \tau,\tau')&=\frac{1}{2\pi N_0\tau_{\text{i}}}\hat{\tau}_3\hat{G}(\mathbf{R},\mathbf{R}; \tau,\tau')\hat{\tau}_3,\\ \frac{1}{2\pi N_0\tau_{\text{i}}}&=\frac{n_{\text{i}}}{2p_0^2}\int_0^{2p_0} dp\, p|u(p)|^2.\end{aligned} \quad (3.6)$$

Here, n_{i} is the density of the impurities, $u(p)$ the Fourier transform of their potential, and N_0 the density of states. Any effects resulting from an anisotropy of $\hat{G}(\mathbf{p},\mathbf{R}; \tau,\tau')$ have been neglected; this is equivalent to the assumption of isotropic scattering (i.e., we do not distinguish between the collision time τ_{i} and the transport time τ_{tr}).

Equations (3.1), (3.5), and (3.6) together with the constituent relation

$$\hat{G}^{-1}=\hat{G}_{\text{ni}}^{-1}-\hat{\Sigma}, \quad (3.7)$$

where $\hat{G}_{\text{ni}}$ is the Green's function of noninteracting particles, form the basic set of equations.

IV. EQUILIBRIUM GREEN'S FUNCTIONS IN A CURRENT-CARRYING STATE

We assume a homogeneous current-carrying state, in which the c.m. of the Cooper pairs has a finite momentum $\mathbf{q}$. Consequently, the elements $\Sigma_{12}(\mathbf{R}; \tau,\tau')$ and $\Sigma_{21}(\mathbf{R}; \tau,\tau')$ of the self-energy (the order parameter and its complex conjugate if $\tau=\tau'$) depend on the space variable as $e^{i\mathbf{q}\cdot\mathbf{R}}$ and $e^{-i\mathbf{q}\cdot\mathbf{R}}$, respectively. In such a case, it is convenient to introduce new (primed) quantities by means of a unitary transformation:

$$\begin{aligned}\hat{G}'(\mathbf{r},\mathbf{r}'; \tau,\tau')&=\hat{U}^\dagger(\mathbf{r})\hat{G}(\mathbf{r},\mathbf{r}'; \tau,\tau')\hat{U}(\mathbf{r}'),\\ \hat{\Sigma}'(\mathbf{R}; \tau,\tau')&=\hat{U}^\dagger(\mathbf{R})\hat{\Sigma}(\mathbf{R}; \tau,\tau')\hat{U}(\mathbf{R}),\end{aligned} \quad (4.1)$$

where

$$\hat{U}(\mathbf{r})=\begin{pmatrix} e^{i\mathbf{q}\cdot\mathbf{r}/2} & 0 \\ 0 & e^{-i\mathbf{q}\cdot\mathbf{r}/2}\end{pmatrix}. \quad (4.2)$$

As a result of this transformation, the self-energy is independent of the space coordinates, and the Green's function depends only on the difference of the coordinates. Thus, the Fourier transforms take the simple form $\hat{\Sigma}'(\omega_\nu)$ and $\hat{G}'(\mathbf{p}; \omega_\nu)$, where $\omega_\nu=\pi T(2\nu+1)$ are the odd Matsubara frequencies.

The equations of Sec. III are covariant with respect to the transformations (4.1) and (4.2). Note that in $(\hat{G}'_{\text{ni}})^{-1}$, the differentiation $(1/i)\nabla$ is replaced by $(1/i)\nabla \pm\frac{1}{2}\mathbf{q}$. Therefore, the Fourier transform

$$[\hat{G}'_{\text{ni}}(\mathbf{p}; \omega_\nu)]^{-1}=(i\omega_\nu-sx)\hat{1}-\epsilon\hat{\tau}_3, \quad (4.3)$$

where

$$\epsilon=\mathbf{p}^2/2m-\mu, \quad s=p_0q/2m=v_0q/2,$$

and

$$x=\cos\sphericalangle(\mathbf{p},\mathbf{q}).$$

The term proportional to $\mathbf{q}^2$ has been absorbed in the definition of μ. The Green's function in the superconducting state is, according to Eq. (3.7), of the following general structure:

$$\hat{G}'(\mathbf{p}; \omega_\nu)=\frac{(i\omega_\nu-\Sigma_{11}-sx)\hat{1}+\epsilon\hat{\tau}_3+\Sigma_{12}\hat{\tau}_1}{(i\omega_\nu-\Sigma_{11}-sx)^2-\epsilon^2-\Sigma_{12}^2}, \quad (4.4)$$

where the symmetry $\Sigma_{11}=\Sigma_{22}$, $\Sigma_{12}=\Sigma_{21}$ has been used. Note that Σ_{11} and Σ_{12} are odd and even functions of ω_ν, respectively.

States of supercurrent flow in the presence of large impurity scattering have been extensively studied by Maki.[17] In order to facilitate a comparison, we introduce the notations

$$\begin{aligned}i\tilde{\omega}_\nu&=i\omega_\nu-\Sigma_{11}(\omega_\nu), & i\bar{\omega}_\nu&=i\omega_\nu-\Sigma_{11}^{\text{ph}}(\omega_\nu),\\ \tilde{\Delta}&=\Sigma_{12}(\omega_\nu), & \Delta(\omega_\nu)&=\Sigma_{12}^{\text{ph}}(\omega_\nu),\end{aligned} \quad (4.5)$$

and the connection is made by identifying $i\tilde{\omega}_\nu$ with Maki's ω. Therefore,

