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Derivation of a Quasiparticle Transport Equation for an Impure Fermi Liquid at Low Temperatures*

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The problem of the linear response to a longitudinal driving field of low frequency and long wavelength for a system of interacting fermions at low temperatures in the presence of dilute random impurities is studied by the use of temperature Green's function techniques. A quasiparticle distribution function for this system is defined and its connection with induced quantities, such as the particle and current densities, is determined. It is shown that this distribution function satisfies a transport equation with a nondissipative part of the form suggested by Landau and a dissipative part made up of the sum of impurity and interparticle scattering terms. The quantities entering the theory, among which are the coefficients of the transport equation, are determined to *all orders* in the interparticle and impurity interaction strengths, and, where appropriate, to first order in the impurity density. Many of these results are obtained from the development and use of a generalization to an impure system of Eliashberg's work on pure Fermi liquids.

I. INTRODUCTION

The phenomenological theory of long-wavelength low-frequency transport in a normal Fermi liquid at low temperature was first discussed by Landau[1-3] for a pure system with short-range interparticle interactions and extended to a system with Coulomb interactions by Silin[4]; several other authors[5-7] have reviewed this work, as well as discussing some of its implications. This theory was given in terms of a quasiparticle transport equation with a nondissipative part closely resembling in form that obtained in the weak coupling limit, except that energy and velocity are renormalized, and there is a quasiparticle interaction term. For the dissipative part of the equation, Landau suggested an interquasiparticle scattering term, also of a form analogous to the weak coupling limit, which has the important property that it vanishes in the zero-temperature limit. Silin,[8] Heine,[9] and Heine *et al.*[10] have considered the extension of the theory to the case of a Fermi liquid at zero temperature in the presence of dilute random impurities. This extension essentially amounts to having a transport equation with a nondissipative part of the form suggested by Landau and a dissipative part, appropriately renormalized, in the form of the usual impurity scattering term. It is natural to expect that at nonzero temperatures, the transport equation appropriate to an impure Fermi system has a nondissipative part of the Landau form and a dissipative part, just the sum of the interparticle and impurity scattering terms discussed above. We shall establish this result to *all orders* in perturbation theory in this paper.

Much work has been done to justify the phenomenological theory of transport in a Fermi liquid. A transport equation has been derived for a pure zero-temperature system by Nozières and Luttinger,[11,12] using the zero-temperature limit of the temperature Green's function technique, and by Nozières,[6] using the zero-temperature Green's function technique. Both of these studies depend on Landau's[3] idea for the handling of the type of singularity, crucial to the theory, which comes about because the poles of two propagators are very close to, but on opposite sides of an appropriate integration contour. The derivation of the quasiparticle transport equation for a pure system at low (and possibly zero) temperature has been given to all orders in perturbation theory by Eliashberg,[13,14] whose work is reviewed in the book of Abrikosov, Gor'kov, and Dzyaloshinskii.[15] This calculation required a careful study of the analytic properties of various many-body functions entering in the temperature Green's function technique, and produced a result in agreement with the prediction of Landau[1-3] (including the appropriate dissipative part). Results of a similar form were found by Résibois,[16,17] and Watabe and Dagonnier[18] using a diagram technique developed by Résibois (in which the coefficients of the transport equation for the bare-particle distribution function are found directly). This method has not lent itself to a complete summation with respect to the interactions and thus the calculation was done only to finite order in perturbation theory.

The problem of transport in an impure interacting Fermi liquid has been studied by various authors. Langer,[19-22] in a series of papers, evaluated the current induced in such a system at zero temperature by a static uniform field; he made a start, as well, at obtaining interparticle scattering effects in this system. To make these calculations, Langer developed a systematic technique,[20,22] of which we will make use in our work, for the evaluation of the discontinuities of many-body functions across their branch line singularities. Using zero-temperature Green's function techniques, Betbeder-Matibet and Nozières[23,24] derived a zero-temperature quasiparticle transport equation of the form suggested in the phenomenological theory[8–10] for an impure Fermi liquid. In the work of Sigel and Argyres,[25,26] this problem has been studied by the extension of the techniques of Résibois to an impure system. In these papers, a dissipative part of the form suggested earlier, namely, the sum of impurity and interparticle scattering terms, was found. As in the case of Résibois, this work was done only to finite order in perturbation theory.

In the present paper, we attack the problem of the linear response to a longitudinal driving field of low frequency ω and long wavelength q^{-1} for a normal Fermi liquid at low temperatures in the presence of dilute random impurities. We derive a quasiparticle transport equation with coefficients determined to all orders in the interaction strengths (but to first order in the impurity density), which has the form suggested above. To reach this result, we develop a technique which is the generalization to the case of an impure system of the one given by Eliashberg[13–15] for a pure system. We use this method because of the great ease and clarity with which it allows one to study certain types of singular behavior (including that mentioned above[3]) connected with various properties of the Fermi surface (as, for example, the incomplete degeneracy at nonzero temperatures). The understanding of this behavior, we feel, can be exploited to explain other phenomena (not discussed here) involving, essentially, the features of the Fermi surface.

To complete this section, we indicate briefly the organization of the rest of the paper. In Sec. II, we discuss the impurity averaged bare-particle distribution function and its connection with the two-particle temperature Green's function. In Sec. III, we introduce the relevant features of the diagram technique, and from a study of the analytic properties of the functions relevant to the theory, we obtain integral equations of the Bethe-Salpeter type. Sections IV and V are devoted to a detailed determination of the quantities involved in these equations. In Sec. VI, we collect together our results and show how one of the integral equations obtained in Sec. III can be interpreted as a quasiparticle transport equation; we also discuss here the partial connection of the quasiparticle distribution function to the bare-particle distribution function. To conclude Sec. VI, we point out, very briefly, the modifications necessary when the interactions are Coulombic in nature.

II. BARE-PARTICLE DISTRIBUTION FUNCTION AND GREEN'S FUNCTIONS

In this section, we formulate the problem of transport for a system of interacting fermions in the presence of random impurities, in terms of the impurity averaged bare-particle distribution function. This function determines the linear response of most of the single-particle quantities associated with the system. We further review the connection between this distribution function and the impurity averaged two-particle Green's function.

The unperturbed system, consisting of a set of interacting fermions in the presence of randomly placed spinless impurity centers all confined to unit volume, has a Hamiltonian

$$H=\sum_k \epsilon_k a_k{}^\dagger a_k+\tfrac{1}{4}\sum_{kll'k'} V(kll'k')a_k{}^\dagger a_l{}^\dagger a_{l'} a_{k'}+\sum_{j=1}^{n_i}\sum_{kk'}\left[u(k-k')e^{-i(\vec{k}-\vec{k}')\cdot\vec{R}_j}\right]a^\dagger_k a_{k'} \quad . \tag{2.1}$$

In this equation, $a_k(a_k{}^\dagger)$ is the annihilation (creation) operator for a fermion of momentum $\vec{k}$ and spin σ; ϵ_k is the energy of a bare particle having momentum and spin k [$\epsilon_k=(\vec{k})^2/2m$ in the case of a homogeneous gas and we take $\hbar=1$]. $V(kll'k')$ is the antisymmetrized matrix element of the interparticle interaction, while $u(k-k')\exp[-i(\vec{k}-\vec{k}')\cdot\vec{R}_j]$ is the matrix element of the interaction between an electron and the impurity at position $\vec{R}_j$ [without loss of generality we take $u(0)\equiv 0$]. The sum over j in the third term on the right-hand side of Eq. (2.1) is over all n_i impurities, and the matrix element is proportional to a unit matrix in the spin coordinates. In equilibrium at temperature β^{-1}, the properties of the system are determined by the density matrix

$$\rho_0=\exp[-\beta(H-\mu N)]/\mathrm{Tr}\{\exp[-\beta(H-\mu N)]\}\ , \tag{2.2}$$

where μ is the chemical potential, and N is the fermion number operator.

