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Evaporation from Superfluid Helium†

David S. Hyman and Marlan O. Scully
Physics Department and Center for Materials Science and Engineering,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

Allan Widom
Physics Department, Northeastern University, Boston, Massachusetts 02115

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A simple theory of evaporation from liquid He II is constructed, and it is used to analyze the experimental results of Johnston and King on the velocity distribution of atoms evaporating from He II at low temperatures. The theory suggests that rotons are responsible for shifting the velocity distribution to energies ≈ 1.5°K, in good agreement with experiment. However, multiexcitation processes are needed to explain the complete velocity distribution at high energies. We propose two new experiments that would test our model and provide new information concerning excitations in liquid helium.

I. INTRODUCTION

Some time ago, Johnston and King[1,2] reported measurements of the velocity distribution of atoms evaporating from liquid helium at 0.59–0.7°K. In all cases, the distributions were characteristic of a source approximately 1° hotter than the liquid; that is, they could be fit quite well by a Maxwellian distribution at an effective temperature $T_{\rm eff} \approx 1.5-1.6°{\rm K}$. To date, these interesting results with liquid helium have not received an adequate theoretical explanation.[3]

In this paper, we show that the above results have a plausible explanation in terms of a simple theory of evaporation from He II at low temperatures. We regard evaporation at low temperatures as a quasiparticle tunneling process: Excitations of suitable energies are annihilated near the liquid surface, accompanied by the emission of a free atom into the vacuum. Since in principle multi- as well as single-excitation processes can provide mechanisms for evaporation, we treat the tunneling problem very generally in Sec. II. Using standard methods of tunneling theory,[4] we obtain a general expression for the total evaporation rate as a function of temperature in terms of certain correlation functions describing properties of the liquid. Without making detailed assumptions about the evaporation mechanism, we find that the dominant part of the evaporation rate varies as $\exp(\mu/k_BT)$, where $\mu = -7.15°{\rm K}$ is the chemical potential, T is the absolute temperature, and k_B is Boltzmann's constant.

More detailed results are obtained by considering a specific mechanism for evaporation. We suggest that one can account for a significant fraction of the total evaporation by the direct conversion of single excitations into evaporated atoms. Since it is very difficult to estimate the relevant tunneling matrix elements from first principles, we utilize a phase-space analysis. As shown in Sec. III, phonons with energies $\geq |\mu|$ lead to an energy distribution characteristic of a source at temperature $T \approx 0.4°{\rm K}$. However, as a result of their very high density of states, the rotons lead to a highly peaked energy distribution for evaporated atoms. The peak occurs at an energy $E_{\rm peak} = (\Delta + \mu)$, with the minimum roton energy $\Delta = 8.65°{\rm K}$. This is in good agreement with the experimental results for $T \approx 0.6°{\rm K}$. From phase space considerations, we suggest that the roton excitations will be more important than the phonons, at least in the sense that they should cause an energy distribution shifted to energies $\gtrsim 1.5°{\rm K}$. Insofar as the peak in the energy distribution is due to single roton processes, we find that the dominant temperature dependence of the evaporation rate at the peak is given by $\exp[-\Delta/k_BT]$. These conclusions, and the validity of our specific assumptions, are discussed briefly.

In addition to the above results, we show in Sec. IV that two important experiments are suggested by our theory. If, as seems to be the case, the peak in the energy distributions of evaporated atoms at low temperatures is a measure of Δ, then a similar experiment with thin He II films would measure the value of $(\Delta + \mu)$ in such systems. This would constitute a valuable addition to existing information on the properties of He II films.

Moreover, assuming the general form of our tunneling Hamiltonian to be correct, we show that one would expect correlations in the arrival times at a detector of evaporating $\rm He^4$ atoms. That is, if the number of atoms at the peak of the distribution is counted in a time interval $\lesssim t_{\rm r}$, where $t_{\rm r}$ is the roton lifetime, then one should find a counting rate described by Bose-Einstein statistics. In effect, this would constitute an atomic analog of a photon counting experiment. As such, it would yield directly a good estimate of excitation lifetimes at low temperatures.

II. EVAPORATION AS A TUNNELING PROCESS

We will treat evaporation semiphenomenologically, by assuming that the process is governed by a Hamiltonian of the form

$$H=H_L+H_g+V\ , \tag{2.1}$$

where H_L is the Hamiltonian of the liquid, H_g is the free-particle Hamiltonian of the evaporated atoms, and V is a standard[4] tunneling interaction. We write V in second-quantized notation

$$V=\sum_{\vec{p}}(s_{\vec{p}}a_{\vec{p}}^{\dagger}+s_{\vec{p}}^{\dagger}a_{\vec{p}})\ ; \tag{2.2}$$

$a_{\vec{p}}(a_{\vec{p}}^{\dagger})$ denotes the annihilation (creation) operator for an evaporated atom of momentum $\vec{p}$, and $s_{\vec{p}}$ is an operator which acts only on states of the liquid. For now, we make no particular assumption about the form of the operators $(s_{\vec{p}})$, apart from the fact that they should describe the removal of an atom from the liquid. In addition, we do not need any explicit form for H_L.