$$\begin{aligned}&\tilde{\omega}_\nu/\tilde{\Delta}=\sinh\phi\,\cos\chi, \quad s/\tilde{\Delta}=\cosh\phi\,\sin\chi,\\ &\Delta/\tilde{\Delta}=(1-\Delta\tau_i\zeta)\cosh\phi(\cosh\phi+\tfrac{1}{2}\tau_{\text{i}}\Delta)^{-1},\\ &\chi=(s/\Delta)[1+2(\tau_i\Delta)^2\zeta\cosh\phi](\cosh\phi+\tfrac{1}{2}\tau_i\Delta)^{-1},\\ &\bar{\omega}/\Delta=\sinh\phi\,(1-\zeta/\cosh\phi), \quad \zeta=\tfrac{2}{3}(\tau_{\text{i}}\Delta)(s/\Delta)^2,\end{aligned} \quad (4.6)$$

provided $\tau_i\Delta\ll1$. In this approximation, all terms of order $(\tau_i\Delta)$ are neglected unless they appear in the combination $\zeta=\frac{2}{3}(\tau_{\text{i}}\Delta)(s/\Delta)^2$. We stress the fact that this simplification has to be done in the final result only.

[15] A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 1438 (1965) [English transl.: Soviet Phys.—JETP **7**, 996 (1958)].

[16] Cf. Ref. 17.

[17] M. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 10 (1963); **29**, 333 (1963). Note that the quantity τ in these papers corresponds to $2\tau_{\text{i}}$ in the present notation.

In the relaxation phenomena we are studying, the implications of $\Delta(\omega_\nu)$ being frequency-dependent are unimportant, thus, Δ can be assumed to be a constant. The important feature is that the analytical continuation of $i\tilde{\omega}_\nu$ has an imaginary part on the imaginary ω_ν (=real frequency) axis. This means that the single-particle states have a finite lifetime because of collision with phonons. Near the transition temperature ($\Delta \ll T$), $i\tilde{\omega}_\nu$ can be approximated by the expression which it has in the normal state. Therefore,[18]

$$i\tilde{\omega}_\nu = a[i\omega_\nu + i\,\mathrm{sgn}(\mathrm{Re}\omega_\nu)/2\tau_c], \tag{4.7}$$

provided that $|\omega_\nu| \lesssim T$ (which is the most important region here). By using properly renormalized quantities, e.g., the Fermi velocity, the renormalization constant a can be put equal to 1. The level width $1/2\tau_c$, where τ_c is the electron-phonon collision time, is proportional to T^3/Θ^2 (Θ=Debye temperature) which indicates that it is proportional to the number of thermally excited phonons.[19]

V. RESPONSE TO AN ELECTRIC FIELD: GENERAL RELATIONS

In the following section, we are interested only in the linear response. We assume a homogeneous electric field $\mathbf{E} = -\partial\mathbf{A}/\partial t$ in the direction of the superflow, i.e., parallel to $\mathbf{q}$. Correspondingly, there is a perturbation in the single-particle Hamiltonian

$$\delta h = -(e/m)\mathbf{A}(1/i)\nabla, \tag{5.1}$$

which causes a change in the free Green's function

$$\delta(\hat{G}'_{\mathrm{ni}})^{-1} = e/m\mathbf{A}(1/i)\nabla\hat{1}. \tag{5.2}$$

A term $\propto \mathbf{q}\cdot\mathbf{A}$ has been neglected, since it has no influence on the current. The current itself can be found from the change $\delta\hat{G}'$ of the Green's function,[20] which is given by

$$\delta\hat{G}' = \hat{G}_{\mathrm{e}}'[-\delta(\hat{G}'_{\mathrm{ni}})^{-1} + \delta\hat{\Sigma}']\hat{G}_{\mathrm{e}}', \tag{5.3}$$

where $\hat{G}_{\mathrm{e}}'$ is the equilibrium Green's function of Sec. IV.

Equation (5.3) indicates that we first have to determine the change in the self-energy $\delta\hat{\Sigma}$. This has to be done self-consistently, according to the relations of Sec. III. For convenience, we put $\delta\hat{\Sigma} = \hat{\phi}$. Since the electric field is homogeneous, $\hat{\phi}$ is independent of its c.m. coordinate $\mathbf{R}$.

According to Eqs. (3.5) and (5.3), the phonon contribution $\hat{\phi}^{\mathrm{ph}}$ is given by [in Fourier transforms, $\hat{\phi}(\tau,\tau') \propto e^{-i\omega_\nu\tau + i\omega_{\nu'}\tau'}$]

$$\hat{\phi}^{\mathrm{ph}}(\omega_\nu,\omega_\nu') = -T\sum_{\omega_n} B(\omega_n)\int \frac{d^3p}{(2\pi)^3}\hat{\tau}_3\hat{G}_{\mathrm{e}}(\mathbf{p},\omega_\nu+\omega_n) \times[-(e/m)\mathbf{A}\cdot\mathbf{p}\hat{1} + \hat{\phi}(\omega_\nu+\omega_n,\omega_{\nu'}+\omega_n)] + \hat{G}_{\mathrm{e}}(\mathbf{p},\omega_{\nu'}+\omega_n)\hat{\tau}_3, \tag{5.4}$$

where

$$B(\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n\tau}B(\tau) \simeq -g^2\left[1 - \frac{\omega_n^2}{\theta^2}\ln\left(1 + \frac{\theta^2}{\omega_n^2}\right)\right] \tag{5.5}$$

and $\omega_n = 2\pi nT$ are even Matsubara frequencies. The last expression is an approximation and corresponds to the Debye model. Similarly, the impurity contribution is given by

$$\hat{\phi}^{\mathrm{imp}}(\omega_\nu,\omega_{\nu'}) = \frac{1}{2\pi N_0\tau_{\mathrm{i}}}\int \frac{d^3p}{(2\pi)^3}\hat{\tau}_3\hat{G}_{\mathrm{e}}(\mathbf{p},\omega_\nu) \times[-(e/m)\mathbf{A}\cdot\mathbf{p}\hat{1} + \hat{\phi}(\omega_\nu,\omega_{\nu'})]\hat{G}_{\mathrm{e}}(\mathbf{p},\omega_{\nu'})\hat{\tau}_3. \tag{5.6}$$