We suppose the system described above is subjected to an adiabatically switched weak time and space varying longitudinal field with potential

$$\equiv \Phi_{\rm ext}(\vec{r}, t) = \phi_{q\omega} \exp[i(\vec{q}\cdot\vec{r} - \omega t)] + \text{c. c.} \quad , \tag{2.3}$$

where $\vec{q}, \omega$ are the wave vector and frequency of the disturbance, and ω is assumed to have an infinitesimal imaginary part $i\eta$ with η positive. Assuming the system to be in equilibrium at temperature β^{-1} in the remote past, we find the density matrix at time t, up to terms linear in the external field, to be

$$\rho(t) - \rho_o = i \int_{-\infty}^{t} dt' \exp(-i\omega t') \exp[-iH(t-t')][\rho_o, \rho_{-q}\phi_{q\omega}] \exp[iH(t-t')] + \text{H. a.} \quad , \tag{2.4}$$

where H. a. stands for "Hermitian adjoint" and the particle-density operator is defined by

$$\rho_{-q} \equiv \sum_k \rho^k_{-q} \equiv \sum_k a^\dagger_{k+\vec{q}/2}\, a_{k-\vec{q}/2} \;. \tag{2.5}$$

The impurity averaged bare-particle distribution function linear in the external field of wave vector $\vec{q}$ and frequency ω is given by

$$f_k(q,\omega) e^{-i\omega t} = \phi_{q\omega} e^{-i\omega t} \langle \text{Tr}\{ i \int_o^\infty dt'\, e^{i\omega t'} \rho_o\, [\rho_{-q}, e^{iHt'} \rho_q^k e^{-iHt'}]\}\rangle_i \;, \tag{2.6}$$

where the angular bracket indicates an average over the positions of the impurities, i. e.,

$$\langle\cdots\rangle_i \equiv \int \prod_{j=1}^{n_i} d^3R_j (\cdots) \quad .$$

Since the impurities are assumed to be randomly distributed, the averaged function $f_k(q, \omega)$ determines the linear response of the physically interesting one-particle quantities as, for example, the current and particle densities. Thus,

$$\vec{j}_{q\omega} = \sum_k (\vec{k}/m) f_k(q,\omega) \,, \qquad \rho_{q\omega} = \sum_k f_k(q,\omega) \;. \tag{2.7}$$

We now review the connection between the function $f_k(q, \omega)$ and the two-particle temperature Green's function. We follow the argument of Luttinger and Nozières.[12] Writing the trace in Eq. (2.6) in terms of the eigenstates $\{|n\rangle\}$ of the Hamiltonian H, we have

$$\frac{f_k(q,\omega)}{\phi_{q\omega}} = \left\langle \sum_{nn'} (\langle n|\rho_o|n\rangle - \langle n'|\rho_o|n'\rangle) \langle n|\rho_{-q}|n'\rangle\langle n'|\rho_q^k|n\rangle (E_n - E_{n'} - \omega)^{-1} \right\rangle_i \quad , \tag{2.8}$$

where we take $(H - \mu N)|n\rangle = E_n|n\rangle$, and we recall that ω has an infinitesimal positive imaginary part.

We define the two-particle temperature Green's function averaged over impurities as

$$K_{klk'l'}(u, v, u', v') \equiv \delta_{k+l,\,k'+l'} \langle \text{Tr}\{\rho_o T(a_k^\dagger(u) a_l^\dagger(v) a_{k'}(u') a_{l'}(v'))\}\rangle_i \;, \tag{2.9}$$

where, for instance,

$$a_k^\dagger(u) \equiv \exp[u(H-\mu N)]\, a_k^\dagger \exp[-u(H-\mu N)] \,, \tag{2.10}$$

with $0 \leq u, u', v, v' \leq \beta$. T is the imaginary time-ordering operator which takes into account the sign of fermion-operator permutations. (For a discussion of the temperature Green's function technique, we refer the reader to the book of Abrikosov *et. al.*[15]) As can easily be seen,

$$K_{klk'l'}(0, v, u', v') = -K_{klk'l'}(\beta, v, u', v') \;; \tag{2.11}$$

similar relationships are true for the variables v, v', and u' as well. This means that K can be expanded in Fourier series; thus, one finds

$$K_{klk'l'}(u, v, u', v') = \frac{1}{\beta^2} \sum_{nn'm} \exp[\mathcal{E}_n(u-u') + \mathcal{E}_{n'}(v-v') + \omega_m(u-v')] K_{klk'l'}(\mathcal{E}_n, \mathcal{E}_{n'}, \omega_m), \tag{2.12}$$

where $\mathcal{E}_n = (2\pi i/\beta)(n+\frac{1}{2})$, and $\omega_m = (2\pi i/\beta)m$. The dependence of the Fourier transform on only three variables is a result of the time translation properties of K.

Next, consider the function $h_k(q, \omega_m)$ defined as follows:

$$h_k(q, \omega_m) \equiv - \int_0^\beta du \exp(-u\omega_m) \langle \mathrm{Tr}[\rho_o \rho_{-q}(u) \rho_q{}^k] \rangle_i$$

$$= \sum_l \frac{1}{\beta} \sum_{nn'} K_{l+q/2,\, k-q/2,\, l-q/2,\, k+q/2}(\mathcal{E}_{n'}, \mathcal{E}_n, \omega_m), \tag{2.13}$$

where $\rho_{-q}(u)$ is defined with respect to ρ_{-q} by an equation like (2.10), and we have used definitions (2.9) and (2.12) to arrive at the second line of (2.13). Expanding the trace in the first line of (2.13) in terms of the eigenstates of the Hamiltonian H, we arrive at the formula

$$h_k(q, \omega_m) = \left\langle \sum_{nn'} (\langle n|\rho_0|n\rangle - \langle n'|\rho_0|n'\rangle) \langle n|\rho_{-q}|n'\rangle \langle n'|\rho_q{}^k|n\rangle (E_n - E_{n'} - \omega_m)^{-1} \right\rangle_i . \tag{2.14}$$

Analytically continuing this expression[12,15] $\omega_m \to \omega$ (m positive – recall that ω has a positive imaginary part), we find by reference to (2.8) that

$$f_k(q, \omega)/\phi_{q\omega} = h_k(q, \omega). \tag{2.15}$$

Equations (2.13) and (2.15) constitute the connection between the distribution function f and the two-particle temperature Green's function.

We have made this connection because of the availability of diagrammatic and analytic techniques for the study of the temperature Green's function. We shall use these techniques in the following sections to study the properties of h thereby obtaining an integral equation for a function, intimately related to h, which can be interpreted in terms of the distribution function for quasiparticles discussed by Landau.[1]

III. DIAGRAM TECHNIQUE AND ANALYTIC PROPERTIES; BETHE-SALPETER EQUATIONS

In the first half of this section, we briefly review the features of the diagram technique which will be of importance to us. We shall relate, as well, the way in which the fermion-impurity interaction is included in the diagrammatic expansion of impurity averaged temperature Green's functions. For detailed descriptions of the diagrammatic technique as applied to the study of temperature Green's function, we refer to the works of Abrikosov, Gor'kov, and Dzyaloshinskii[27] and Luttinger and Ward[28]; and for some of the important results discussed in this section, the paper by Nozières and Luttinger.[11] The inclusion of the fermion-impurity interaction in the diagram technique has been discussed by Edwards[29] and Langer[19] and a good description is found in the work of Betbeder-Matibet and Nozières.[23,24] The second half of this section is taken up by a discussion, following Eliashberg,[13–15] of some of the analytic properties of the functions entering into the theory.

In the diagram technique, a particle-particle interaction is represented by the crossing of two fermion lines, while an impurity interaction is depicted by a dashed line of momentum $\vec{k} - \vec{k}'$ emanating from a fermion line which up to the dashed line has momentum $\vec{k}'$ spin σ and beyond has momentum $\vec{k}$ spin σ (the orientation of a fermion line is from creation to annihilation operator). We take into account the average over the positions of the impurities diagrammatically by having each dashed line belong to some bunch of such lines, all of which end at the same point. In each diagram, there can be a number of such bunches, and all diagrams which represent distinct ways of bunching the lines must be considered separately. Mathematically each bunch of n lines represents a factor $u(\vec{q}_1) \cdots u(\vec{q}_n) n_i \delta(\sum \vec{q}_i, 0)$ in the evaluation of the diagram. [Here, $\delta(\cdots, \cdots)$ is just the usual Kronecker δ, and $\vec{q}_i$ represents the momentum transfer to a fermion line via the impurity interaction.] We note that diagrams having any bunch consisting of a single dashed line vanish since $u(0) = 0$; also a diagram having any closed set of fermion lines (i.e., no external lines) connected to the rest of the diagram by only bunches of impurity lines must never be considered.