With V treated in first-order perturbation theory, the transition rate into the state $\vec{p}$ is given by

$$W_{i\to f}(\vec{p},n_{\vec{p}})=(2\pi/\hbar)|\langle n_{\vec{p}}+1,\Phi_f^{(n-1)}$$
$$\times|V|n_{\vec{p}},\Phi_i^{(n)}\rangle|^2\delta(E_p+\epsilon_f^{(n-1)}-\epsilon_i^{(n)})\ . \tag{2.3}$$

$\Phi_i^{(n)}$ denotes the initial state of the liquid containing n atoms, $\Phi_f^{(n-1)}$ denotes the final state of the liquid containing $(n-1)$ atoms, and the state $\vec{p}$ was initially occupied by $n_{\vec{p}}$ atoms.

We may rewrite Eq. (2.3)

$$W_{i\to f}(\vec{p},n_{\vec{p}})=(2\pi/\hbar)(n_{\vec{p}}+1)|\langle\Phi_f^{(n-1)}$$
$$\times|s_{\vec{p}}|\Phi_i^{(n)}\rangle|^2\delta(E_p+\epsilon_f^{(n-1)}-\epsilon_i^{(n)})\ . \tag{2.4}$$

If the liquid is in thermal equilibrium at temperature T, the rate is given by

$$W_{\mathrm{em}}(\vec{p},n_{\vec{p}})=\sum_{i,f}P_iW_{i\to f}(\vec{p},n_{\vec{p}})\ , \tag{2.5}$$

where $P_i=\exp[(F_n-\epsilon_i^{(n)})]/kT$ is the canonical occupation probability of state $\Phi_i^{(n)}$. Defining

$$D^{<}(p,E)=\sum_{i,f}P_i|\langle\Phi_f^{(n-1)}|s_p|\Phi_i^{(n)}\rangle|^2$$
$$\times\delta(E+\epsilon_f^{(n-1)}-\epsilon_i^{(n)})\ ,$$

Eq. (2.5) can be written

$$W_{\mathrm{em}}(p,n_{\vec{p}})=(2\pi/\hbar)(n_{\vec{p}}+1)D^{<}(\vec{p},E_p)\ . \tag{2.6}$$

$D^{<}(\vec{p},E)$ is related in an obvious way to thermodynamic Green's functions[5]

$$\int D^{<}(\vec{p},E)\exp(-iEt/\hbar)dE=\sum_{i,f}P_i|$$
$$\times\langle\Phi_f^{(n-1)}|s_{\vec{p}}|\Phi_i^{(n)}\rangle|^2\exp\{-i(\epsilon_i^{(n)}-\epsilon_f^{(n-1)})t/\hbar\}$$
$$=\sum_{i,f}P_i\langle\Phi_i^{(n)}|s_{\vec{p}}^{\dagger}(0)s_{\vec{p}}(t)|\Phi_i^{(n)}\rangle$$
$$=\langle s_{\vec{p}}^{\dagger}(0)s_{\vec{p}}(t)\rangle\ ; \tag{2.7}$$

the brackets denote a thermal average.

Proceeding in a very similar way, we find that the absorption rate for an atom of momentum $\vec{q}$ is given by

$$W_{\mathrm{abs}}(\vec{q},n_q)=(2\pi/\hbar)n_{\vec{q}}D^{>}(\vec{q},E_{\vec{q}})\ , \tag{2.8}$$

where $\int D^{>}(\vec{q},E)e^{(-iEt/\hbar)}dE=\langle s_{\vec{q}}(t)s_{\vec{q}}^{\dagger}(0)\rangle$.

It follows directly from their definitions that

$$D^{<}(\vec{p},E)=\mathrm{e}^{(\mu-E)/kT}D^{>}(\vec{p},E)\ , \tag{2.9}$$

which is a statement of detailed balance.[5] We now specialize to the situation of experimental interest, namely, evaporation into a vacuum. In this case, we wish to calculate

$$W_{\mathrm{em}}(p,0)=\exp\{(\mu-E_p)/kT\}W_{\mathrm{abs}}(p,1). \tag{2.10}$$

We remark that in the temperature region where the number of excitations in the liquid is much less than the number of particles, $W_{\mathrm{abs}}(p,1)$ is essentially temperature-independent. In practice, this occurs for $T<0.7°$ K. Therefore, no matter what processes take place when an incident atom is absorbed by the liquid, we would expect the absorption probability to be temperature-independent in this region.

From Eq. (2.10), the total rate is then obtained by summing over all states

$$R=\text{total rate}=\sum_{\vec{p}}W_{\mathrm{em}}(\vec{p},0)=e^{\mu/kT}\sum_{\vec{p}}e^{-E_p/kT}$$
$$\times W_{\mathrm{abs}}(\vec{p},1)=e^{\mu/kT}\eta(T)\ , \tag{2.11}$$

where $\eta(T)$ is presumably a slowly varying function of T, compared to $e^{\mu/kT}$. Eq. (2.11) is our basic expression for the evaporation rate.[6]

III. CONNECTION BETWEEN EXCITATIONS AND EVAPORATION

We now wish to argue for a specific form for the operators $\{s_{\vec{p}}\}$. In doing so, we must look at the properties of He II in some detail.