If one writes Eqs. (5.4) and (5.6) in detail using Eq. (4.4) and the relation

$$\int \frac{d^3p}{(2\pi)^3} = \tfrac{1}{2}N_0\int_{-\infty}^{+\infty} d\epsilon \int_{-1}^{+1} dx,$$

one finds that the solution is of the form

$$\begin{aligned} \hat{\phi} &= \phi_1\hat{1} + \phi_2\hat{\tau}_1, \\ \phi_1(-\omega_\nu, -\omega_{\nu'}) &= -\phi_1(\omega_\nu,\omega_\nu'), \\ \phi_2(-\omega_\nu, -\omega_{\nu'}) &= \phi_2(\omega_\nu,\omega_{\nu'}). \end{aligned} \tag{5.7}$$

Equation (5.6) will be used to eliminate $\hat{\phi}^{\mathrm{imp}}$ from Eq. (5.4); the modifications ensuing from this procedure are known as vertex corrections in the theory of alloys.[9] We obtain

$$\phi_j^{\mathrm{ph}}(\omega_\nu,\omega_{\nu'}) = T\sum_{\omega_n} B(\omega_n)\left[L_j(\omega_\nu+\omega_n,\omega_{\nu'}+\omega_n)\frac{p_0 e}{m}A - \sum_{k=1}^{2} M_{jk}(\omega_\nu+\omega_n,\omega_{\nu'}+\omega_n)\phi_k^{\mathrm{ph}}(\omega_\nu+\omega_n,\omega_{\nu'}+\omega_n)\right], \tag{5.8}$$

where $j=1, 2$. The functions L_j and M_{jk} are various

[18] The form of Σ_{11} in the normal state has been discussed in Sec. 21 of Ref. 14. The transition to the superconducting state introduces a change in $\mathrm{Im}\Sigma_{11}(-i\omega+0)$ of relative order $(\Delta^2/T^2)\times\ln(T/\Delta)$, $(\Delta^2/T^2)\ln(\omega/\Delta)(|\omega|>\Delta)$ etc., which is negligible if $\Delta^2 \ll T^2$. Note that Δ, as defined in this paper by $\Delta = \Sigma_{12}^{\mathrm{ph}}$, is approximately constant for $|\omega| \lesssim T$, which is not the case if defined as usual by $\Delta = \omega_\nu\Sigma_{12}^{\mathrm{ph}}/\tilde{\omega}_\nu$.

[19] In the constant-collision-time approximation, Eq. (4.7), the electronic thermal conductivity limited by electron-phonon collisions is given by the relation $\kappa = (\pi^2/3)nk^2T\tau_c$, provided the metal is in the normal state. Using measured values of κ, we find the collision times at $T=T_c$ to be equal 0.6 10^{-9} sec and 2.8 10^{-11} sec for tin and lead, respectively. The exceptional small value of τ_c in the last case reflects the fact that lead has a high transition temperature and a low Debye temperature.

[20] The currents calculated from the primed and unprimed Green's functions differ only by an irrevelant constant part.

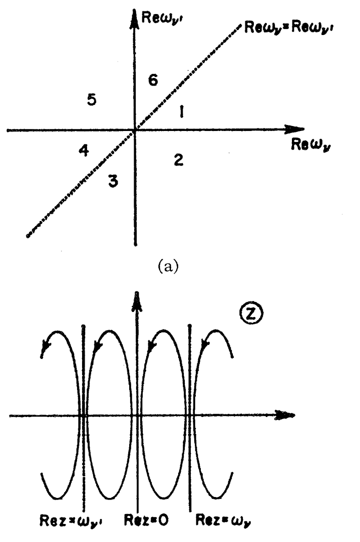

FIG. 1. (a) Domains of analyticity of $\phi_{jk}(\omega_\nu,\omega_{\nu'})$. (b) Contour C of the integral in Eq. (5.11).

combinations of the quantities

$$
\begin{aligned}
P_n(\omega_\nu,\omega_{\nu'}) &= \frac{i\pi}{2}\int_{-1}^{+1} dx\, x^n \frac{1}{W(\omega_\nu)+W(\omega_{\nu'})}\,,\\
Q_n(\omega_\nu,\omega_{\nu'}) &= \frac{i\pi}{2}\int_{-1}^{+1} dx\, x^n \frac{(i\tilde{\omega}_\nu - sx)(i\tilde{\omega}_{\nu'}-sx)+\tilde{\Delta}\tilde{\Delta}'}{[W+W']WW'}\,,\\
R_n(\omega_\nu,\omega_{\nu'}) &= \frac{i\pi}{2}\int_{-1}^{+1} dx\, x^n \frac{(i\tilde{\omega}_\nu - sx)\tilde{\Delta}'+(i\tilde{\omega}_{\nu'}-sx)\tilde{\Delta}}{[W+W']WW'}\,,\\
W(\omega_\nu) &= [(i\tilde{\omega}_\nu - sx)^2-\tilde{\Delta}^2]^{1/2}, \quad \mathrm{Im}W>0\,.
\end{aligned}
\tag{5.9}
$$

