Normal-Fluid Densities in Liquid Helium II under Pressure*

R. H. Romer† and R. J. Duffy
Amherst College, Physics Department, Amherst, Massachusetts 01002
(Received 1 April 1969)

Measurements of the normal-fluid fraction ρ_n/ρ in superfluid helium have been made by observing the period of torsional oscillation of a stack of disks, at pressures up to 25 atm, at temperatures from 1.45 K to the λ line. At all pressures, the temperature dependence of ρ_n/ρ is qualitatively similar to the previously observed behavior at the vapor pressure. At all temperatures, ρ_n/ρ is a monotonically increasing function of pressure. Comparison is made with values calculated by means of the two-fluid model from second-sound velocities and thermodynamic data; good agreement is observed, but it is necessary to take into account the fact that a commonly used approximation $C_p \cong C_v$ is seriously incorrect at high pressures.

I. INTRODUCTION

Measurements of the normal and superfluid densities of liquid helium by Andronikashvili[1] played a crucial role in establishing the essential correctness of the two-fluid model. In particular, the generally good agreement between the values obtained directly from Andronikashvili's stack-of-disks experiment, and those inferred indirectly from second-sound velocities u_2 and thermodynamic data, using the equations of the two-fluid model, provided strong quantitative support for the basic features of the model. The two-fluid model still provides a useful description of many aspects of the behavior of liquid helium. In view of the importance of the disk measurements and of the fact that the measurements by Andronikashvili and others,[2-4] like most measurements on superfluid helium, have all been carried out in helium under its own saturated vapor pressure, we have made similar measurements at pressures up to 25 atm. Our experiments were intended to provide values of ρ_n and ρ_s over a wide range of conditions, and to make possible a comparison with values calculated from second-sound velocities, thereby subjecting the two-fluid model to a test over a wide range of conditions. The approximation $C_p \cong C_v$, which is usually made when deriving an expression for u_2, is seriously incorrect at higher pressures. A more detailed analysis which takes into account the inequality of the specific heats leads to a more general expression for u_2, valid over the entire pressure range of this experiment.

II. EXPERIMENTAL DESIGN

An over-all view of the apparatus is shown in Fig. 1. The disk stack, a rigid support rod A, and the torsion fiber C are enclosed in a pressurized chamber, from the top of which connections are made to a pressure gauge [Heise Bourdon Tube Co., 0–500ψ (gauge), accuracy 0.5ψ,] ballast volume, and helium supply. (Gauge pressures are converted to absolute pressures by adding the atmospheric pressure.) The torsion fiber and the apparatus for starting and recording the oscillations of the system are at room temperature.

The support rod is enclosed in a 4.5-mm-inside diam tube B, which connects the room-temperature space to the sample chamber. When operating with the sample under pressure at a temperature below that of the λ line, there must be a column of superfluid in this tube, with a temperature difference between the liquid in the sample chamber and the liquid at some higher point along the tube which is at the λ temperature. Under some conditions, the resulting heat flow can cause an excessively large rate of evaporation of the bath and, more seriously, can produce effects which interfere with the taking of

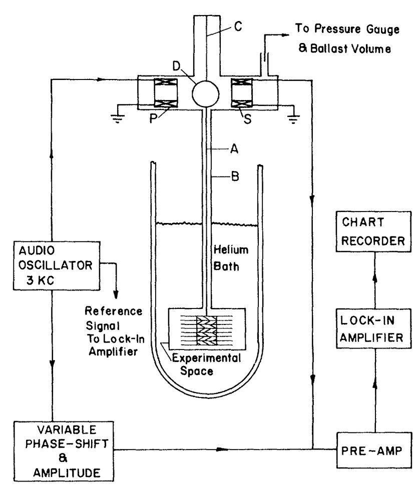

FIG. 1. Schematic diagram of the apparatus.

meaningful data, as discussed later. If these experiments were to be extended to lower temperatures, it would be highly desirable to use a closed system,[5] with disks, suspension fiber, and detection apparatus at the low temperature, and only a pressurizing capillary leading to room temperature. Even with such improvements, however, the disk method becomes a poorer and poorer means of measuring ρ_n as one goes to lower temperatures; it is most valuable at the higher temperatures where ρ_n is relatively large.

The design of the disk stack and the suspension represents a compromise between various conflicting criteria. The disks should be closely spaced so that the normal fluid between the disks will be almost completely set into rotation. On the other hand, the spacing between the disks should be large, relative to the disk thickness, so that the moment of inertia of the disks themselves will not be overwhelmingly large in comparison with the moment of inertia of the fluid.