It is an experimental fact that for temperatures well below the λ point, He II can be regarded as a gas of weakly interacting excitations. The energy-versus-momentum curve of these excitations is shown in Fig. 1.[7] To understand their relevance to evaporation, we must remember that it takes a finite amount of energy to remove an atom from the liquid. At low temperatures, this binding energy is[8] approximately 7.15° K; we show this energy as a dashed horizontal line in Fig. 1. Thus, the problem is to ascertain the way in which the energy of these excitations is transferred to evaporating atoms.

The simplest dynamical process that could lead to evaporation is the annihilation of an excitation near the liquid surface, accompanied by the emission of a free atom. However, since we require energies $\gtrsim |\mu|$, Fig. 1 shows that only rotons and high-energy phonons would contribute. Nevertheless, we believe that such processes should account for a significant fraction of the total evaporation. In fact, reasonable support for this conjecture comes from experimental results of neutron scattering from He II.[7] It is known that a substantial fraction of the inelastic scattering is accounted for by the production of a single excitation in the liquid. For wave vectors $q \lessapprox 0.6$ Å^{-1}, almost all the scattering takes place via single phonon production. In the region of the roton minimum, Miller *et al.*[9] estimate that about 60% of the scattering is accounted for by roton production. If we now imagine an experiment in which cold He^4 atoms are used as a probe instead of neutrons, then the neutron results would suggest that a similar fraction of the absorbed atoms would create single excitations. Since evaporation is essentially the inverse of this process, we would then expect from detailed balance, [Eq. (2.10)], that our single excitation mechanism accounts for roughly 50% of the total evaporation. We will, therefore, examine evaporation in the context of this assumption.

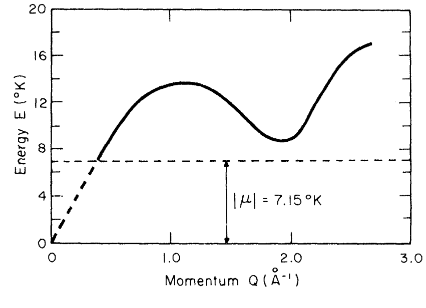

FIG. 1. Experimental excitation spectrum of He II, obtained by Henshaw and Woods (Ref. 7). The horizontal dotted line at $E = 7.15$ °K represents the chemical potential, or binding energy per atom.

Introducing the destruction operator $b_{\vec{k}}$ for an excitation of momentum $\vec{k}$, we write the operator $s_{\vec{p}}$ in the form

$$s_{\vec{p}}(t) = \sum_{\vec{k}} \lambda_{\vec{p}\vec{k}} b_{\vec{k}}(t) e^{-i\mu t/\hbar} ; \tag{3.1}$$

the $\{b_{\vec{k}}\}$ satisfy the usual boson commutation relations

$$[b_{\vec{p}}, b_{\vec{q}}^{\dagger}] = \delta_{\vec{p}\vec{q}} ,$$

and the factor $e^{-i\mu t/\hbar}$ expresses the fact that $s_{\vec{p}}$ couples states of N and $N-1$ particles. We then find that

$$\langle s_{\vec{p}}(t) s_{\vec{p}}^{\dagger}(0)\rangle = \sum_{\vec{q}} |\lambda_{\vec{p}\vec{q}}|^2 \exp[-i\mu t/\hbar] \langle b_{\vec{q}}(t) b_{\vec{q}}^{\dagger}(0)\rangle , \tag{3.2}$$

and $$D^{>}(\vec{p}, E) = \sum_{\vec{q}} |\lambda_{\vec{p}\vec{q}}|^2 G^{>}(\vec{q}, E-\mu), \tag{3.3}$$

where $G^{>}(\vec{q}, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} \langle b_{\vec{q}}(t) b_{\vec{q}}^{\dagger}(0)\rangle \, dt$.

Therefore, the emission rate into the state $\vec{p}$ is given by

$$W_{\text{em}}(\vec{p}, 0) = \exp\{(\mu - E_p)/k_B T\} (2\pi/\hbar) \times \sum_{\vec{k}} |\lambda_{\vec{p}\vec{k}}|^2 G^{>}(\vec{k}, E_p - \mu). \tag{3.4}$$

Equation (3.4) is our formal expression for the emission rate as a function of momentum. In the absence of a detailed theory of the matrix elements $\{\lambda_{\vec{p}\vec{k}}\}$, we will analyze the respective contributions of rotons and phonons to $W_{\text{em}}(\vec{p})$ from phase-space considerations.