For instance, the quantities we need the most later are

$$
\begin{aligned}
L_2(\omega_\nu,\tilde{\omega}_{\nu'}) &= -2\pi\tau_i N_0 \frac{(2\pi\tau_i - P_0+Q_0)R_1 - R_0(P_1-Q_1)}{(2\pi\tau_i-P_0)^2-Q_0^2+R_0^2}\,,\\
&\qquad\qquad (5.10)\\
M_{22}(\omega_\nu,\omega_{\nu'}) &= -2\pi\tau_i N_0 \frac{2\pi\tau_i(P_0+Q_0)-P_0^2+Q_0^2-R_0^2}{(2\pi\tau_i-P_0)^2-Q_0^2+R_0^2}\,,
\end{aligned}
$$

We assume that the vertex corrections do not introduce changes in the analytical properties of the functions L_j and M_{jk}; i.e., we assume that they are holomorphic functions of ω_ν and $\omega_{\nu'}$ in the four regions $\mathrm{Re}\omega_\nu \gtrless 0$, $\mathrm{Re}\omega_{\nu'} \gtrless 0$. Then, it is a consistent assumption that $\phi_j^{\mathrm{ph}}(\omega_\nu,\omega_{\nu'})$ is a holomorphic function in the six regions shown in Fig. 1. [The discontinuity at $\mathrm{Re}\omega_\nu = \mathrm{Re}\omega_{\nu'}$ arises from the singularities of $B(\omega_n)$ at $\mathrm{Re}\omega_n = 0$.] The following arguments are given essentially in Ref. 6. In order to find the analytical continuation into the region of "real" frequencies, one changes the summation in (5.8) into a contour integral

$$
T\sum_{\omega_n} \to \frac{1}{4\pi}\oint dz\, \coth\frac{iz}{2T}\,. \tag{5.11}
$$

The contour C is illustrated in Fig. 1(b). The transition to real frequencies is then made by letting $\omega_\nu \to -i\omega_1$ and $\omega_{\nu'} \to -i\omega_2$. At the same time, we introduce the c.m. frequency $\Omega = \omega_1 - \omega_2$, and the internal frequency $\omega = \frac{1}{2}(\omega_1+\omega_2)$. In order to obtain the correct retarded expression, we chose $\mathrm{Im}\Omega > 0$. Let the superscript $\rho = 1, \ldots, 6$, denote the analytical continuations according to the six regions of Fig. 1(a), and put

$$
\Phi_j^{(\rho)}(\omega,\Omega) = \phi_j^{\mathrm{ph}(\rho)}(-i\omega - i\Omega/2,\ -i\omega + i\Omega/2)\,. \tag{5.12}
$$

Then, the symmetry of the functions $L_j^{(\rho)}$ and $M_{jk}^{(\rho)}$ allows us to seek a solution with the following properties:

$$
\begin{aligned}
[\Phi_j^{(\rho)}(\omega,\Omega)]^* &= (-)^j \Phi_j^{(\rho)}(-\omega,\ -\Omega)\,,\\
\Phi_j^{(1)}(-\omega,\ \Omega) &= (-)^j \Phi_j^{(3)}(\omega,\Omega)\,,\\
\Phi_j^{(2)}(-\omega,\ \Omega) &= (-)^j \Phi_j^{(2)}(\omega,\Omega)\,.
\end{aligned}
\tag{5.13}
$$

In particular, $\Phi_1^{(2)}(0,\Omega) = 0$. It can be shown that the difference $\Phi_j^{(1)}(0,\Omega) - \Phi_j^{(2)}(0,\Omega)$ is negligible (its relative order of magnitude is Δ/θ). This also means that $\Phi_1^{(1)}(0,\Omega) \simeq 0$. Since we only expect significant changes in the functions $\Phi_j^{(\rho)}$ over the range $\omega \simeq \theta$, we are allowed to neglect the quantities $\Phi_1^{(\rho)}$ altogether.

If one puts $A = s = 0$ in Eq. (5.8), the ensuing homogeneous equation has a solution $\Phi_2^{(1)}(\omega) = \chi(\omega)$ at $T = T_c$ (second-order phase transition). This homogeneous equation will be subtracted from Eq. (5.8); at the same time we try the ansatz $\Phi_2^{(1)}(\omega,\Omega) = \chi(\omega)$. After some consistent simplifications (for details, we refer to Ref. 6), we arrive at the following equation:

$$
\begin{aligned}
0 = B(i\omega - 0)\lambda(\Omega)(p_0 e/m)A(\Omega)&\\
- B(i\omega-0)\mu(\Omega)\chi(0)\,,&
\end{aligned}
\tag{5.14}
$$

where

$$
\begin{aligned}
\lambda &= \lambda_r + \lambda_s\,,\\
\lambda_r &= \frac{2}{4\pi i}\int dy\, \tanh\left(\frac{y-\Omega/2}{2T}\right) \times L_2^{(1)}\left(-iy - i\frac{\Omega}{2},\ -iy + i\frac{\Omega}{2}\right),\\
\lambda_s &= \frac{1}{4\pi i}\int dy\left(\tanh\frac{y+\Omega/2}{2T} - \tanh\frac{y-\Omega/2}{2T}\right) \times L_2^{(2)}\left(-iy - i\frac{\Omega}{2},\ -iy + i\frac{\Omega}{2}\right),
\end{aligned}
\tag{5.15}
$$

and

$$\mu=\mu_r+\mu_s,$$

$$\mu_r=-\frac{2}{4\pi i}\int dy\tanh\left(\frac{y-\Omega/2}{2T}\right)\times M_{22}{}^{(1)}\left(-iy-i\frac{\Omega}{2},\,-iy+i\frac{\Omega}{2}\right),$$

$$\mu_s=-\frac{1}{4\pi i}\int dy\left(\tanh\frac{y+\Omega/2}{2T}-\tanh\frac{y-\Omega/2}{2T}\right)\times M_{22}{}^{(2)}\left(-iy-i\frac{\Omega}{2},\,-iy+i\frac{\Omega}{2}\right). \quad (5.16)$$

Starting from Gorkov's equation,[21] one obtains almost the same result. In that case, $\chi(0)$ stands for the change of the order parameter $\delta\Delta$, and $-N_0B(i\omega-0)$ stands for the interaction parameter N_0V. However, there is one difference in that the self-energy $\Sigma_{11}{}^{\mathrm{ph}}$ does not appear in the Green's function, i.e., there are no electron-phonon collisions included.