Because of the low density of liquid helium, this consideration is of dominant importance in the design of the experiment. This can be seen by taking the following simplified view of the experiment. Let us suppose that all of the normal fluid within the geometrical region marked out by the disk stack is dragged along with the disks – thereby increasing the rigid-body moment of inertia of the pendulum – and that none of the superfluid, and none of the normal fluid outside this region, is affected. If we further suppose that the total density is approximately temperature-independent at a given pressure, it follows that:

$$\rho_n(T)/\rho_\lambda = [\tau(T)^2 - \tau_0^2]/(\tau_\lambda^2 - \tau_0^2), \qquad (1)$$

where $\rho_n(T)/\rho_\lambda$ is the ratio of the normal-fluid density at temperature T to the density at the λ temperature for that pressure, $\tau(T)$ is the corresponding period, τ_0 is the vacuum period, and τ_λ is the period at and above the λ line. Although a more refined analysis is necessary, as discussed below, any analysis of the data requires – in one guise or another – the subtraction of the vacuum period. It is obvious that the disk method of determining normal-fluid densities is best at relatively high temperatures, and becomes poorer and poorer at lower temperatures, where $\tau(T) \cong \tau_0$. It would be desirable to make the change in period due to the presence of the liquid as large as possible, but it is difficult to improve significantly on the value which applies to our apparatus, $[\tau(T) - \tau_0]/\tau_0 \lesssim 0.1$, without violating other criteria.

In order to enhance the effect of the normal fluid on the period, the disks should be thin and light; but they should also be rigid and as nearly plane as possible, in order to avoid any warping, which may cause some of the superfluid to be dragged[3,4] along with the disks and the normal fluid. The period of oscillation determines the viscous penetration depth Λ. ($\Lambda^2 = \tau\eta/\pi\rho_n$, where τ is the period, η is the normal-fluid viscosity, and ρ_n is the normal-fluid density.) This length should be large in comparison with the disk spacing, so as to approximate complete dragging of the normal fluid; but this length should be fairly small to minimize corrections for the liquid dragged around the periphery of the disk stack. The stationary walls of the surrounding container should be sufficiently far from the disk stack, in comparison with the penetration depth, to eliminate the need for wall corrections.

The disk stack used consisted of 120 mica disks (3.175 cm diam, 0.0025 cm thick) mounted on an aluminum spindle, separated by mica washers (1.27 cm diam, 0.018 cm thick). This assembly was attached to the bottom of a 112 cm length of stainless-steel tubing (0.046 cm outside diam) serving as a support rod A, leading to the top of the cryostat at room temperature. At the top of the support tube was mounted a small metallic disk D, used in starting the pendulum and observing its motion. This complete assembly was suspended from a 51-cm-long, 0.051-mm-diam platinum suspension C. The disk system itself is enclosed in a copper can (6 cm high, 6 cm diam), and the support rod is enclosed in a 4.5-mm-inside diam tube B.

The resulting torsion pendulum had a period in vacuum of approximately 32.9 sec, with a damping time constant of about 2500 sec. When immersed in liquid helium at a pressure of 25 atm, at temperatures at and above the λ line, the period increased to approximately 36.4 sec. The geometry and the viscous penetration depths were such that under the worst conditions (near the λ line), the normal fluid was set into motion with at least 99.9% efficiency, according to the hydrodynamic calculations of Dash and Taylor.[3] Hollis-Hallett[2] has observed an increase in period at large amplitudes of oscillation, apparently due to the fact that the superfluid can be set into motion if a critical velocity is exceeded. To avoid such effects, we kept the amplitude of oscillation smaller than 5°, and no dependence of period on amplitude was observed.

The angular position of the torsion pendulum is recorded continuously with the aid of the metallic disk D at the upper end of the support rod. This disk is located midway between two coils, which serve as primary P and secondary S of a variable-coupling 3-kHz transformer. The amplitude of the voltage induced in the secondary depends on the angular position of the disk. This voltage is combined with a 3-kHz voltage of adjustable phase and amplitude from the same oscillator which excites the primary, so that the sum is nearly zero when the pendulum is in its rest position. This signal is then fed to a lock-in amplifier, whose reference voltage is also derived from the same oscillator. The lock-in output is then put on a strip-chart recorder, thus providing a continuous record of the angular position of the pendulum. The recorder deflection is an inherently nonlinear function of the position of the pendulum. This nonlinearity, however, has little effect on the determination of a period. In addition, the voltage-angle relationship is known, permitting corrections to be made if necessary. The rest position of the pendulum is chosen to make this relationship as nearly linear as possible over the operating range, and excursions from the rest position are kept small. The amplitude of the exciting voltage on the primary is kept far below the level at which eddy-current torques on the disk have an observable effect on the period or damping of the disk. The period determinations depend on the constancy of the chart speed, which is checked against a local quartz-crystal frequency standard. Chart speed variations contribute negligibly to the over-all error.