If we regard phonons as having infinite lifetimes at low temperatures, then we may write

$$G^{>}_{\text{phonon}}(\vec{k}, \omega) = \{1 + f(\omega)\} (2\pi) \delta(\omega - ck),$$

where $f(\omega) = (e^{\omega/k_B T} - 1)^{-1}$

and c is the speed of sound in the liquid. Since we need $\omega \gtrsim |\mu|$, it is clear that $\exp(\omega/k_B T) \gg 1$ at low temperatures. Therefore, we may write

$$G^{>}_{\text{phonon}}(\vec{k},\omega)\simeq(2\pi)\delta(\omega-ck)$$

with sufficient accuracy. We use this expression to evaluate the phonon contribution to Eq. (3.4). After replacing the matrix elements $\{\lambda_{pk}\}$ by some constant λ, and converting the sum to an integral, we find

$$\begin{aligned} W_{\text{em}}(p,o) &\propto e^{(\mu-E_p)/k_BT}\int dk(2\pi k^2)\delta(E_p-\mu-ck) \\ &\propto e^{(\mu-E_p)/k_BT}\int \epsilon^2\delta(E_p-\mu-\epsilon)d\epsilon \\ &\propto e^{(\mu-E_p)/k_BT}(E_p-\mu)^2. \end{aligned} \tag{3.5}$$

To obtain the emission rate into an energy interval dE_p around E_p, we multiply (3.5) by the free-particle density of states $N(E_p)$:

$$W_{\text{em}}(\vec{p},o)N(E_p)dE_p = \text{constant} \times e^{(\mu-E_p)/k_BT}(E_p-\mu)^2\sqrt{E_p}\,dE_p. \tag{3.6}$$

The energy-dependent part of (3.6) is shown as the broad curve in Fig. 2. It can be seen that the phonon contribution resembles somewhat a thermal distribution peaked at an energy ≈ 0.4° K. This analysis shows, therefore, that phonons alone cannot account for the form of the distribution measured in Ref. 1. Rather, they can only lead to appreciable evaporation at energies less than 1.5–1.6° K.

To evaluate the roton contribution to $W_{\text{em}}(\vec{p})$, we need to include explicitly the finite (but small) roton linewidth. In the absence of the latter, the infinite roton density of states at $\epsilon_{\text{roton}}=\Delta$ would lead to a divergent result for $W_{\text{em}}(\vec{p})$. Thus, assuming a Lorentzian line shape for rotons, we may write

$$\begin{aligned} G^{>}_{\text{roton}}(p,\omega) &= \{1+f(\omega)\}\{\gamma/[(\omega-E_p)^2+\tfrac{1}{4}\gamma]\} \\ &\simeq \gamma/[(\omega-E_p)^2+\tfrac{1}{4}\gamma^2] \;, \end{aligned} \tag{3.7}$$

where γ is an average roton linewidth. An approximate theoretical calculation of γ is given in Appendix A. With this Lorentzian line shape

$$\begin{aligned} W_{\text{em}}(E,o) &\propto e^{(\mu-E)/k_BT}\int_{\text{rotons}} p^2\,\frac{\gamma}{(E-\mu-E_p)^2+\frac{1}{4}\gamma^2}\,dp \\ &\propto e^{(\mu-E)/k_BT}p_o^{\,2}\int_{\Delta}^{\infty} d\epsilon\,\frac{(\epsilon-\Delta)^{-1/2}\gamma}{(E-\mu-\epsilon)^2+\frac{1}{4}\gamma^2}\;. \end{aligned}$$

We evaluate this integral in Appendix C. The result is

$$W_{\text{em}}(E,o)\propto\exp\{(\mu-E)/k_BT\}\{(E-\mu-\Delta)^2+\tfrac{1}{4}\gamma^2\}^{-3/4}F(E), \tag{3.8}$$

where $F(E)=\frac{1}{2}\gamma$, for $(E-\mu-\Delta)\leqslant 0$,

and $F(E)=\{2(E-\mu-\Delta)^2+\frac{1}{4}\gamma^2\}^{1/2}$, for $(E-\mu-\Delta)>0$.

This distribution is the very narrow spike shown in Fig. 2; we have chosen $\gamma/k_B=0.001$°K as a reasonable estimate of the roton linewidth for $T\approx 0.6$ °K (cf. Appendix A). The roton contribution is seen to be sharply peaked at an energy ≈ 1.5 °K. This is a direct consequence of the very high density of states of rotons for $\epsilon\sim\Delta$. However, single-roton processes do not account for the major part of the evaporation at energies $\neq 1.5$ °K. Nevertheless, our analysis suggests that, at least for $T\approx 0.6$ °K, the peak in the experimentally measured velocity distribution is due to direct conversion of rotons into evaporated atoms at the surface of the liquid. We expect the rotons to dominate near the peak because for $T\gtrsim 0.6$°K, there are many more rotons available compared to high-energy phonons. Even at $T=0.6$°K, $n_r/n_{\text{ph}}\sim 9$, where n_r is the number density of rotons, and n_{ph} is the number density of phonons with energies $\geqslant 7.15$ °K. Thus, if the peak is due to single-

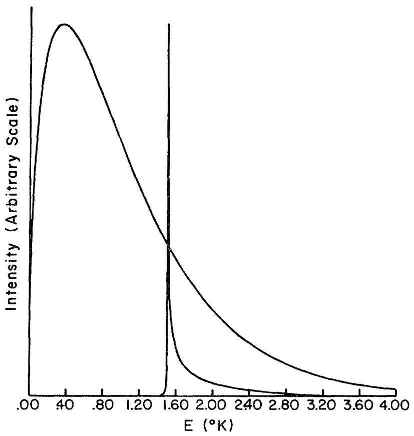

FIG. 2. Broad curve is the evaporation rate due to high-energy phonons [Eq. (3.6)]. The thin spike is the rate due to rotons [Eq. (3.8)]. Both curves were calculated for $T=0.6$° K, and the vertical scale of each is in arbitrary units.

roton processes, then our analysis leads to the following predictions: (a) The energy of the peak in the velocity distribution is given by $\bar{E} = \Delta + \mu$. (b) The evaporation rate into an energy interval dE around $\bar{E}$ should have a temperature dependence given essentially by $\exp[-\Delta/kT]$.