For general information, we might add that the regular quantities λ_r and μ_r can be expanded in powers of (Δ^2/T^2) and (q^2/T^2) (terms of order Ω/T can be neglected). The reason for this is that $L_2{}^{(1)}$ and $M_{22}{}^{(1)}$ are holomorphic in the upper y half-plane. In contrast to that, $L_2{}^{(2)}$ and $M_{22}{}^{(2)}$ have singularities on both sides of the real y axis; and the evaluation of the singular quantities λ_s and μ_s is the most difficult task. Fortunately, the linear approximation in Ω is sufficient. As the coefficient of Ω is quite large (this reflects the fact that τ_{R} is large, too), small contributions to this coefficient can be discarded.

It is convenient to split the induced current density $\mathbf{j}$ into a paramagnetic part $\mathbf{j}_p$ and a diamagnetic part $\mathbf{j}_d$. Considering that $\phi_1=0$, we obtain from Eqs. (5.3) and (5.9)

$$j_p{}^{(1)}(\omega_n)=\frac{e^2n}{m}3T\sum_{\omega_n}\{-[P_2(\omega_\nu,\omega_\nu-\omega_n)-Q_2(\omega_\nu,\omega_\nu-\omega_n)]A\},$$

$$j_p{}^{(2)}(\omega_n)=\frac{e^2n}{m}3T\sum_{\omega_\nu}\left[-\frac{m}{ep_0}R_1(\omega_\nu,\omega_\nu-\omega_n)\times\phi_2(\omega_\nu,\omega_\nu-\omega_n)\right]. \quad (5.17)$$

Note that $\mathbf{j}$ is parallel to $\mathbf{A}$. The diamagnetic part $\mathbf{j}_d$ is the same here as in the normal state. Therefore, we obtain the total current if we subtract from Eq. (5.17) the corresponding expression in the normal state and add the current density in the normal state, namely, $i\omega\sigma_nA$.

According to the arguments given in Ref. 6, the dependence of $\phi_2(\omega_\nu,\omega_\nu-\omega_n)$ on ω_ν is weak [i.e., of order $(\omega_\nu{}^2/\Theta^2)$], whereas R_1 is concentrated mainly in the region of small ω_ν. Therefore, ϕ_2 can be considered to be independent of ω_ν. (The c.m. frequency Ω has to be identified with $i\omega_n$.)

The analytical continuation will be done in a manner similar to the treatment above. First, one replaces

$$T\sum_{\omega_\nu}\rightarrow\frac{1}{4\pi}\oint dz\tanh\frac{iz}{2T}. \quad (5.18)$$

Then, the contour C is deformed so that, e.g.,

$$T\sum_{\omega_\nu}P_2(\omega_\nu,\omega_\nu-\omega_n)=\frac{1}{4\pi i}\int dy\tanh\frac{y}{2T}[P_2{}^{(1)}(-iy;\,-iy-i\Omega)+P_2{}^{(1)}(-iy-i\Omega;\,-iy)]+\frac{1}{4\pi i}\int dy\left(\tanh\frac{y}{2T}-\tanh\frac{y-\Omega}{2T}\right)\times P_2{}^{(2)}(-iy;\,-iy+i\Omega). \quad (5.19)$$

Contributions of the type displayed by the first and second integral in Eq. (5.19) will be called regular and singular, respectively. We will find that the singular contributions are small and do not show any frequency dependence in the range of interest.

VI. CLEAN-METAL: RESULTS AND DISCUSSION

If $1/\tau_i=0$, we have from Eq. (5.10)

$$L_2=-N_0R_1;\quad M_{22}=-N_0(P_0+Q_0). \quad (6.1)$$

Upon replacing $i\tilde{\omega}_\nu$ by $y\pm i/2\tau_c$, we see that y and $s=v_0q/2$ occur essentially in the combination $y-sx$. If one absorbs $-sx$ in the integration variable y, it appears only in the argument of tanh, i.e., $\tanh[(y+sx\pm\Omega/2)/2T]$. From this it is clear that λ and μ can be expanded in powers of s^2/T^2.