The vertical disk D (0.95 cm diam) represents a potentionally serious source of error. During the measurements under pressure, this disk is surrounded by compressed room-temperature gas, which is partially set into motion. This gas contributes to the effective moment of inertia of the system; if this is a large effect, its neglect could lead to spuriously high values of ρ_n. It is difficult to calculate the size of this effect, and a separate set of measurements was made to learn how large a source of error this might be. For this purpose, the stack of disks was replaced by a brass cylinder of approximately the same moment of inertia (1.72 cm high, 0.858 cm diam), the low-temperature part of the apparatus was replaced by a long 3-in.-diam closed tube, and the apparatus was filled with room-temperature helium gas. As the gas pressure was increased from 0 to 25 atm, the period of oscillation increased gradually, the total change being approximately 0.009 sec. An effect of this magnitude would change the values of ρ_n/ρ by at most 0.002, a value considerably smaller than other inaccuracies in the results.

The system is set into oscillation in a reproducible fashion, with no mechanical disturbance of the apparatus, by means of two small chips of steel attached to the periphery of the disk D. To start the oscillation, the 3-kHz voltage is temporarily removed, the two coils are connected in series, and a slowly increasing current (derived from a single cycle of a sawtooth voltage of 20-sec period) is passed through the coils. The pendulum is then released to swing freely, and the 3-kHz voltage is then turned on.

Each run consisted of a series of checks of the vacuum period, followed by a series of measurements at a particular pressure. About 10 full periods were averaged to yield a single point on a period-versus-temperature plot. In some runs, the temperature was made to drift very slowly upwards, over a period of 5 to 7 h, from the lowest attainable temperature to a temperature well above the λ line. Under these conditions, the temperature varied during the time necessary to obtain 10 full oscillations, over a range which was as large as 15 mK at the lowest temperatures and which was reduced to about 2 mK near the λ line. In other runs, a series of measurements at selected discrete temperatures was made with the aid of an electronic bath regulator, which held the temperature constant to within less than 1 mK. In each run, the high-pressure pass was followed by a similar pass at the vapor pressure (with the sample chamber opened to the bath).

Temperatures are determined from carbon and germanium resistance thermometers; some inside the can and others in the outer bath. During the second half of each run, a series of period data was taken at the vapor pressure (with the sample chamber short-circuited to the bath), and the resistors were checked against each other and against the bath vapor pressure. By having resistors both in the sample chamber and in the bath, it was possible to be on guard against a significant temperature difference between the two regions. Such a temperature difference was observed under some conditions, as discussed below.

The temperature range over which useful data at pressures above the vapor pressure could be obtained was limited by the pumping speed of the bath pump, and also by the following effect: At a pressure of 5 atm, for instance, as the temperature was lowered below a temperature of approximately 1.7 K, there was a sudden increase in the heat leak to the bath, due to heat conducted along the column of superfluid helium and the sudden appearance of a significant temperature difference between the sample chamber and the bath. (At 5 atm and 1.5 K, this temperature difference was as much as 0.01 K.) At the same time, the damping of the torsional oscillations became significantly larger, and the behavior of the system became sufficiently erratic that useful period data could not be obtained. These effects are not understood in detail, and they impose a limitation on the temperature range for the data at the lower pressures. Except at the vapor pressure (where the effect does not occur), the lower the pressure, the higher the temperature at which these effects set in. At 25 atm, no such effects were seen down to 1.45 K, the lowest temperature at which observations were made.