Experiments that would test these predictions are currently in progress.[10]

It is clear from our phase-space analysis that much of the evaporation at high energies must be due to multiexcitation processes. In order to estimate graphically the contribution of the neglected processes, we compare the experimental emission rate with a linear superposition of the phonon-roton contributions of Fig. 2; such a comparison is shown in Fig. 3. The outer curve is a Maxwellian distribution; $W(E) = Ee^{-E/k_BT_e}$, with $T_e = 1.5$ °K. The inner curve is the linear superposition, adjusted so that the height of the peak at 1.5 °K matches that of the Maxwellian. Again, since this is a phase-space analysis, the curves represent evaporation rates in arbitrary units. Figure 3 thus suggests that approximately 65% of the evaporation arises from multiexcitation processes. Prominent among these are probably roton-phonon interactions leading to evaporated atoms. That is, one might be able to account for more of the evaporation at energies ~ 1.5 °K by processes in which a roton decays at the surface of the liquid into a low-energy phonon and an evaporated atom. However, we have not investigated this mechanism in detail. In addition, surface excitations do exist,[11] and they undoubtedly play some role as well. These, and other more complicated processes become difficult to estimate precisely. Nevertheless, we feel that our analysis in terms of the simplest processes does offer a reasonable explanation for the relative abundance of evaporated atoms at energies ~ 1.5–1.6 °K.

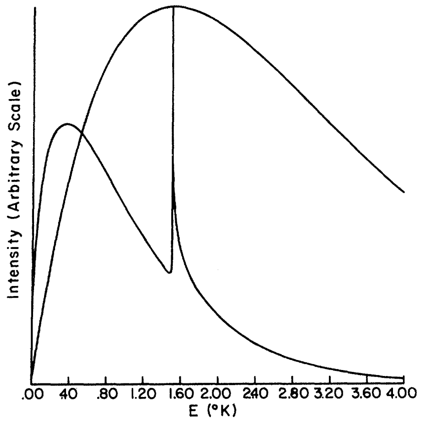

FIG. 3. Comparison of Maxwellian distribution at 1.5° K (outer curve) with linear combination of evaporation rates at $T = 0.6$° K due to phonons and rotons (inner curve). Vertical scale is in arbitrary units.

IV. SUGGESTED EXPERIMENTS

On the basis of the preceding discussion, we suggest two new experiments that would test our model and provide new information concerning excitations in liquid helium. The first would involve repeating the Johnston-King experiment with thin He II films. If, in a suitable temperature range, the peak in the evaporation curve is due to single-roton annihilation, then the value of the corresponding energy at the peak will give the value of $(\Delta + \mu)$ for the film. Since μ can be obtained from existing thermodynamic data, the value of Δ for thin films could then be determined.

In addition, the general form of the interaction Hamiltonian [Eq. (2.2)] suggests a novel extension of photon counting experiments to the case of evaporating He^4 atoms. One expects (cf. Appendix B) that those atoms which are ejected due to the annihilation of a roton will obey Bose-Einstein statistics. That is,

$$P_k(n,t) = [\bar{n}_k(t)]^n / [1 + \bar{n}_k(t)]^{n+1} \quad , \tag{4.1}$$

where $P_k(n,t)$ is the probability that n atoms of momentum $\vec{k}$ evaporated in a time interval t, and $\bar{n}_k(t)$ is the mean number of such atoms.

Equation (4.1) is recognized as a Bose-Einstein distribution; as such, it implies correlations in the arrival times of He atoms at a detector. This phenomenon, which was first observed by Hanbury Brown and Twiss[12] with photon beams, is fully discussed in the literature.[13] The correlations can be seen experimentally by measuring $P_{\vec{k}}(2,t)$, which is the probability that two evaporated atoms of momentum $\vec{k}$ will be counted in a time interval t. In the range of energies in which single-roton processes give a large contribution to the evaporation, $P_{\vec{k}}(2,t)$ will be twice as large as the value expected from the Poisson statistics of random events, provided $t \lesssim$ (roton lifetime). At temperatures for which the roton lifetime is approximately 10^{-8}–10^{-9} sec (corresponding to a linewidth $\approx 10^{-3}$–10^{-4} °K), such a correlation experiment would be feasible. Apart from the intrinsic interest of seeing "boson bunching" with He^4 atoms instead of photons, this experiment would provide a direct measure of the roton lifetime at low temperatures.

The above experiment would tend to confirm the assumptions of this model. To the extent that Eq. (3.1) is valid, the above correlation effect is a rigorous result. That is, it can be obtained as an exact solution to the equations of motion for the system.