Consider now the leading term of μ_s,

$$\mu_s=\frac{N_0}{4}\frac{\Omega}{2T}\int dy\frac{1}{\cosh^2(y/2T)}\times\left[1+\frac{(y+i/2\tau_c)(y-i/2\tau_c)+\Delta^2}{[(y+i/2\tau_c)^2-\Delta^2]^{1/2}[(y-i/2\tau_c)^2-\Delta^2]^{1/2}}\right]\times\frac{1}{[(y+i/2\tau_c)^2-\Delta^2]^{1/2}+[(y-i/2\tau_c)^2-\Delta^2]^{1/2}}; \quad (6.2)$$

where the imaginary parts of the square roots in the denominators are positive. Since for $|y|\gtrsim\Delta$

$$\left[\left(y\pm\frac{i}{2\tau_c}\right)^2-\Delta^2\right]^{1/2}\simeq\pm\operatorname{sgn}y\,(y^2-\Delta^2)^{1/2}\times\left[1\pm\frac{iy}{2\tau_c(y^2-\Delta^2)}\right], \quad (6.3)$$

[21] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl. Soviet Phys.—JETP **7**, 505 (1958)].

the sum in the denominator of Eq. (6.2) is very small. Therefore, the region $|y|\gtrsim\Delta$ is the most important one (provided $\Delta\gg 1/\tau_c$). Thus,

$$\mu_s=-N_0\frac{i\Omega\tau_c\Delta^2}{4T}\int_{|y|>\Delta}dy\frac{1}{\cosh^2(\frac{1}{2}y/T)}\frac{1}{|y|[y^2-\Delta^2]^{1/2}}=-\frac{i\pi}{4}N_0\frac{\Omega\tau_c\Delta}{T_c}. \quad (6.4)$$

Note that this expression is greater by the factor $2\tau_c\Delta$ than the corresponding one in the gapless case.[6] Corrections to the result (6.4) are of the relative order $\tau_c\Omega$, $(\tau_c\Delta)^{-2}$, and s^2/T^2, which are small quantities.

Calculating μ_r, we find that the term linear in Ω does not contain the large factor $\tau_c\Delta$; this allows us to put $\Omega=0$. Then, we obtain in standard approximation

$$\mu_r=-N_0\frac{T_c-T}{T_c}+N_0\frac{7\zeta(3)}{8}\frac{3\Delta^2+\frac{1}{6}v_0^2q^2}{\pi^2T_c^2}. \quad (6.5)$$

Consider the Ginsburg-Landau equation in the case where the order parameter has an $e^{i\mathbf{q}\mathbf{r}}$ dependence,[22]

$$\left(-N_0\frac{T_c-T}{T_c}+N_0\frac{7\zeta(3)}{8}\frac{\Delta^2+\frac{1}{6}v_0^2q^2}{\pi^2T_c^2}\right)\Delta=0. \quad (6.6)$$

One recognizes that μ_r is the derivative with respect to Δ of the left-hand side of Eq. (6.6).

The main contribution to λ is the simple term,

$$\lambda=N_0[7\zeta(3)/12](\Delta v_0q/\pi^2T_c^2). \quad (6.7)$$

The remaining contributions are by a factor Δ^2/T^2, s^2/T^2, $1/\tau_cT$, and Ω/T (from λ_s) smaller.

The equation $\mu=0$ defines a relaxation time $\tau_R=(\mathrm{Im}\Omega)^{-1}$,

$$\tau_r=\tau_c\frac{\pi^2}{8}\left[\frac{8}{7\zeta(3)}\right]^{1/2}\left[\frac{T_c}{T_c-T}\right]^{1/2}[1-\tfrac{1}{3}q^2/q_M^2]^{-1/2}, \quad (6.8)$$

where Eq. (6.6) has been used, and where

$$q_m^2=\frac{4}{7\zeta(3)}\frac{\pi^2T_c^2}{v_0^2}\frac{T_c-T}{T_c}=\tfrac{1}{3}\xi_{GL}^{-2} \quad (6.9)$$

is the momentum corresponding to maximal current as introduced in Sec. II. Note that relation (6.8) agrees with the corresponding expression in Ref. 6, except for the last factor which expresses the dependence of Δ on q.

The preceding results allow us to put Eq. (5.14) in the following form:

$$2\Delta\chi(0)=\tfrac{2}{3}ev_0^2qA\frac{1}{1-i\Omega\tau_R}. \quad (6.10)$$

[22] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 37, 1407 (1959) [English transl.: Soviet Phys.—JETP 10, 998 (1960)].

The left-hand side can be considered to be equal to $\delta\Delta^2$. From Eq. (6.6) and $A=-(i/\Omega)E$, it follows that this relation corresponds exactly to Eq. (2.4).

The contribution $j_p^{(2)}$ of the current involves the quantity R_1, which in the present case is equal to $-(1/N_0)L_2$. Therefore, the frequency sum (apart from a factor of proportionality), leads to λ; in particular

$$j^{(2)}=j_p^{(2)}=-\frac{e^2n}{m}\frac{7\zeta(3)}{12}\frac{v_0^2q^2}{\pi^2T_c^2}A\frac{1}{1-i\Omega\tau_R}. \quad (6.11)$$

Considering $j^{(1)}$, one finds that the singular part does not contribute and one obtains the result

$$j^{(1)}=-\frac{e^2n}{m}\frac{7\zeta(3)}{4}\frac{\Delta^2}{\pi^2T_c^2}A+i\Omega\sigma_nA. \quad (6.12)$$

Adding up the total current, one recovers Eq. (2.5) (except for the contribution which involves the constant conductivity σ_n). In particular,

$$\sigma_{s0}=i\frac{e^2n}{m}2\frac{T_c-T}{T_c}\frac{1}{\Omega}. \quad (6.13)$$

VII. DIRTY-METAL: RESULTS AND DISCUSSION

Case (a). The dc is zero; $s=0$. This is an introduction to the general case. Considering Eq. (5.9), we find the relation

$$P_0^2-Q_0^2+R_0^2=0. \quad (7.1)$$

This allows us to simplify considerably the expression (5.10) for M_{22}. Furthermore, we obtain from Eq. (4.6)

$$\tilde{\omega}_\nu=\bar{\omega}_\nu\eta_\nu,\quad \tilde{\Delta}=\Delta\eta_\nu,\quad \eta_\nu=1+(i/2\tau_i)[(i\bar{\omega}_\nu)^2-\Delta^2]^{-1/2}. \quad (7.2)$$

Inserting these relations, we find that M_{22} is the same function of $\bar{\omega}_\nu$, $\bar{\omega}_{\nu'}$ and Δ as in the clean case.