III. ANALYSIS OF DATA

Values of ρ_n/ρ were obtained from the period-versus-temperature data in the following way. If the disks are perfectly smooth and plane, so that no superfluid is set into motion, we may write[6]

$$\tau^2 = \tau_0{}^2 + m\rho_n\,(1 + c_1\Lambda + c_2\Lambda^2). \qquad (2)$$

Here, τ and τ_0 denote the period and the vacuum period; m is a constant of the apparatus, depending on the torsion constant of the suspension and the moment of inertia of the pendulum; c_1 and c_2 are geometrical constants of the apparatus[3]; and Λ is the viscous penetration depth. Because of imperfections in the disk system, a small fraction of the superfluid may be set into motion. To allow for this possibility, we modify Eq. (2) by adding a term proportional to the superfluid density

$$\tau^2 = \tau_0{}^2 + m\,[\rho_n(1 + c_1\Lambda + c_2\Lambda^2) + \alpha\rho_s], \qquad (3a)$$

$$\tau^2 = (\tau_0{}^2 + m\alpha\rho) + m\rho_n\,(1 - \alpha + c_1\Lambda + c_2\Lambda^2). \qquad (3b)$$

Here, ρ is the total density, and α (to be determined empirically) takes into account the superfluid which is set into motion, so as to contribute to the moment of inertia of the disk stack. Such a partial dragging of the superfluid has been treated in a similar fashion by Dash and Taylor,[4] and by Mehl and Zimmermann.[7]

The terms $c_1\Lambda$ and $c_2\Lambda^2$ account for the liquid around the periphery of the disk system (approximately the liquid within a distance Λ of the disks) which is partially set into motion.[3] For our geometry, the calculated values are: $c_1 = 2.015$ cm^{-1}, $c_2 = 0.807$ cm^{-2}. The penetration depth Λ is not itself directly known, as $\Lambda^2 = \tau\eta/\pi\rho_n$. The most complete set of data available pertaining to the normal-fluid viscosity is that of Goodwin and McCormick,[8] who have determined the product $\eta\rho_n$ over a wide range of temperatures and pressures. With this information, it is possible to use the period data to extract the value of ρ_n from Eq. (3b) if, in addition to c_1 and c_2, the parameters τ_0, α, and m are known.

The vacuum period τ_0 was determined in separate low-temperature runs, in which a small amount of helium exchange gas was present in the sample chamber. The exchange gas was sufficient to ensure thermal equilibrium, but had no significant effect on the period. The period was measured over the temperature range 1.5–2.1 K, at gas pressures from 5 to 50 μ Hg. The variation in period over this range of conditions was less than 0.002 sec.

The constant m is determined by the following considerations. In each run, the λ temperature is identified by a sharp corner on a plot of period versus temperature. The λ line thus determined (see Table I) agrees within 2 mK with the measurements of Elwell and Meyer.[9] From the value of the period at the λ line and the known values[8,9] of ρ and η, the constant m is thus determined from Eq. (3a). In other words, we take advantage of the sharp corner on the period-temperature plot to normalize our data to give $\rho_n = \rho = \rho_\lambda$ at the λ line; there is no further adjustment of the data. [If the correction terms involving α, c_1, and c_2 were negligibly small, this procedure would yield Eq. (1).]

The coefficient α was measured in runs in which the vacuum period was first measured as described above, and the sample chamber was then filled with liquid helium at the vapor pressure. A series of period-versus-temperature measurements was then made, primarily at low temperatures to emphasize the term in $\alpha\rho_s$. Making use of Dash and Taylor's vapor-pressure results[3,4] for ρ_n and η, we extrapolate Eq. (3b) to $T = 0$ K to find the value of $\tau_0{}^2 + m\alpha\rho$. From the vapor pressure value[9] of ρ and from the previously determined values of τ_0 and m, α was determined to be 0.03 ± 0.01. Our results are not very sensitive to the exact value of α, because the vapor-pressure part of each run provides a check on the value of the important combination $\tau_0{}^2 + m\alpha\rho$. The vapor-pressure data actually serve to check the value of this combination, with ρ representing the total density of the liquid at the vapor pressure, whereas it is this combination with ρ representing the total density under pressure which enters into the calculation of our results for ρ_n under pressure. To the extent that changes in the total density[9] are small relative to changes in ρ_n, the vapor-pressure part of each run checks the desired quantity directly. Analysis of the data with α varying over the range 0.01–0.05 shows that uncertainty in α contributes negligibly to the error of the experiment.