V. CONCLUSIONS

We have shown in the preceding sections that evaporation from superfluid helium can be treated as a quasiparticle tunneling process. Our calculation of the evaporation rate from phase-space considerations, although not complete, does suggest a plausible explanation of the experimental results of Johnston and King. In addition, it suggests that evaporation can serve as a useful tool for investigating rotons in He II. Toward this end, we feel that our approach offers a useful starting point for more detailed calculations.

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APPENDIX A

We indicate here the essential steps in a calculation of the roton lifetime at low temperatures. For $T \lesssim 1°$, the dominant processes leading to a finite roton lifetime are roton-phonon collisions. Following Landau and Khalatnikov,[14, 15], we assume that the relevant interaction is given by

$$V = \tfrac{1}{2}(\vec{\mathrm{p}}\cdot\vec{\mathrm{v}} + \vec{\mathrm{v}}\cdot\vec{\mathrm{p}}) + \tfrac{1}{2}\left[\frac{\partial^2\Delta}{\partial\rho^2} + \left(\frac{1}{\mu_0}\right)\left(\frac{\partial p_0}{\partial\rho_0}\right)^2\right][\delta\rho(\vec{\mathrm{r}})]^2$$

$$\equiv V_1 + V_2 \quad , \tag{A1}$$

where $\vec{\mathrm{p}}$ is the roton momentum, $\vec{\mathrm{v}}$ is the velocity field associated with a long-wavelength phonon; p_0, Δ, and μ_0 are parameters characterizing the roton spectrum; and $\delta\rho(\vec{\mathrm{r}})$ is the density fluctuation associated with the phonon. In this quantum hydrodynamic approach, $\delta\rho(\vec{\mathrm{r}})$ and $\vec{\mathrm{v}}(\vec{\mathrm{r}})$ are related to the phonon creation and destruction operators $(b_{\vec{\mathrm{k}}}^{\pm}, b_{\vec{\mathrm{k}}})$ by

$$\delta\rho(\vec{\mathrm{r}}) = \sum_{\vec{\mathrm{k}}}(b_{\vec{\mathrm{k}}}e^{i\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}} + b_{\vec{\mathrm{k}}}^{\dagger}e^{-i\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}) \; ,$$

$$\vec{\mathrm{v}}(\vec{\mathrm{r}}) = \sum_{\vec{\mathrm{k}}}(v_{\vec{\mathrm{k}}}e^{i\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}} + v_{\vec{\mathrm{k}}}^{\dagger}e^{-i\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}) \; ; \tag{A2}$$

$\vec{\mathrm{v}}(\vec{\mathrm{k}}) = (c\vec{\mathrm{k}}/\rho_0 k)b_{\vec{\mathrm{k}}}$, ρ_0 is the uniform liquid density, c is the speed of sound in the liquid, and

$$[b_{\vec{\mathrm{k}}}, b_{\vec{\mathrm{p}}}^{\dagger}] = \delta_{\vec{\mathrm{k}}\vec{\mathrm{p}}} \quad .$$

For the scattering process $\vec{\mathrm{p}}_1 + \vec{\mathrm{q}}_1 \rightarrow \vec{\mathrm{p}}_1' + \vec{\mathrm{q}}_1'$, in which $\vec{\mathrm{p}}_1(\vec{\mathrm{p}}_1')$ refers to the initial (final) roton and $\vec{\mathrm{q}}_1(\vec{\mathrm{q}}_1')$ refers to the initial (final) phonon, the first term on the right-hand side of (A1) contributes only in second order, and the second term contributes in first order. The relevant matrix elements are given by

$$\langle f|V_1|i\rangle = \sum_n \frac{\langle\vec{\mathrm{p}}_1'\mathrm{q}_1'|V_1|n\rangle\langle n|V_1|\vec{\mathrm{p}}_1\vec{\mathrm{q}}_1\rangle}{E_i - E_n} = (c/2\rho_0)$$

$$\times(q_1q_1')^{-1/2}\{[p_0^2q_1q_1'(\vec{\mathrm{m}}\cdot\vec{\mathrm{n}})(\vec{\mathrm{m}}\cdot\vec{\mathrm{n}}')$$

$$+p_0^2q_1^2q_1'(\vec{\mathrm{m}}\cdot\vec{\mathrm{n}})(\vec{\mathrm{n}}\cdot\vec{\mathrm{n}}')][(cq_1)^{-1}$$

$$+(\vec{\mathrm{m}}\cdot\vec{\mathrm{n}})^2/2\mu_0c^2] + [-(cq_1')^{-1} + (\vec{\mathrm{m}}\cdot\vec{\mathrm{n}}')^2/2\mu_0c^2]$$

$$\times[p_0^2q_1q_1'(\vec{\mathrm{m}}\cdot\vec{n})(\vec{\mathrm{m}}\circ\vec{\mathrm{n}}') - p_0q_1(q_1')^2(\vec{\mathrm{m}}\cdot\vec{\mathrm{n}}')(\vec{\mathrm{n}}\cdot\vec{\mathrm{n}}')]\}$$

$$\times\delta(\vec{\mathrm{p}}_1 + \vec{\mathrm{q}}_1 - \vec{\mathrm{p}}_1' - \vec{\mathrm{q}}_1') \quad , \tag{A3}$$

and

$$\langle f|V_2|i\rangle = (\rho_2/2c)(9_19_1')^{1/2}\left[\frac{2^2\Delta}{2\rho^2} + \frac{1}{\mu_0}\left(\frac{2p^0}{2\rho}\right)^2\right]$$

$$\times\delta(\vec{\mathrm{p}}_1 + \vec{\mathrm{q}}_1 - \vec{\mathrm{p}}_1' - \vec{\mathrm{q}}_1) \equiv A(q_1q_1')^{1/2}\delta(\vec{\mathrm{p}}_1 + \vec{\mathrm{q}}_1 - \vec{\mathrm{q}}_1') \, . \tag{A4}$$