Just as for a system not having impurity interactions, in which case the important functions are determined in terms of the exact single-particle propagators and skeleton diagrams (those having no self-energy parts in any internal lines), so for the case under consideration, the impurity averaged functions of interest are related to appropriately defined skeleton diagrams and impurity averaged exact single-particle propagators

$$G_k(\mathcal{E}_n) \equiv \int_0^\beta du \exp(-u\mathcal{E}_n)\langle \mathrm{Tr}[\rho_o a_k^\dagger(u) a_k]\rangle_i \,. \tag{3.1}$$

[In this definition, $\mathcal{E}_n = (2\pi i/\beta)(n+\frac{1}{2})$.]

In the impurity averaged case, we define a proper self-energy part of momentum-spin k and frequency $\mathcal{E}_n$ to be a diagram with a stub for one entering and one exiting fermion line each of momentum spin k and frequency $\mathcal{E}_n$, and with no internal line constrained to have the same values for all of these parameters (see, for example, Fig. 1). As usual, the relation between the propagators and the sum of all proper self-energy parts, denoted by $M_k(\mathcal{E}_n)$, is given by the equation

$$G_k^{-1}(\mathcal{E}_n) = \mathcal{E}_n - (\epsilon_k - \mu) - M_k(\mathcal{E}_n)\,. \tag{3.2}$$

A skeleton diagram, which is unambiguously defined for diagrams having either external lines or stubs for such lines, is one in which no line has any self-energy part inserted in it. [Figures 1(a) and 1(b) are skeleton self-energy diagrams.]

We now turn to a consideration of the diagrammatic representation of the two-particle Green's function. We shall also discuss other functions which are related to the two-particle Green's function as they will be useful for obtaining later results. Graphically, the two-particle Green's function can be represented by a free part in which two lines propagate independently, and a part in which two lines enter and two lines leave a central core called the scattering function Γ [see Fig. 2(a)]. These remarks are represented by the equation (for $\omega_m \neq 0$)

$$K_{k'+q/2,\,k-q/2,\,k'-q/2,\,k+q/2}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m) \equiv K_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m) = G_{k+q/2}(\mathcal{E}_n+\omega_m)G_{k-q/2}(\mathcal{E}_n)$$
$$\times [\delta_{nn'}\delta_{kk'} + (\beta)^{-1}\Gamma_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m)\, G_{k'+q/2}(\mathcal{E}_{n'}+\omega_m)G_{k'-q/2}(\mathcal{E}_{n'})]\,. \tag{3.3}$$

By extracting the factors multiplying the square bracket on the right-hand side of (3.3), we can define the so-called vertex functions Λ^α

$$\Lambda_{q\omega_m}^{\alpha}(k,\mathcal{E}_n) \equiv \lambda_\alpha + \beta^{-1}\sum_{k'n'}\lambda'_\alpha G_{k'+q/2}(\mathcal{E}_{n'}+\omega_m)G_{k'-q/2}(\mathcal{E}_{n'})\Gamma_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m)\,, \tag{3.4}$$

where $\lambda_\alpha = \vec{k}_\alpha/m$ for $\alpha = 1, 2, 3$ and $\lambda_4 = 1$. Diagrammatically, the functions Λ^α are represented by Fig. 2(b). In the case of a system with a vanishing particle-impurity interaction, there is no restriction on the value of $\mathcal{E}_n$, with respect to $\mathcal{E}_{n'}$ in any diagram for Γ; however, in the case under consideration, there are diagrams [see, for example, Fig. 3(a)] which have a value proportional to $\beta\delta_{nn'}$. As we shall see, these

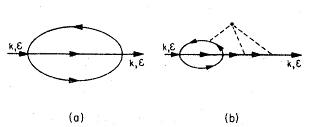

FIG. 1. Skeleton diagrams for the proper self-energy part. (a) Diagram contributing to $M_k^{(o)}(\mathcal{E})$, the part of the self-energy not directly dependent on the impurity density n_i. Diagrams for $M^{(o)}$ are the same as those for the self-energy part in a pure system. (b) Diagram contributing to $M_k^{(1)}(\mathcal{E})$, the part of the self-energy proportional to powers of n_i. Deletion of the factor n_i in the contribution of this diagram (there is just one bunch of impurity lines) gives a contribution to the diagonal t matrix element of momentum k.

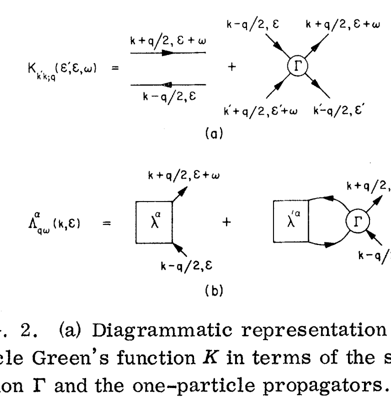

FIG. 2. (a) Diagrammatic representation of the two-particle Green's function K in terms of the scattering function Γ and the one-particle propagators. (b) Representation of the vertex function in terms of Γ, the one-particle propagators, and λ^α.

diagrams are crucial in determining the impurity scattering term in the transport equation.[23,24]

In the standard way,[11] the function Γ can be decomposed into irreducible parts I, which have no pairs of lines of the form $G_{k''+q/2}(\mathcal{E}_{n''}+\omega_m)G_{k''-q/2}(\mathcal{E}_{n''})$, connected together by such pairs of lines. Examples of diagrams for I are found in Fig. 3. In terms of I, we have the Bethe-Salpeter equations

$$\Gamma_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m)=I_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m)+\frac{1}{\beta}\sum_{k'',n''}I_{k'k'';q}(\mathcal{E}_{n'},\mathcal{E}_{n''},\omega_m)$$

$$\times G_{k''+q/2}(\mathcal{E}_{n''}+\omega_m)G_{k''-q/2}(\mathcal{E}_{n''})\Gamma_{k''k;q}(\mathcal{E}_{n''},\mathcal{E}_n,\omega_m), \tag{3.5}$$

and $$\Lambda^{\alpha}_{q\omega_m}(k,\mathcal{E}_n)=\lambda_\alpha+\frac{1}{\beta}\sum_{k'n'}\Lambda^{\alpha}_{q\omega_m}(k',\mathcal{E}_{n'})G_{k'+q/2}(\mathcal{E}_{n'}+\omega_m)G_{k'-q/2}(\mathcal{E}_{n'})\,I_{k'k;q}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m). \tag{3.6}$$

Later, when discussing the analytic properties of Γ and Λ we shall find it useful to decompose these functions in a manner somewhat different from the above.

As has been discussed by Eliashberg[13] and later by others,[30,31] the scattering function $\Gamma(\mathcal{E}',\mathcal{E},\omega)$ has important analytic properties as a function of the variables $\mathcal{E}',\mathcal{E},\omega$ when continued away from the discrete points on which it was originally defined. The continuation is not unique,[14] however, this presents no problem in the present context. Restricting ourselves to the case $\mathrm{Im}\,\omega>0$, we find the singularities of the function Γ are as represented by Figs. 4(a) and 4(b). For the part of Γ for which n' is not restricted to be equal to n, there are 16 analytic parts with discontinuities at $\mathrm{Im}\,\mathcal{E}'=0$, $\mathrm{Im}(\mathcal{E}'+\omega)=0$, $\mathrm{Im}(\mathcal{E}+\omega)=0$, $\mathrm{Im}\mathcal{E}=0$, $\mathrm{Im}(\mathcal{E}'-\mathcal{E})=0$, and $\mathrm{Im}(\mathcal{E}+\mathcal{E}'+\omega)=0$. For the part of Γ for which $n=n'$, there are three branches with discontinuities at $\mathrm{Im}\,\mathcal{E}=0$ and $\mathrm{Im}(\mathcal{E}+\omega)=0$; note that $\beta\delta_{nn'}\to\beta\delta(\mathcal{E}-\mathcal{E}')$ upon continuation. We also observe that Λ has the same analytic structure as depicted in Fig. 4(b).