Because the data are normalized to give $\rho_n = \rho$ at $T = T_\lambda$, the experiment leads directly to values of ρ_n/ρ_λ, where ρ_λ is the total density at the λ line for a particular pressure. [Note, however, that the total density ρ in Eq. (3b) is $\rho(P, T)$, not $\rho(P, T_\lambda)$. The actual value[9] $\rho(P,T)$ is used in this correction term, though this correction itself is small enough that, for this purpose, the difference between ρ and ρ_λ is of no importance.] In order to facilitate comparison with second-sound results, the experimental results of this paper are presented as values of $\rho_n(P, T)/\rho(P, T)$. The density data of Elwell and Meyer[9] were used to convert values of ρ_n/ρ_λ to values of ρ_n/ρ. [At a given pressure, the density at the λ line is typically 1–1.5% higher than at lower temperatures ($T \simeq 1.6$ K).]

TABLE I. Smoothed ρ_n/ρ data at five pressures: C_V: calculated values, using Eq. (4). C_P: calculated values, using Eq. (9).

T	ρ_n/ρ(%) Expt	C_V	C_P
	P=5.28 atm		
1.75	31.1	31.4	31.2
1.80	36.0	36.4	36.2
1.85	42.3	42.2	42.0
1.90	48.5	48.7	48.5
1.95	56.2	56.4	56.2
2.00	65.0		
2.02	69.1		
2.04	73.6		
2.06	78.5		
2.07	81.1		
2.08	84.0		
2.09	87.1		
2.10	90.7		
2.11	95.1		
2.116	100.0		
	P=10.37 atm		
1.68	29.7		
1.70	31.7	31.4	31.0
1.75	36.5	36.6	36.1
1.80	42.5	42.7	42.0
1.85	49.5	49.5	48.8
1.90	57.5	58.1	57.4
1.95	66.0	67.4	66.7
1.98	73.2		
2.00	78.2		
2.01	81.0		
2.02	83.9		
2.03	87.1		
2.04	90.8		
2.05	95.6		
2.055	100.0		
	P=15.04 atm		
1.63	30.8		
1.65	33.0		
1.70	38.8	38.4	37.5
1.75	45.3	45.1	43.9
1.80	52.0	52.0	50.5
	P=15.04 atm (*Con't.*)		
1.85	60.0	60.1	58.4
1.90	69.5	70.8	69.1
1.95	80.0	83.9	82.1
1.96	84.0		
1.97	87.3		
1.98	91.1		
1.99	96.2		
1.994	100.0		
	P=20.02 atm		
1.58	31.6		
1.60	33.5	35.0	34.1
1.65	39.2		
1.70	45.7	47.8	46.1
1.75	52.5	55.9	53.8
1.80	61.0	64.5	61.8
1.85	72.0	76.2	72.7
1.87	77.3		
1.88	80.2		
1.89	83.3		
1.90	86.8		
1.91	90.8		
1.92	96.0		
1.924	100.0		
	P=25.10 atm		
1.45	25.5		
1.50	30.0		
1.55	35.5		
1.60	41.0	43.4	41.7
1.65	46.8		
1.70	55.0	59.0	56.0
1.75	64.6	69.6	66.0
1.78	72.0		
1.80	78.0	83.1	79.2
1.81	81.4		
1.82	85.1		
1.83	89.4		
1.84	94.9		
1.845	100.0		

IV. RESULTS AND DISCUSSION

Smoothed results for ρ_n/ρ at five pressures are shown by the solid curves in Figs. 2 and 3 and in Table I. Error bars are not shown. Individual data points yield ρ_n/ρ points which scatter within about ±0.01 of the curves shown. This scatter, the origin of which is not completely understood, is sufficiently large that uncertainties in the temperature and pressure contribute a negligible amount to the uncertainties of the results. Errors can also arise from errors in the vacuum period τ_0, the period at the λ line τ_λ, the geometrical coefficients α, c_1, and c_2, and the data[8] $\eta\rho_n$ pertain-

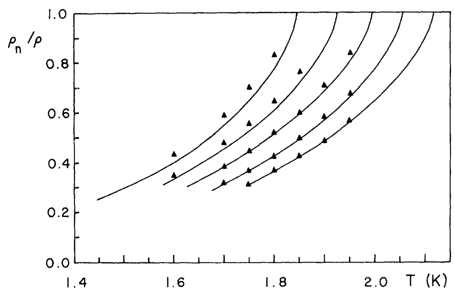

FIG. 2. Curves: smoothed values of ρ_n/ρ versus temperature from this experiment, at 5.28, 10.37, 15.04, 20.02, and 25.10 atm absolute pressure. Pressure increases from curve at lower right to curve at upper left. Triangles: values calculated from Eq. (4) for the same pressures.

ing to the viscosity. Analysis of the data with α, c_1, c_2, and $\eta\rho_n$ varying over conservatively large ranges produces little change in the calculated values of ρ_n/ρ. Uncertainties in τ_0 and τ_λ are the dominant sources of possible systematic error, with τ_0 being the more important at low values of ρ_n/ρ, and τ_λ being the major offender at high values of ρ_n/ρ. We estimate the possible systematic errors in ρ_n/ρ from these sources to be about ± 0.005 over the whole range of conditions studied. We thus conservatively estimate our values of ρ_n/ρ to be accurate to within about ± 0.015.