Since $L_2=0$, we have $\chi(0)=j^{(2)}=0$ and do not expect a particular frequency dependence for the induced current. Indeed, the current is found to be ($\tau_iT\ll 1$)

$$j=-\frac{e^2n}{m}\frac{2\pi^2}{7\zeta(3)}T_c\tau_i\frac{T_c-T}{T_c}A+\frac{e^2n\tau_i}{m}\left\{1-\frac{7\zeta(3)}{2}\frac{\Delta^2}{\pi^2T_c^2}+\frac{\Delta}{T}\left[2\ln\left(\frac{1}{\tau_c\Delta}-i\frac{\Omega}{\Delta}\right)+\frac{1}{2}\right]\right\}(i\Omega A). \quad (7.3)$$

The response of the Cooper pairs (first line) is reduced by a factor $[2\pi^3/7\zeta(3)]T\tau_i$ as compared to expression (6.13). For intermediate values of τ_iT, one finds this factor to be Gorkov's $\chi(\rho)$ function[22] (as it should be). In the expression on the second line, we observe that the finite collision time τ_c prevents a singular behavior for small Ω. Note that the ratio of the superconductive response to the resistive one is of the order $(T_c/\Omega)(T_c-T/T_c)$, which is quite large except under extreme conditions.

Case (b). The dc is nonzero; $s\neq 0$. The terms λ_s and μ_s require particular care. In the last relation of Eq. (4.6), we put $\bar{\omega}=-iy\pm 0$ (assume $1/\tau_c$ negligible), and $\sinh\phi=-i(t/\Delta)$. Then,

$$y=t\left[1-i\zeta\frac{\Delta}{(t^2-\Delta^2)^{1/2}}\right],\quad \mathrm{Im}[t^2-\Delta^2]^{1/2}>0. \tag{7.4}$$

The general solution of this equation is complicated; fortunately, we do not need it. For $|y|\gg\Delta$, we have

$$y\pm i0=t\pm i\Delta\zeta. \tag{7.5}$$

Furthermore, the nearest distance of the complex variable t to $(\Delta,0)$ is of the order $\Delta\zeta^{2/3}$. Therefore, $M_{22}^{(2)}(-iy-i\Omega/2;\,-iy+i\Omega/2)$ can be expanded in powers of Ω, if $\Omega\ll\Delta\zeta^{2/3}$. (In the clean case, this condition was $\Omega\ll 1/\tau_c$). The same holds for $L_2^{(2)}$. In view of this and the fact that λ_s and μ_s already contain a factor $\propto\Omega$, we are allowed to put $\Omega=0$ in $M_{22}^{(2)}$ and $L_2^{(2)}$.

It is important to evaluate the denominator in $M_{22}^{(2)}$ and $L_2^{(2)}$ very carefully, because there are large canceling terms. Under suitable circumstances, it is a good approximation to put $W(\omega_\nu)+W(\omega_{\nu'})=i/\tau_i$ [Eq. (5.9); note that $i\omega_\nu=y+i0$ and $i\omega_{\nu'}=y-i0$] for instance,

$$\pi\tau_i-P_0=\frac{\pi\tau_i}{2}\int dx\,\frac{W+W'-i/\tau_i}{W+W'}$$

$$\simeq-\frac{i\pi\tau_i^2}{2}\int dx\left(W+W'-\frac{i}{\tau_i}\right). \tag{7.6}$$

Finally, one obtains for the denominator

$$2(\pi\tau_i-P_0)+P_0^2-Q_0^2+R_0^2$$

$$=4\pi^2\Delta\tau_i^3\left[\cosh\phi+\cosh\phi'+\tfrac{1}{2}\zeta\left(\frac{1}{\cosh\phi}-\frac{1}{\cosh\phi'}\right)^2-2\zeta+0(\tau_i\Delta)\right]. \tag{7.7}$$

For $y=0$ and $y=\Delta$, the denominator is of the order $4\pi^2\Delta\tau_i^3$ and $4\pi^2\Delta\tau_i^3\zeta^{1/3}$, respectively. Using Eqs. (7.4) and (7.5), we find for $|y|\gg\Delta$ that (7.7) becomes

$$10\pi^2\Delta\tau_i^3(\Delta^6/y^6)\zeta^3. \tag{7.8}$$

In the following, $\zeta\ll 1$. This means that the region $|y|\gg\Delta$ contributes the most to μ_s, since there the denominator is small. This allows us to use approximation (7.8), and to put $t=y$ in the numerator of $M_{22}^{(2)}$. Thus, we obtain

$$\mu_s=N_0^2\frac{\pi^4}{5}\frac{i\Omega}{T}\left(\frac{T}{\Delta}\right)^5\zeta^{-3}. \tag{7.9}$$

The calculation of μ_r poses no particular problem; one obtains the result

$$\mu_r=-N_0\frac{T_c-T}{T_c}+N_0\frac{7\zeta(3)}{8}\frac{3\Delta^2+\frac{1}{6}\chi(\rho)v_0^2q^2}{\pi^2T^2}, \tag{7.10}$$

which is the same as Eq. (6.5) except for the factor $\chi(\rho)\simeq[2\pi^3/7\zeta(3)]T_c\tau_i$, if $T_c\tau_i\ll 1$. A corresponding modification has to be made in the Ginsburg-Landau equation (6.6).