For our purposes it is also important to know the analytic properties of $G_k(\mathcal{E})$. It is easily seen from the spectral representation of $G_k(\mathcal{E})$[32] that there are two analytic branches with a discontinuity at $\mathrm{Im}\mathcal{E}=0$, moreover, $[G_k(\mathcal{E})]^*=G_k(\mathcal{E}^*)$ so that the discontinuity across the real axis occurs in the imaginary part of $G_k(\mathcal{E})$. The function $G_k(\mathcal{E})$ in the upper (lower) half-plane is denoted by $G_k^{R(A)}(\mathcal{E})$ [$R(A)$ stands for retarded (advanced)]. It is useful to note that the combination $G_{k+q/2}(\mathcal{E}_n+\omega_m)G_{k-q/2}(\mathcal{E}_n)$ when continued away from the discrete points $\mathcal{E}_n$, ω_m has three branches

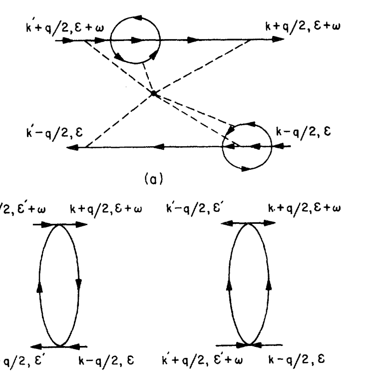

FIG. 3. Diagrams for the scattering function Γ; more specifically these are all diagrams for the irreducible scattering function I. (a) Contribution to ${}^iI$, proportional to n_i. This type of diagram is important for the determination of impurity scattering effects. (b) Contribution to ${}^pI$ typical of those having a discontinuity at $\mathrm{Im}(\mathcal{E}'-\mathcal{E})=0$. (c) Contribution to ${}^pI$ typical of those having a discontinuity at $\mathrm{Im}(\mathcal{E}+\mathcal{E}'+\omega)=0$.

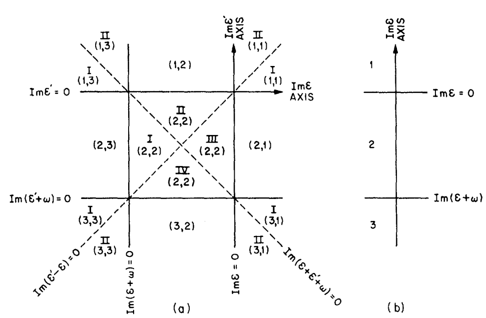

FIG. 4. (a) Analytic structure for $\mathrm{Im}\omega>0$, of the functions ${}^p\Gamma(\mathcal{E}',\mathcal{E},\omega)$, ${}^pI(\mathcal{E}',\mathcal{E},\omega)$, ${}^p_0\Gamma(\mathcal{E}',\mathcal{E},\omega)$. This diagram is a representation in the plane of the variables $\mathrm{Im}\,\mathcal{E}$, $\mathrm{Im}\,\mathcal{E}'$ of the lines of discontinuity and regions of analyticity of these functions. The part of one of these functions analytic in one of these regions is labeled in the same way as that region. (b) Analytic structure, for $\mathrm{Im}\omega>0$, of the functions ${}^i\Gamma(\mathcal{E}',\mathcal{E},\omega)$, ${}^iI(\mathcal{E}',\mathcal{E},\omega)$, ${}^i_0\Gamma(\mathcal{E}',\mathcal{E},\omega)$ (with frequency δ functions deleted). The structure of $G(\mathcal{E}+\omega)G(\mathcal{E})$ and $\Lambda^\alpha(\mathcal{E},\omega)$ is also represented here.

$$G^R_{k+q/2}(\mathcal{E}+\omega)G^R_{k-q/2}(\mathcal{E}) \equiv g_1(k,q,\mathcal{E},\omega), \quad \mathrm{Im}\,\mathcal{E}>0, \quad \mathrm{Im}(\mathcal{E}+\omega)>0$$

$$G^R_{k+q/2}(\mathcal{E}+\omega)G^A_{k-q/2}(\mathcal{E}) \equiv g_2(k,q,\mathcal{E},\omega), \quad \mathrm{Im}\,\mathcal{E}<0, \quad \mathrm{Im}(\mathcal{E}+\omega)>0 \tag{3.7}$$

$$G^A_{k+q/2}(\mathcal{E}+\omega)G^A_{k-q/2}(\mathcal{E}) \equiv g_3(k,q,\mathcal{E},\omega), \quad \mathrm{Im}\,\mathcal{E}<0, \quad \mathrm{Im}(\mathcal{E}+\omega)<0\,.$$

Following Eliashberg,[13,15] we shall also define

$$\Lambda_{q\omega}(k,\mathcal{E}) = {}^1\Lambda_{q\omega}(k,\mathcal{E}), \quad \mathrm{Im}\,\mathcal{E}, \quad \mathrm{Im}(\mathcal{E}+\omega)>0$$

$$\Lambda_{q\omega}(k,\mathcal{E}) = {}^2\Lambda_{q\omega}(k,\mathcal{E}), \quad \mathrm{Im}\,\mathcal{E}<0, \quad \mathrm{Im}(\mathcal{E}+\omega)>0 \tag{3.8}$$

$$\Lambda_{q\omega}(k,\mathcal{E}) = {}^3\Lambda_{q\omega}(k,\mathcal{E}), \quad \mathrm{Im}\,\mathcal{E}, \quad \mathrm{Im}(\mathcal{E}+\omega)<0\,.$$

For the part of Γ proportional to $\delta_{n,n'}$, which we call ${}^i\Gamma$, there are three branches ${}^i\Gamma^{1,2,3}$ defined as in (3.8). The part of Γ depending on the three variables $\mathcal{E}_{n'}$, $\mathcal{E}_n$, and ω_m, which we call ${}^p\Gamma$ has, as we said, 16 branches which are labeled as in Fig. 4(a) in a way consistent with the labeling of the functions g in (3.7). We note that the discussion of the analytic properties of ${}^i\Gamma$ and ${}^p\Gamma$ and the labeling of these functions holds equally well for functions ${}^iI$ and ${}^pI$ which are the parts of I, the irreducible scattering function, analogous to ${}^i\Gamma$ and ${}^p\Gamma$, the parts of Γ.

The usefulness of the analytic continuations discussed above comes from the fact that sums over discrete variables can be turned into integrations. We choose contours of integration lying within the different domains of analyticity of the functions in question and surrounding those discrete points which are summed over in each domain. The integral is taken of the functions in question multiplied by a function which has poles, with appropriate residues, at the discrete points. For a variable such as $\mathcal{E}_n = (2\pi i/\beta)(n+\frac{1}{2})$, an appropriate function is $(\frac{1}{2}\beta)\tanh\frac{1}{2}\beta\mathcal{E}$ which has residues, at the points $\mathcal{E}_n$, equal to unity. (For a discussion of this technique see Luttinger and Ward[28] and Abrikosov *et al.*[15,27])