In (A3) and (A4), we have set $|\vec{\mathrm{p}}_1| = p_0$; $\vec{\mathrm{m}}$ is a unit vector in the direction of the initial roton, and likewise $\vec{\mathrm{n}}$ and $\vec{n}'$ refer to the incident and scattered phonon, respectively. In addition, since the process is similar to the scattering of a light particle by a heavy one, the direction of the roton, and the energy of the phonon, are left practically unchanged. This is reflected in the simplified form of the matrix elements given in (A2).

The lifetime is now obtained by averaging over the incident phonon momentum, and summing over final states

$$\tau_{\mathrm{roton}}^{-1} = (2\pi/\hbar)\int[\exp(cq_1/k_bT) - 1]|\langle f|V_1 + V_2|i\rangle|^2$$

$$\times\delta[cq_1 - cq_1' - (p_1' - p_0)^2/2\mu_0](2\pi\hbar)^{-9}d\vec{\mathrm{q}}_1 d\vec{\mathrm{q}}_1' d\vec{\mathrm{p}}_1 \, . \tag{A5}$$

Substituting (A3) and (A4) into (A5), and letting

$$\delta[cq_1 - cq_1' - (p_1' - p_0)^2/2\mu_0] \approx \delta[cq_1 - cq_1'] \, ,$$

we find the result of the integration to be

$$\tau_{\mathrm{roton}}^{-1} = (h^3c)^{-1}6!\,\zeta(7)[k_bT/c]^7$$

$$\times\{A^2 + (p_0/\rho_0)^2[(p_0/\mu_2c)^2 + 2]\} \, , \tag{A6}$$

where $\zeta(7)$ is the Riemann ζ function. Expressing this as an energy width in °K, and evaluating it for $T = 0.6°$ K, we find

$$\delta\epsilon_{\mathrm{roton}} = (\hbar/k_b)\tau_{\mathrm{roton}}^{-1} \simeq 4\times10^{-4}\,°\mathrm{K}.$$

Since this kind of calculation is probably accurate only to a power of 10, the above number is only a guide, but nevertheless a useful one.

APPENDIX B

Systems of coupled oscillators have been well studied in the context of radiation theory and photon statistics.[16] For the sake of completeness, however, we will outline here the proof of the result stated in Sec. IV. Furthermore, to reduce the algebra, we will treat a simpler version of our Hamiltonian; this involves no loss of generality, and it enables us to see the point clearly.

We consider two coupled oscillator modes described by the Hamiltonian

$$H=\omega b^{\dagger}b+\epsilon a^{\dagger}a+\lambda(a^{\dagger}b+ab^{\dagger})\;; \tag{B1}$$

the operators $(a, a^{\dagger})$ describe an evaporated atom, and the $(b, b^{\dagger})$ describe an excitation in the liquid. We introduce the coherent states $|\alpha\rangle$ of an oscillator,[17] defined by

$$a|\alpha\rangle=\alpha|\alpha\rangle\;,$$

and given by $|\alpha\rangle=e^{\alpha a^{\dagger}-\alpha^{*}a}|0\rangle$.

The parameter α is an arbitrary complex number, and $a|0\rangle=0$. If, at time t=0, the b mode is in thermal equilibrium, and the a mode is in its ground state, we can write the initial statistical operator for the system in the form

$$\hat{\rho}(0)=\pi^{-1}\int d^2\beta\,|\beta\rangle\,|0\rangle\langle 0|\langle\beta|\,P(\beta)\quad, \tag{B2}$$

where $P(\beta)$ is the appropriate statistical weight, and the integration is over the two-dimensional complex β plane. The time evolution of the system is then described by

$$\hat{\rho}(t)=e^{-iHt}\hat{\rho}(0)e^{iHt}\;\hat{\rho}(0)e^{iHt}\;.$$

To evaluate this, we make use of a theorem due to Glauber,[18] which states that for the kind of Hamiltonian we are considering, a state which is initially coherent remains coherent in time. Thus the state given by

$$|\Psi(0)\rangle=|0\rangle\,|\beta\rangle \tag{B3}$$

evolves in time as $|\Psi(t)\rangle=|\alpha(t)\rangle\,|\beta(t)\rangle$, where $\alpha(t)$ and $\beta(t)$ are easily calculated in terms of $\alpha(0)$ and $\beta(0)$. It then follows that:

$$\hat{\rho}(t)=(\pi)^{-1}\int d^2\beta\,P(\beta)\,|\alpha(t)\rangle\,|\beta(t)\rangle\langle\beta(t)|\langle\alpha(t)|\,. \tag{B4}$$