The same procedure as outlined above will be applied in the calculation of λ. The contribution of λ_s is, in the frequency range of interest, smaller by a factor $10^{-3}\times(\Delta/T)^4$ than λ_r. The final result is

$$\lambda=N_0[7\zeta(3)/12]\chi(\rho)(\Delta v_0q/\pi^2T_c^2). \tag{7.11}$$

Proceeding as in the previous section, we define a relaxation time

$$\tau_R=\frac{27\pi^5}{640}\left(\frac{7\zeta(3)}{8}\right)^2\frac{\hbar}{kT_c}\left(\frac{T_c}{T_c-T}\right)^5\frac{q_m^6/q^6}{(1-\frac{1}{3}q^2/q_m^2)^2}, \tag{7.12}$$

where

$$q_m^2=\frac{4}{7\zeta(3)}\frac{\pi^2T_c^2}{\chi(\rho)v_0^2}\frac{T_c-T}{T_c}=\tfrac{1}{3}\xi_{GL}^{-2}. \tag{7.13}$$

Note that

$$\zeta_m=\frac{7\zeta(3)}{4\pi^3}\frac{\Delta}{T}\ll 1.$$

No difficulty arises in the calculation of the contribution $j^{(2)}$ to the current. The "singular" part is negligible, and one obtains

$$j^{(2)}=-\frac{e^2n}{m}\frac{7\zeta(3)}{12}\chi^2(\rho)\frac{v_0^2q^2}{\pi^2T_c^2}A\frac{1}{1-i\Omega\tau_R}. \tag{7.14}$$

As far as the contribution $j^{(1)}$ is concerned, we note that its singular part is difficult to evaluate exactly. However, an estimate shows that it is small and that it leads to a conductivity σ which varies only slowly with frequency (the range is $\Delta\zeta^{2/3}$ or $1/\tau_c$, whichever is larger). The regular part can easily be evaluated. As in the clean metal, it has to be interpreted as due to the direct response of Cooper pairs. We have

$$j^{(1)} = -(e^2n/m)\tfrac{1}{4}7\zeta(3)\chi(\rho)(\Delta^2/\pi^2T_c^2)A + \sigma i\Omega A. \quad (7.15)$$

These results confirm again Eq. (2.5). Note that quite generally,

$$\sigma_{s0} = i(e^2n/m)2\chi(\rho)(T_c - T/T_c)(1/\Omega). \quad (7.16)$$

Consider now the case where $1/\tau_c$ is not negligible. One obtains a result which can be considered as an interpolation between the two limits, Eqs. (6.8) and (7.12). Let these expressions be $\tau_R^{(c)}$ and $\tau_R^{(i)}$, respectively. Then, this interpolation can approximately be represented by the relation

$$\frac{1}{\tau_R} = \frac{1}{\tau_R^{(c)}} + \frac{1}{\tau_R^{(i)}}. \quad (7.17)$$

It conclusion it can be said that the results of the microscopic calculation agree with the phenomenological analysis given in Sec. II. In addition, quantitative expressions have been derived for the relaxation time τ_R [Eqs. (7.17), (7.12), and (6.8)] which hold for temperature close to the transition temperature. The relaxation time will be determined mainly by inelastic electron-phonon collisions. One expects impurity scattering to influence noticeably the relaxation time only when the superconductor has such a low transition temperature that the number of excited electrons and phonons is small. However, if one considers the dependence on the temperature difference $(T_c - T)$ of these two processes, one is tempted to assume that for low temperatures (which are outside the scope of the present considerations) impurity scattering may give rise to an effective relaxation mechanism in a superconductor of short mean free path and considerable supercurrent flow.

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Electron Correlation in Narrow Energy Bands. II. One Reversed Spin in an Otherwise Fully Aligned Narrow S Band

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In a previous paper, a new Green's-function decoupling scheme was applied to the Hubbard Hamiltonian, and an improved version of Hubbard's first approximation was obtained. That result did not reduce to the correct low-density limit as obtained by Kanamori. In the present article, the theory is improved for the special case of a single reversed spin in an otherwise fully aligned band, and the improved theory is correct in the low-density limit. Numerical results are presented for the simple cubic lattice. If we define an effective exchange-interaction parameter $U_{\rm eff}$ as the $k=0$ reversed-spin self-energy for $U \to \infty$, divided by the number $n_\uparrow$ of up-spin electrons per site, we find that the present result departs rather rapidly from the Kanamori result as $n_\uparrow$ is increased, and it is concluded that the Kanamori result overestimates the increase in $U_{\rm eff}$ with $n_\uparrow$, at least in the present case. For intermediate values of $n_\uparrow$, the two-pole approximation of the previous article and the present calculation give very similar results for this quantity.

INTRODUCTION

In the first paper in this series,[1] a new Green's-function decoupling scheme[2,3] was applied to the Hubbard model of a narrow nondegenerate band governed by the Hamiltonian

$$H = U\sum_i n_{i\uparrow}n_{i\downarrow} + \sum_{ij\sigma} t_{ij}c_{i\sigma}^\dagger c_{j\sigma}, \quad (1)$$

where c_i annihilates an electron on the ith Wannier site. This model includes in its simplest form the competition between the intra-atomic Coulomb energy and the electron kinetic energy. This was solved in an improved version of Hubbard's first approximation[4] in which the one-particle Green's function is assumed to have two poles on the real axis.

In I, an improvement of the theory was suggested which would lead to the correct low-density limit for the electron self-energy. In the present article, we shall apply the improved version of the theory to the case of one reversed spin in an otherwise aligned nondegenerate band. The advantage of this special case is that in the fully aligned state, the electrons are noninteracting so that the wave function is known. An argument will be given to show that the approximation is good for finite densities in this particular case. There are two purposes

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