As an example of these techniques, we obtain by use of Eqs. (2.13), (3.3), and (3.4)

$$h_k(q,\omega_m) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\mathcal{E}\,\tanh\tfrac{1}{2}(\beta\mathcal{E})\Big[{}^1\Lambda^4_{q,\omega_m}(k,\mathcal{E})g_1(k,q,\mathcal{E},\omega_m) - {}^2\Lambda^4_{q,\omega_m}(k,\mathcal{E})g_2(k,q,\mathcal{E},\omega_m)$$
$$+{}^2\Lambda^4_{q,\omega_m}(k,\mathcal{E}-\omega_m)g_2(k,q,\mathcal{E}-\omega_m,\omega_m) - {}^3\Lambda^4_{q,\omega_m}(k,\mathcal{E}-\omega_m)g_3(k,q,\mathcal{E}-\omega_m,\omega_m)\Big] \tag{3.9}$$

which upon continuing $\omega_m \to \omega$ (ω has a positive infinitesimal imaginary part) we can write, after some change in integration variables,

$$h_k(q,\omega) = (4\pi i)^{-1}\int_{-\infty}^{\infty} d\mathcal{E}\,\Big\{\tanh(\tfrac{1}{2}\beta\mathcal{E})\,{}^1\Lambda^4_{q\omega}(k,\mathcal{E})g_1(k,q,\mathcal{E},\omega) + \Big[\tanh\Big(\frac{\beta(\mathcal{E}+\omega)}{2}\Big) - \tanh\Big(\frac{\beta\mathcal{E}}{2}\Big)\Big]$$
$$\times\,{}^2\Lambda^4_{q\omega}(k,\mathcal{E})g_2(k,q,\mathcal{E},\omega) - \tanh\Big(\frac{\beta(\mathcal{E}+\omega)}{2}\Big)\,{}^3\Lambda^4_{q\omega}(k,\mathcal{E})g_3(k,q,\mathcal{E},\omega)\Big\}\,. \tag{3.10}$$

We observe here the important fact that in the term involving g_2 in (3.10) the integration over the frequency is essentially limited by the factor $\{\tanh[\frac{1}{2}\beta(\mathcal{E}+\omega)] - \tanh(\frac{1}{2}\beta\mathcal{E})\}$. Integration of this factor alone over $\mathcal{E}$ gives a result 2ω; thus it appears, at first glance, that for small ω, the contribution of the term involving g_2 is small compared with the rest of (3.10). Under conditions we now describe, this conclusion is false. To most simply state these conditions, let us consider a parameter s collectively characterizing qv_F, ω, γ^i, and γ^p; here v_F is the Fermi velocity, γ^i the width due to impurity scattering is proportional to n_i, and γ^p the width due to interparticle scattering is proportional to $\beta^{-2}\mu^{-1}$. When, as we shall assume in this paper, the parameters characterized by s are small, the contribution from the term involving g_2 is as important as the rest of the terms in (3.10). The reason for this is as follows: Since the two quasiparticle poles multiplied together in g_2 lie on either side of the real axis, the contribution of this function has a part which is essentially the inverse of the (complex) difference in the positions of the two poles. (This is shown explicitly in Sec. IV when we integrate g_2 over $|k|$.) As this difference is linear in the quantities characterized by s, it follows that when s is small, the term in (3.10) involving Sec. II is just as important as the rest of (3.10) because the large contribution of the poles of g_2 just counterbalances the small contribution of the tanh factors. (In this connection, note that the poles of g_1 and g_3 cannot introduce large factors of order s^{-1} because they lie on the same side of the real axis; on the other hand, there is no limitation in the frequency integral due to the tanh factors involved with these functions.) The situation described here is quite general: Terms with a Sec. II, involving g_2, also always involve some function which, either because it effectively limits the range of the frequency integration or because it is proportional to n_i, introduces a factor s; this factor, however, is offset by the poles of g_2 which contribute a factor s^{-1}. As is evident from what has been said, the contributions from Sec. II have the characteristic, unique to them, that they are very sensitive to the four quantities characterized by s; it will thus be necessary in what follows, to treat separately these sections so that their delicate s dependence can be handled properly.

In order to segregate the sections involving factors g_2 from the other sections, we define new functions ${}_0\Gamma$, ${}_0\Lambda^\alpha$, and $h_k{}^0(q,\omega)$; these are, respectively, the sum of all diagrams having no factors g_2 for the scattering function Γ, the vertex function Λ^α, and the function $h_k(q,\omega)$. As before, we can identify a contribution to ${}_0\Gamma$ proportional to $\beta\delta_{nn'} \to \beta\delta(\mathcal{E}-\mathcal{E}')$ which we call ${}^i_0\Gamma$; the rest of ${}_0\Gamma$ we call ${}^p_0\Gamma$. It is easy to see that

$$ {}^i_0\Gamma^{1,3} = {}^i\Gamma^{1,3}, \quad {}^i_0\Gamma^2 = {}^iI^2 ; \tag{3.11} $$

a graphical representation of ${}^p_0\Gamma$ is given in Fig. 5. Let us note the simple relationship between ${}_0\Lambda^\alpha$ and $h_k{}^0(q,\omega)$, namely,

$$ h_k{}^0(q,\omega) = \int \frac{d\mathcal{E}}{4\pi i}\left\{\tanh\left(\frac{\beta\mathcal{E}}{2}\right){}^1_0\Lambda^4{}_{q\omega}(k,\mathcal{E})g_1(k,q,\mathcal{E},\omega) - \tanh\left(\frac{\beta(\mathcal{E}+\omega)}{2}\right){}^3_0\Lambda^4{}_{q\omega}(k,\mathcal{E})g_3(k,q,\mathcal{E},\omega)\right\} . \tag{3.12} $$

A convenient way to write the contribution to $h_k(q,\omega)$ of terms having factors g_2 is to single out the first factor g_2 which is found as one follows the appropriate diagram from the lines having momentum $k \pm q$.[15] Doing this we get

$$ h_k(q,\omega) = h_k{}^0(q,\omega) + \sum_{k'} \int \frac{d\mathcal{E}}{4\pi i}\frac{d\mathcal{E}'}{4\pi i} Q_{q\omega}(k,k',\mathcal{E},\mathcal{E}')\, g_2(k',q,\mathcal{E}',\omega) $$
$$ \times\left[\tanh\left(\frac{\beta(\mathcal{E}'+\omega)}{2}\right) - \tanh\left(\frac{\beta\mathcal{E}'}{2}\right)\right]{}^2\Lambda^4{}_{q\omega}(k',\mathcal{E}') . \tag{3.13} $$

Here,

$$ Q_{q\omega}(k,k',\mathcal{E},\mathcal{E}') \equiv 4\pi i\delta(\mathcal{E}-\mathcal{E}')\delta_{kk'} + \tanh(\tfrac{1}{2}\beta\mathcal{E})\; {}_0\Gamma_{(2,1)k'k;q}(\mathcal{E}',\mathcal{E},\omega)g_1(k,q,\mathcal{E},\omega) $$
$$ - \tanh[\tfrac{1}{2}\beta(\mathcal{E}+\omega)]\; {}_0\Gamma_{(2,3)k'k;q}(\mathcal{E}',\mathcal{E},\omega)g_3(k,q,\mathcal{E},\omega) , \tag{3.14} $$

and we note that

$$ \int \frac{d\mathcal{E}}{4\pi i} \sum_k \lambda_\alpha Q_{q\omega}(k,k',\mathcal{E},\mathcal{E}') = {}^2_0\Lambda^\alpha{}_{-q,-\omega}(k',\mathcal{E}'+\omega) . \tag{3.15} $$

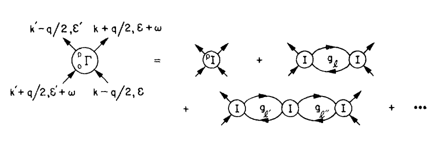

FIG. 5. Graphical representation of the function ${}^p_0\Gamma$. In the second and subsequent terms on the right-hand side, at least one of the factors I must be of the form ${}^pI$. The subscripts l, l', l'', etc., on the factors g can take on the values 1 or 3; they should be summed over where appropriate.