We have also calculated indirect values of ρ_n/ρ from second-sound velocities[10,11] u_2 and thermodynamic data,[12,13] using the equations of the two-fluid model. The commonly used relationship between these quantities is

$$u_2^2 = (\rho_s/\rho_n)\, TS^2/C_v . \tag{4}$$

Figure 2 shows the comparison between our results and some values of ρ_n/ρ calculated from Eq. (4). This comparison is also given in Table I. Allowing for the stated uncertainties in the input data, and for the existing discrepancies in values calculated from alternative sources (for example, u_2 data from Ref. 10 or 11), we estimate the uncertainties in the calculated values of ρ_n/ρ to be about ± 0.015, that is, about the same as those in our own directly measured values. Values of ρ_n/ρ at higher temperatures than those indicated by the triangles in Fig. 2, were not calculated, because the quantities used in the calculation begin to vary rapidly with temperature, and possible temperature errors in combining data from several sources become progressively more important. After due allowance is made for errors both in our values and in the calculated values, there remains a clear discrepancy, at least at the higher pressures.

This apparent discrepancy is not an indication of a failure of the two-fluid model; in fact, it provides a confirmation of one feature of the two-fluid model which was pointed out many years ago,[14] but has been almost universally ignored since that time. One assumption used in the derivation of Eq. (4) is that $C_p = C_v$, i.e., that the thermal expansion coefficient is zero. Under these conditions, the two wave modes in liquid helium are pure density oscillations u_1 and pure thermal oscillations u_2, and the velocity of the second-sound branch is given by Eq. (4). If, however, $C_p \neq C_v$, the two types of oscillation become coupled,[15] and the expressions for the velocities are modified, as discussed below.

Two-fluid hydrodynamics[14] leads to the following equation for the two-wave velocities in bulk liquid helium:

$$2u^2 = u_{10}^2 + u_{20}^2 \pm (u_{10}^2 - u_{20}^2)\left(1 + \frac{4\Theta u_{10}^2\, u_{20}^2}{(u_{10}^2 - u_{20}^2)^2}\right)^{1/2}, \tag{5}$$

where $\Theta = 1 - C_v/C_p$ (proportional to the square of the thermal expansion coefficient), and u_{10} and u_{20} denote the usual expressions for the velocities of first and second sound, valid if $C_p = C_v$:

$$u_{10}^2 = \left(\frac{\partial P}{\partial \rho}\right)_S , \tag{6}$$

$$u_{20}^2 = (\rho_s/\rho_n)\, TS^2/C_v . \tag{7}$$

For all the temperatures and pressures at which we have used the two-fluid model to calculate values of ρ_n/ρ, Θ is not larger than 0.2, and $u_{20}^2/u_{10}^2 \lesssim 0.005$. Therefore, we make the approximation $u_{20}^2 \ll u_{10}^2$ and expand the square root in Eq. (5). The two roots for u then become, for first sound

$$u_1^2 = u_{10}^2\,[1 + \Theta\, u_{20}^2/u_{10}^2] , \tag{8}$$

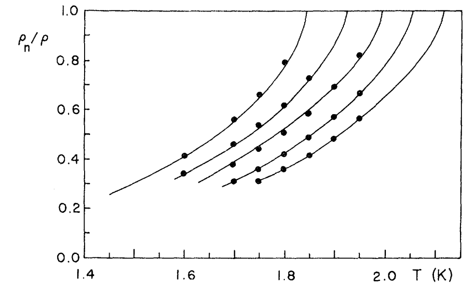

FIG. 3. Same as Fig. 2, but circles represent values calculated from Eq. (9).

and for second sound

$$u_2^2 = u_{20}^2 [1 - \Theta]$$
$$= (\rho_s/\rho_n)\, TS^2/C_p. \qquad (9)$$

Although to this approximation, the magnitudes of the corrections to the squared velocities are the same for first and second sound, the fractional correction is much less important in the case of first sound. Eq. (9) is the expression which we use henceforth to calculate indirect values of ρ_n/ρ. It is remarkable that simply replacing C_v by C_p in the usual expression for u_2 yields a result which is highly accurate under all conditions, including, of course, the case $C_p \cong C_v$.