Now, $\alpha(t)$ and $\beta(t)$ satisfy the coupled equations

$$i\hbar\,\frac{d\alpha}{dt}=\epsilon\alpha(t)+\gamma\beta(t)\;,$$

$$i\hbar\,\frac{d\beta}{dt}=\omega\beta(t)+\gamma\alpha(t)\;.$$

These have the solution[16]

$$\begin{aligned}\alpha(t)&=\mu(t)\alpha(0)+\zeta(t)\beta(0)\;,\\ \beta(t)&=\nu(t)\alpha(0)+\eta(t)\beta(0)\;,\end{aligned} \tag{B5}$$

where μ, ζ, ν, and η are coefficients whose explicit form we do not need. In our problem, $\alpha(0)=0$, and $\beta(0)$ is arbitrary. For a thermal ensemble,

$$P(\beta)=f(\omega)\,[f^{-1}(\omega)-1]\exp\{|\beta|^2[f(\omega)-1]\}\;, \tag{B6}$$

with $\qquad f(\omega)=e^{-\omega/k_BT}$.

Using (B1), we may calculate the probability that the gas contains n atoms at time t

$$P(n,t)=\sum\nolimits_b\langle n,b|\hat{\rho}(t)|n,b\rangle\;, \tag{B7}$$

and we sum over all b states. This gives

$$P_n(t)=f(\omega)[f^{-1}(\omega)-1]^2\,[\zeta(t)]^{2n}/[1-f(\omega)+\zeta^2(t)]^{n+1}. \tag{B8}$$

However, defining

$$\bar{n}(t)=\mathrm{Tr}\{\hat{\rho}(t)a^{\dagger}a\}\;,$$

we find $\bar{n}(t)=[\zeta(t)]^2/[1-f(\omega)]$.

This enables us to rewrite $P(n,t)$ as

$$P(n,t)=[\bar{n}(t)]^{n}/[1+\bar{n}(t)]^{n+1}\;.$$

This we recognize as the result characteristic of Bose-Einstein statistics, and it implies the correlations first observed by Hanbury Brown and Twiss.[12] The generalization of this derivation to the case of an arbitrary set of $\{b_j\}$ and $\{a_i\}$ is straightforward. Moreover, if a given mode b_j has a natural lifetime, then our above result is essentially correct for time $t\lesssim$(excitation lifetime). For $t\gg$ the lifetime, Poisson statistics are reproduced.

APPENDIX C

We wish to evaluate the integral

$$I(E)=\int_{\Delta}^{\infty}d\epsilon\,(\epsilon-\Delta)^{-1/2}\left[\frac{a}{a^2+(E-\mu-\epsilon)^2}\right]\;. \tag{C1}$$

With the substitution $\epsilon-\Delta=y^2$, (C1) becomes

$$I(E)=(2a)\int_0^{\infty}dy\;[a^2+(E-\mu-\Delta-y^2)^2]^{-1}$$

$$=a\int_{-\infty}^{\infty}dy\;[a^2+(\zeta-y^2)^2]^{-1}, \tag{C2}$$

where $\zeta = E - \mu - \Delta$.

The denominator in the integrand of (C2) can be factored into the form

$$(y + z_1^{1/2})(y - z_1^{1/2})(y + z_2^{1/2})(y - z_2^{1/2}) \ ,$$

where $z_{1,2} = \zeta \pm i a$.

Therefore, (C2) is readily evaluated as a contour integral, in which the path encloses the upper-half z plane

$$I(E) = \oint a[(z + z_1^{1/2})(z - z_1^{1/2})(z + z_2^{1/2})(z - z_2^{1/2})]^{-1} \cdot dz. \tag{C3}$$

The positions of the poles of the integrand depend on the sign of ζ. For $\zeta > 0$, $\sqrt{z_1}$ and $(-\sqrt{z_2})$ are in the upper-half z plane. We then obtain

$$\begin{aligned} I(E) &= a(2\pi i)\left\{\frac{1}{2z_1^{1/2}(z_1 - z_2)} + \frac{1}{(-2z_2^{1/2})(z_2 - z_1)}\right\} \\ &= \pi(a^2 + \zeta^2)^{-1/4}\cos\tfrac{1}{2}[\tan^{-1}(a/\zeta)] \\ &= (\pi/\sqrt{2})(a^2 + 2\zeta^2)^{1/2}(a^2 + \zeta^2)^{-3/4} \ . \end{aligned}$$

For $\zeta < 0$, $\sqrt{z_1}$ and $\sqrt{z_2}$ are in the upper-half z plane. Therefore,

$$\begin{aligned} I(E) &= a(2\pi i)\left\{\frac{1}{2z_1^{1/2}(z_1 - z_2)} + \frac{1}{2z_2^{1/2}(z_1 - z_2)}\right\} \\ &= -\pi(a^2 + \zeta^2)^{-1/4}\sin\tfrac{1}{2}[\tan^{-1}(a/\zeta)] \\ &= (\pi/\sqrt{2})\, a(a^2 + \zeta^2)^{-3/4} \ . \end{aligned}$$

It is clear that the two contributions are equal for $\zeta = 0$, as they should be.

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