[In (3.14), ${}_0\Gamma(2,l)^{l=1,3}$ is that branch of ${}_0\Gamma$ analytic for Im$\mathcal{E}'$ negative, Im$(\mathcal{E}'+\omega)$ positive, and both Im$\mathcal{E}$, Im$(\mathcal{E}+\omega)$ positive (negative) for $l=1$ (3), see Fig. 4(a).] It proves useful to make a similar decomposition for ${}^2\Lambda^\alpha$, and we are led to the Bethe-Salpeter type equation,

$$ {}^2\Lambda^\alpha_{q\omega}(k,\mathcal{E}) = {}^2_0\Lambda_{q\omega}{}^\alpha(k,\mathcal{E}) + \sum_{k'} \int_{-\infty}^{\infty} \frac{d\mathcal{E}'}{4\pi i}\, {}^2\Lambda^\alpha_{q\omega}(k',\mathcal{E}')g_2(k',q,\mathcal{E}',\omega) $$

$$ \times[4\pi i\delta(\mathcal{E}-\mathcal{E}')\,{}^iI^2_{k'k;q}(\mathcal{E}',\omega) + \mathcal{L}^0_{k'k;q}(\mathcal{E}',\mathcal{E},\omega)] , \tag{3.16} $$

where
$$ \mathcal{L}^0_{k'k;q}(\mathcal{E}',\mathcal{E},\omega) \equiv {}^p_0\Gamma^{\rm IV}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega)\tanh[\tfrac12\beta(\mathcal{E}'+\omega)] - {}^p_0\Gamma^{\rm II}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega)\tanh[\tfrac12\beta\mathcal{E}'] $$

$$ + \coth[\tfrac12\beta(\mathcal{E}'-\mathcal{E})]\,[{}^p_0\Gamma^{\rm II}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega) - {}^p_0\Gamma^{\rm III}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega)] + \coth[\tfrac12\beta(\mathcal{E}+\mathcal{E}'+\omega)] $$

$$ \times[{}^p_0\Gamma^{\rm III}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega) - {}^p_0\Gamma^{\rm IV}_{(2,2)k'k;q}(\mathcal{E}',\mathcal{E},\omega)] . \tag{3.17} $$

[See Fig. 4(a) for the regions of analyticity of the various functions ${}^p_0\Gamma$ in (3.17).] We remark here that the steps leading to this equation are appropriate only for Im$(\mathcal{E}+\frac12\omega)>0$, for Im$(\mathcal{E}+\frac12\omega)<0$ the correct steps lead to a function different from $\mathcal{L}^0$; when, as in our case, $\mathcal{E}$ and ω are continued to the real axis these functions coincide, and the use of $\mathcal{L}^0$ in (3.16) is justified.

Our goal in the next sections will be to obtain more explicit forms for (3.13) and (3.16); this will enable us to define a function which can be interpreted as a quasiparticle distribution function obeying a transport equation of the Landau form and having appropriate relationships with such quantities as the current and particle densities.

IV. EVALUATION OF TERMS INVOLVING SMALL PARAMETER (s)

In Sec. III, we indicated qualitatively that the dependence on the small parameters qv_F, ω, γ^i, and γ^p, characterized by s, is all found in terms involving Sec. II with factors g_2. To make more precise what we mean, we point out that when $s/\mu \ll 1$ (as we assume throughout this paper) we can accurately find the response of such quantities as the current and particle densities by determining them to lowest order in the parameter s. For this purpose, we consider the function $h_k(q,\omega)$ to lowest nonvanishing order in s, namely, the zeroth. (Actually the physical quantities of interest are of order s^{-1}; this is understood by the observation that the distribution function $f=h\phi$, but if the external field of force $-i\vec{q}\phi$ is finite then $\phi\sim 1/s$ and thus, $f\sim 1/s$.) It is when we consider the dependence on s of the zeroth-order contributions of functions such as $h_k(q,\omega)$ and ${}^2\Lambda_{q\omega}(k,\mathcal{E})$ that we find that all s dependence resides in Sec. II. In the rest of this paper, we shall focus our attention on these zeroth-order contributions of the functions mentioned, and from this study, we shall be able to obtain a quasiparticle transport equation. Our purpose in this section is to obtain, explicitly, the zeroth-order dependence of Sec. II; specifically, we calculate the contribution of order s^{-1} of g_2 and the contributions of order s of γ^i, γ^p, ${}^iI^2$, and $\mathcal{L}^0$.

We have indicated that the poles of g_2 give a contribution of order s^{-1}; we now see how this contribution is evaluated. Note first that we only have to consider $g_2(k,q,\mathcal{E},\omega)$ for $|\mathcal{E}|\lesssim \max(\beta^{-1},\omega)\ll\mu$. In (3.13), this is obvious because of the tanh factors; for the same reason, it is necessary to consider (3.16) only for $|\mathcal{E}|\ll\mu$, and since the contribution of $\mathcal{L}^0(\mathcal{E}',\mathcal{E},\omega)$ is important only for small values of $\mathcal{E}'$ when $\mathcal{E}$ is small, it follows that in (3.16), as well, $g_2(k',q,\mathcal{E}',\omega)$ is needed just for $|\mathcal{E}'|\ll\mu$. Writing

$$ g_2(k,q,\mathcal{E},\omega) = [\mathcal{E}+\omega-(\epsilon_{k+q/2}-\mu)-{\rm Re}M_{k+q/2}(\mathcal{E}+\omega)-i\,{\rm Im}M_{k+q/2}(\mathcal{E}+\omega+i\eta)]^{-1} $$

$$ \times[\mathcal{E}-(\epsilon_{k-q/2}-\mu)-{\rm Re}M_{k-q/2}(\mathcal{E})+i\,{\rm Im}M_{k-q/2}(\mathcal{E}+i\eta)]^{-1}, \tag{4.1} $$

and noting, as will be shown, that $|{\rm Im}M(\mathcal{E})|$ is of order s for $|\mathcal{E}|\lesssim\max(\beta^{-1},\omega)$, we see that g_2 is important for values k such that the real parts of the denominators in (4.1) are of order s. Consider then the equation

$$ \mathcal{E}-(\epsilon_k-\mu)-{\rm Re}M_k(\mathcal{E})=0; \tag{4.2} $$

for small $\mathcal{E}$, we assume this equation defines a one-to-one continuous relationship between $\mathcal{E}$ and $|k|$ with $(|k| - k_F)/k_F \ll 1$. Letting E_k (the quasiparticle energy) be the value of $\mathcal{E}$ determined by (4.2) for fixed $|k|$ (as we are going to integrate over $|k|$, we need keep only the part of E_k independent of impurities), we have

$$\mathcal{E} - (\epsilon_k - \mu) - \mathrm{Re}M_k(\mathcal{E}) = \mathcal{E} - E_k - \mathrm{Re}[M_k(\mathcal{E}) - M_k(E_k)] \cong (\mathcal{E} - E_k)\left(1 - \frac{\partial \mathrm{Re}M_k(\mathcal{E})}{\partial \mathcal{E}}\bigg|_{E_k}\right). \tag{4.3}$$

For the use of (4.3), it is assumed that the coefficient of $\mathcal{E} - E_k$ is of order 1 and only varies significantly as a function of $|k|$, over a range of order k_F. This coefficient is the inverse of the renormalization constant, e.g.,

$$z_k = \left(1 - \frac{\partial\,\mathrm{Re}\,M_k(\mathcal{E})}{\partial \mathcal{E}}\bigg|_{E_k}\right)^{-1} \tag{4.4}$$