By carrying further terms in the approximation, it is found that the error in ρ_n/ρ, if Eq. (9) is used, is at most

$$\Theta (\rho_n/\rho)\,(1-\rho_n/\rho)\,(u_{20}/u_{10})^2. \qquad (10)$$

For all the points at which we have calculated values of ρ_n/ρ, this error in ρ_n/ρ is at most 0.35×10^{-4}, or in terms of the percentage effect on ρ_n/ρ, at most 0.007%. From Eq. (10), it is clear that the remarkable accuracy of Eq. (9) results from the fact that in those regions near the λ line where the Θ term in Eq. (10) becomes large, the factors $1-\rho_n/\rho$ and $(u_{20}/u_{10})^2$ are, at the same time, becoming small.

Under the conditions of our experiments, there is often a significant difference between the values of ρ_n/ρ calculated from Eq. (4) and those calculated from Eq. (9). The change produced in the calculated values of ρ_n/ρ upon changing from Eq. (4) to Eq. (9), can be approximately expressed by

$$\Delta(\rho_n/\rho) \cong -(\rho_n/\rho)\,(1-\rho_n/\rho)\,(C_p/C_v - 1). \qquad (11)$$

Thus, for example, at $p = 25$ atm, $T = 1.75$ K, where $\rho_n/\rho \cong 0.65$ and $C_p/C_v \cong 1.17$, $\Delta(\rho_n/\rho) \cong -0.04$, which represents the major part of the discrepancy shown in Fig. 2.

The fact that Eq. (4) is only approximate was pointed out by Dingle,[14] who made the comment that in practice the corrections are negligible. Dingle cited as an example the liquid at the vapor pressure and 1.5 K, where $\rho_n/\rho \cong 0.1$, $C_p/C_v \cong 1.0005$, and thus $\Delta(\rho_n/\rho) \cong -0.45 \times 10^{-4}$. Even at temperatures much closer to the λ point, use of Eq. (4) does not lead to serious errors in the calculation of ρ_n/ρ at low pressures. At the vapor pressure, at 2.12 K, for instance, $\Delta(\rho_n/\rho) \cong -0.001$.

Probably because the estimated corrections in the example used by Dingle are so very small, the possible failure of Eq. (4) has been almost[16] universally ignored. It is clear, however, that at elevated pressures, where the rise in C_p/C_v sets in[17] at temperatures much farther below the λ line, there is a marked difference between Eqs. (4) and (9). Anyone who wishes to calculate values of ρ_n and ρ_s under pressure indirectly (i.e., from u_2 and thermodynamic data) should be aware of this fact. Furthermore, it is clear that any future experiments intended as investigations of the coupling between the two-sound modes could best be carried out at those pressures where there is a significant difference between C_p and C_v.

Figure 3 shows our results compared with values calculated from Eq. (9). (See also Table I.) The data needed to convert from C_v to C_p were obtained from Grilly's[17] comparison of the isothermal and adiabatic compressibilities, and alternatively, from the data of Elwell and Meyer[9] on the thermal expansion coefficient. Within the various experimental errors, there is no discrepancy between our directly measured values of ρ_n/ρ and those calculated from Eq. (9). It is not surprising but certainly reassuring that the two-fluid model, which has been used so successfully at the vapor pressure, can be used with equal confidence at elevated pressures. It was interesting to find that the apparent discrepancy exhibited in Fig. 2 is due to the use of a common but invalid approximation, and not to the basic equations of the model. Our results provide a confirmation of the more general expression for the second-sound velocity.

ACKNOWLEDGMENTS

We wish to thank Donald Martin, George Smith, and Glen Truehart for their part in the construction of the apparatus, and Dr. D. L. Elwell for his work on a preliminary version of this experiment.

---

*Supported in part by the National Science Foundation and the Office of Naval Research. Brief reports of this work have been given previously: D. L. Elwell and R. H. Romer, Bull. Am. Phys. Soc. 8, 90 (1963); R. H. Romer and R. J. Duffy, *ibid.* 13, 111 (1968).

†On leave from Amherst College, Mass., 1969–70; present address: Voorhees College, Denmark, S.C. 29042.

[1]E. Andronikashvilli, J. Phys. USSR 10, 201 (1946).