Note that, for our purposes, impurity contributions to z_k can be neglected as they introduce terms of higher order in s. From (4.3), (4.4), and (4.1), for small $|k| - k_F$, we have, to the desired accuracy,

$$g_2(k, q, \mathcal{E}, \omega) = \frac{(z_k)^2}{\{[\mathcal{E} + \omega - \frac{1}{2}(\vec{q}\cdot\vec{v}_k) - E_k] + i\gamma_k(\mathcal{E}+\omega)\}\{[\mathcal{E} + \frac{1}{2}(\vec{q}\cdot\vec{v}_k) - E_k] - i\gamma_k(\mathcal{E})\}}, \tag{4.5}$$

where $\vec{v}_k = \vec{\nabla}_k E_k$ is the quasiparticle velocity of magnitude v_f, and $\gamma_k(\mathcal{E}) = z_k|\mathrm{Im}M_k(\mathcal{E})|$ is the quasiparticle width. From this equation we see that the significant values of $|k|$ lie in a range of order s/v_F; over this range we may ignore the variation of v_k and $\mathrm{Im}M_k(\mathcal{E})$ as functions of $|k|$. A sum over k of g_2 taken with any function B_k (varying slowly over a range of order s/v_F) is now easily evaluated (by use of the method of partial fractions and cognizance of the discontinuity of the logarithm across its branch line) to give

$$\sum_{\vec{k}} B_k g_2(k, q, \mathcal{E}, \omega) = \int_{|k|=k_F} \frac{d\Omega_k}{4\pi} D(0) \frac{2\pi i z_k^{\,2}}{\omega - \vec{v}_k\cdot\vec{q} + i[\gamma_k(\mathcal{E}+\omega) + \gamma_k(\mathcal{E})]} B_k, \tag{4.6}$$

where $D(0) = k_F^2/v_F 2\pi^2$ is the density of states at the Fermi surface. (We have here used the fact that $E_{k_F} = 0$.[28]) For completeness, we remark that the contribution to (4.6) from large values of k does not diverge since $g_2(k) \to m^2/k^4$ as $k \to \infty$; clearly the (finite) contribution from this region does not give terms of order s^{-1} and so may be ignored.

In order to complete the investigation of the contribution of g_2, we now evaluate $\gamma_k(\mathcal{E})$. The evaluation of the imaginary part of the self-energy for small values of $\mathcal{E}$ is most conveniently carried out by use of a technique due to Langer[20,22] for the determination of the branch discontinuities of functions occurring in many-body theory. This method, which is an adaption of one developed by Landau[33] for field theory in high-energy physics, works as follows: To determine the discontinuity of a function in one of its frequency variables, first consider in a skeleton diagram for the function, all sets of fermion lines the appropriate sum (or difference) of frequencies of which equals the external frequency of interest. It is easy to find such a set of lines since the severance of the lines belonging to it cuts the fermion part of the diagram into two pieces (which may still be connected by dashed impurity lines); one of the pieces has external vertices with total incoming (the other with total outgoing) frequency equal to that in which the singularity is being determined. The contribution to the discontinuity from a given set of lines is calculated by assuming that all the singular behavior comes from these lines. All such contributions for each pertinent diagram are added together to get the total discontinuity of the function of interest. (In connection with this, see Ref. 20 for the treatment of overlapping singularities.) We remark that under certain circumstances the use of this method becomes dubious: This happens when one of the internal lines of a diagram, not a member of the set of lines singled out as described above, has values of energy and momentum which are restricted to lie near a propagator pole by the small values of certain external parameters and by the frequency δ function and fermion occupation factors introduced in the evaluation of the discontinuity. Specifically, this situation occurs when there is a pair of lines, with momentum-frequency difference (or sum) equal to externally determined values, one of which is a member of the set of lines used to evaluate the discontinuity; the other member of the pair is then a line of the general type described above. This problem is relevant when the discontinuities across the real axis are being considered for the variables $\mathcal{E}$, $\mathcal{E}+\omega$ in $\Lambda^{\alpha}{}_{q\omega}(k, \mathcal{E})$ and for the variables $\mathcal{E}$, $\mathcal{E}+\omega$, $\mathcal{E}'$, $\mathcal{E}'+\omega$ in $\Gamma_{k'k;\,q}(\mathcal{E}', \mathcal{E}, \omega)$; no difficulty of this sort is encountered in the consideration of the self-energy part or in the irreducible scattering function I[34] (see Fig. 6).

It is useful in the application of the techniques described above to the determination of the imaginary part of the self-energy to distinguish terms which are not directly proportional to the impurity density (and are thus represented by diagrams with no explicit impurity lines) from those which are proportional to nonzero powers of n_i (with diagrams having explicit impurity lines). As we just have to keep terms of order s in calculating $\gamma_k(\mathcal{E})$, we need only evaluate terms of the latter type proportional to n_i. The contribution of such terms can be written [see Fig. 1(b) for a representative diagram]

$$M_k{}^{(1)}(\mathcal{E}\pm i\eta)=n_i\, t(k,k,\mathcal{E}\pm i\eta)\,, \tag{4.7}$$

where $t(k,k,\mathcal{E}\pm i\eta)$ is just the diagonal matrix element of the t matrix for a single impurity in the Fermi liquid. The diagrams for the off-diagonal element $t(k',k,\mathcal{E}\pm i\eta)$ are the same as those for $M^{(1)}$, except that the momentum transfer is restricted to $\vec{k}'-k$ instead of $\vec{0}$ and the factor n_i is dropped. The sets of lines which can cut the fermion part of $M^{(1)}$ in two pieces contain one, three or in general an odd number of lines. The discontinuity obtained from considering a set with three lines gives a factor $\beta^{-2}\mu^{-1}$ or ω^2/μ [as we shall see when considering the other part of $M_k(\mathcal{E})$]; thus, a contribution of this type to the discontinuity of $M^{(1)}$ is at least of order s^2 and can be dropped. Sets with a greater number of lines clearly yield even higher orders in s and need not be considered. The discontinuity from the sum of all sets containing only one line is given by

$$\begin{aligned}\mathrm{Im}M_k{}^{(1)}(\mathcal{E}+i\eta)&=\tfrac{1}{2}[M_k{}^{(1)}(\mathcal{E}+i\eta)-M_k{}^{(1)}(\mathcal{E}-i\eta)]\\&=n_i\sum_{k'} t(k,k',\mathcal{E}+i\eta)\frac{\mathrm{Im}M_{k'}(\mathcal{E}+i\eta)}{[\mathcal{E}-(\epsilon_{k'}-\mu)-\mathrm{Re}M_{k'}(\mathcal{E})]^2+[\mathrm{Im}M_{k'}(\mathcal{E}+i\eta)]^2}\,t(k',k,\mathcal{E}-i\eta)\,.\end{aligned} \tag{4.8}$$

We are interested in this formula for small $\mathcal{E}$, and thus, we again use the fact that $\mathrm{Im}M$ is of order s. Inasmuch as n_i is of order s in smallness, we see that we only have to keep the contribution from the denominator in (4.8) of order s^{-1}. This is done as in the discussion given above for the contribution of g_2. Assuming that $t(k,k',\mathcal{E}\pm i\eta)$ is not rapidly varying as a function of its momentum variables, we have

$$\mathrm{Im}M_k{}^{(1)}(\mathcal{E}+i\eta)\equiv\frac{-1}{z_k}\gamma_k{}^i(\mathcal{E})=-n_i\pi\int\frac{d\Omega_{k'}}{4\pi}D(0)z_{k'}\,|t(k,k',\mathcal{E}+i\eta)|^2\,, \tag{4.9}$$

where we have used the fact that $\mathrm{Im}M_k(\mathcal{E}+i\eta)$ must be negative. [This follows from Eq. (3.2) and the analyticity of $G^R(\mathcal{E})$ in the upper half-plane.] A result of this form for zero temperature Green's function was obtained in Refs. 23 and 24. Note that the assumption that the t matrix is slowly varying in momentum means that $\gamma_k{}^i(\mathcal{E})$ is slowly varying as well.