[2]A. C. Hollis-Hallett, Proc. Roy. Soc. (London) A210, 404 (1951).

[3]J. G. Dash and R. D. Taylor, Phys. Rev. 105, 7

(1957).

[4]J. G. Dash and R. D. Taylor, Phys. Rev. 107, 1228 (1957).

[5]J. A. Tyson, Phys. Rev. 166, 166 (1968).

[6]Reference 3, Eq. (31).

[7]J. B. Mehl and W. Zimmermann, Phys. Rev. 167, 214 (1968).

[8]J. Goodwin and W. D. McCormick (unpublished).

[9]D. L. Elwell and H. Meyer, Phys. Rev. 164, 245 (1967); D. L. Elwee, Ph.D. thesis, Duke University, 1967 (unpublished). Vapor-pressure density data are given by E. C. Kerr and R. D. Taylor, Ann. Phys. (N. Y.) 26, 292 (1964).

[10]V. P. Peshkov and K. N. Zinoveva, Zh. Eksperim. i Teor. Fiz. 18, 438 (1948).

[11]R. D. Maurer and M. A. Herlin, Phys. Rev. 82, 329 (1951).

[12]O. V. Lounasmaa, Cryogenics 1, 212 (1961).

[13]C. J. N. Van den Meijdenberg *et al.*, Physica 27, 197 (1961).

[14]R. B. Dingle, Proc. Phys. Soc. (London) A63, 638 (1950). See also, F. London, Superfluids (John Wiley & Sons, Inc., New York, 1954), Vol. II, p. 85.

[15]Such a coupling has recently been observed directly: G. Jacucci and G. Signorelli, Phys. Letters 26A, 5 (1967); A. Hofmann, K. Keck, and G. U. Schubert, *ibid.* 26A, 598 (1968).

[16]Even at the vapor pressure, Eq. (4) is inadequate in studies of the critical region very near the λ point: C. J. Pearce, J. A. Lipa, and M. J. Buckingham, Phys. Rev. Letters 20, 1471 (1968); J. A. Tyson and D. H. Douglass, *ibid.* 21, 1308 (1968); D. L. Johnson and M. J. Crooks, *ibid.* 27A, 688 (1968).

[17]E. R. Grilly, Phys. Rev. 149, 97 (1966).

PHYSICAL REVIEW VOLUME 186, NUMBER 1 5 OCTOBER 1969

Quantum Corrections to the Square-Well Classical Second Virial Coefficient

Trygve S. Nilsen

Institutt for Teoretisk Fysikk, NTH, 7034 Trondheim, Norway

(Received 1 April 1969)

The first quantum correction to the classical value of the second virial coefficient for the square-well potential is calculated. The result is $B(T)=\frac{2}{3}\pi d^3[\eta^3+(1-\eta^3)e^{-\beta u}]+2^{-1/2}\pi\lambda d^2\{e^{-\beta u}+\eta^2[1+e^{-\beta u}-2e^{-\beta u/2}I_0(\frac{1}{2}\beta u)]\}$, with $\lambda=(h^2/2\pi mkT)^{1/2}$ and $\beta=(kT)^{-1}$. Here, u denotes the depth of the well, d is the diameter of the hard core, and ηd is the range of the potential.

I. INTRODUCTION

The purpose of this paper is to give the first quantum correction to the classical value of the second virial coefficient $B(T)$ for a gas of particles interacting via the square-well potential:

$$\begin{aligned} \psi(r) &= \infty, && \text{for } r<d \\ &= u, && \text{for } d<r<\eta d \\ &= 0, && \text{for } r<\eta d \end{aligned} \qquad (1)$$

where u is a constant. This simple-model potential is supposed to represent the effect of a real intermolecular potential fairly well. To calculate the classical second virial coefficient for this potential is trivial.[1] For light gases, however, the quantum corrections are not completely negligible.[2]

The quantum-mechanical second virial coefficient for the square-well potential has two noteworthy features. First, it cannot be obtained by the usual Wigner-Kirkwood high-temperature expansion[3] because that expansion is essentially an expansion in powers of the gradient operator, and is therefore not applicable to the singular square-well potential.[4] Second, the presence of the hard core in the potential implies that at high temperatures all symmetrization effects are negligible or, more precisely, are exponentially small.[5] In a series expansion of $B(T)$ in powers of the ratio of the thermal deBroglie wavelength $\lambda=(h^2/2\pi mkT)^{1/2}$ to the hard-core diameter d, it therefore suffices to consider the spin-independent part B_{direct}.