We now discuss the contribution to the imaginary part of the self-energy of the terms not directly proportional to the impurity density [see, for example, Fig. 1(a)]. This part which we call $M_k{}^{(0)}(\mathcal{E})$ has, as we indicated, a diagrammatic representation which is precisely the same as that of the self-energy in the case of a pure system. The sets of lines which when cut, separate the skeleton diagrams for $M_k{}^{(0)}(\mathcal{E})$ into

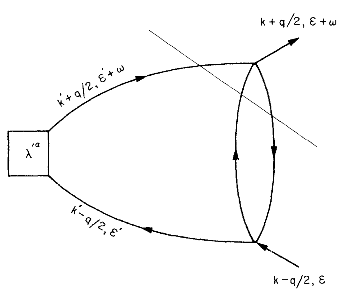

FIG. 6. This diagram is an example of the situation in which the use of the technique described in the text for the evaluation of a discontinuity is in doubt. For the calculation of the discontinuity at $\mathrm{Im}(\mathcal{E}+\omega)=0$ of this contribution to $\Lambda^{\alpha}(\mathcal{E},\omega)$, the method calls for the consideration of the detailed behavior of *only* the three fermion lines cut by the light diagonal line; however, when $q, \mathcal{E}, \omega, \beta^{-1}$ are all small, the behavior of the propagator represented by the line labeled $k'-\frac{1}{2}q$, $\mathcal{E}'$ cannot be ignored because of the proximity of its variables to its quasiparticle pole which is forced when the discontinuity is being evaluated.

two pieces, have an odd number, greater than 3, of lines. The contributions of sets containing five or more lines to the imaginary part of the self-energy for small frequency $\mathcal{E}$ are smaller than those obtained from sets containing three lines by powers of $s\mu^{-1}$,[20,35] and thus, may be ignored. The calculation of the imaginary part of the self-energy from consideration of sets of three lines has been done by Eliashberg[14] (see as well, Ref. 15) and in part by Langer.[22] For $|\mathcal{E}| \lesssim \max(\omega, \beta^{-1})$ and $|k| \sim k_F$, we have to lowest order in s

$$z_k \operatorname{Im} M_k^{(0)}(\mathcal{E}+i\eta) \equiv -\gamma_k^{\,p}(\mathcal{E}) = (-\pi) \sum_{\sigma_1\sigma_2} \int_{|k_i|=k_F} \frac{d\Omega_{k_1}}{4\pi} \frac{d\Omega_{k_2}}{4\pi} D^2(0) z_k z_{k_1} z_{k_2}$$

$$\times z_{k+k_2-k_1} \left| \Gamma_{k,k_2,k_1,k+k_2-k_1} \right|^2 (k_F v_k)^{-1} \delta(|\hat{k}+\hat{k}_2-\hat{k}_1|-1)$$

$$\times \int_{-\infty}^{\infty} d\mathcal{E}_1 d\mathcal{E}_2 \frac{1}{8} \frac{\cosh(\beta\mathcal{E}/2)}{\cosh(\beta\mathcal{E}_1/2)\cosh(\beta\mathcal{E}_2/2)\cosh[\beta(\mathcal{E}+\mathcal{E}_2-\mathcal{E}_1)/2]}\,, \tag{4.10}$$

where $\hat{k}_i$ is a unit vector in the direction of $\vec{k}_i$, and Γ is just the scattering function evaluated at $\mathcal{E}_i = 0$. In order to obtain this result, it must be assumed that Γ is an insensitive function of its arguments [this makes $\gamma_k^{\,p}(\mathcal{E})$ insensitive to $|k|$ as well]; of course, this is not true for angular regions corresponding to such situations as forward scattering, however, as these regions are very small, no difficulty occurs in an integral over all angles. Also, away from these angles, the imaginary parts of the frequency arguments in the Γ's appearing in (4.10) are arbitrary; alteration of these imaginary parts merely introduces factors of $s\mu^{-1} \ll 1$, thus, we may simply write $|\Gamma|^2$ in (4.10). The frequency integrals in (4.10) have been carried out[14] and give

$$\gamma_k^{\,p}(\mathcal{E}) = \tfrac{1}{4}\pi^3 \frac{1}{\beta^2 k_F v_k} \left[1+\left(\frac{\beta\mathcal{E}}{\pi}\right)^2\right] \left[\sum_{\sigma_1\sigma_2} \int \frac{d\Omega_{k_1} d\Omega_{k_2}}{(4\pi)^2} \left| z^2 D(0) \Gamma_{k,k_2,k_1,k+k_2-k_1} \right|^2 \delta(|\hat{k}+\hat{k}_2-\hat{k}_1|-1)\right]. \tag{4.11}$$

In this expression, the factor $z^2 D(0)\Gamma$ is of order unity [in He^3, for example, this factor is of the same order as the ratio of the effective mass $(m^* \equiv k_F/v_F)$ to the bare mass, this ratio is of order 1]; it follows therefore, that for $|\mathcal{E}| \lesssim \max(\beta^{-1}, \omega)$

$$\gamma^p(\mathcal{E}) \sim \beta^{-2}/\mu \sim s, \quad \text{for} \quad \omega \ll \beta^{-1}, \quad \text{and} \quad \gamma^p(\mathcal{E}) \sim \omega^2/\mu \sim s^2/\mu, \quad \text{for} \quad \omega \gtrsim \beta^{-1}. \tag{4.12}$$

The second part of Eq. (4.12) reflects the fact that interparticle scattering is unimportant when the temperature is less than the frequency of the driving field – in the case of a pure system this is the collisionless regime – it is useful to have this result nevertheless, since it essentially gives the (very small) value of the damping for such collective modes as zero sound[14] (in a pure system). Collecting our results we have for $|\mathcal{E}| \lesssim \max(\beta^{-1}, \omega)$

$$-z_k \operatorname{Im} M_k(\mathcal{E}+i\eta) = \gamma_k(\mathcal{E}) = \gamma_k^{\,i}(\mathcal{E}) + \gamma_k^{\,p}(\mathcal{E}) + O(s^2/\mu)\,. \tag{4.13}$$

We have completed the evaluation of the contribution of the factor g_2 in Sec. II. It is now necessary to evaluate the other terms ${}^iI^2$ and $\mathcal{L}$. The term ${}^iI^2$ gives a contribution of order s or higher because it is directly proportional to powers of n_i; we need consider here only that part proportional to n_i which gives [a diagrammatic representation is found in Fig. 3(a)] for

$$\operatorname{Im}\mathcal{E} < 0, \quad \operatorname{Im}(\mathcal{E}+\omega) > 0, \quad {}^iI^2_{k'k;q}(\mathcal{E},\omega) = n_i t(k+\tfrac{1}{2}q, k'+\tfrac{1}{2}q, \mathcal{E}+\omega+i\eta) t(k'-\tfrac{1}{2}q, k-\tfrac{1}{2}q, \mathcal{E}-i\eta)\,, \tag{4.14}$$

where t is the single-impurity t matrix discussed earlier. We have already assumed t to be slowly varying in its momentum variables (compared with the characteristic momentum s/v_F); if also the frequency derivative does not introduce a factor s^{-1}, we may write to order s

$${}^iI^2_{k'k;q}(\mathcal{E},\omega) = n_i |t(k,k',\mathcal{E}+i\eta)|^2\,. \tag{4.15}$$

Note that to the appropriate accuracy

$$\gamma_k^{\,i}(\mathcal{E}) = \pi \int \frac{d\Omega_{k'}}{4\pi} D(0) z_k z_{k'}\, {}^{i}I^2_{k'k;q}(\mathcal{E}, \omega). \tag{4.16}$$

Unlike the impurity term, the contribution of $\mathfrak{L}^0$ to Sec. II is of order s because of the limitation of the frequency integral imposed by the hyperbolic functions of the frequencies. It follows that terms with any direct proportionality to s in $\mathfrak{L}^0$, such as those with factors n_i, may be ignored. We can thus consider the same contributions to $\mathfrak{L}^0$ as analyzed by Eliashberg[14, 15] for the pure case. To determine these contributions, we note from (3.17) that we need the values of the discontinuities across the real axis in the variables $(\mathcal{E}' - \mathcal{E})$ and $(\mathcal{E} + \mathcal{E}' + \omega)$ of the function ${}^p_0\Gamma_{(2,2)}(\mathcal{E}', \mathcal{E}, \omega)$. Since these variables are bosonlike (in their discrete form they are integral multiples of $2\pi i/\beta$), the evaluation of the discontinuity for them involves (in the method previously described) sets of even numbers (≥ 2) of lines. The evaluation of the discontinuity for four or more lines leads to contributions of order s^2 or higher and will not be made here.[22] Consideration of sets of two lines the sum or difference of the frequencies of which is equal to one of the variables $\mathcal{E}' - \mathcal{E}$ or $\mathcal{E} + \mathcal{E}' + \omega$ shows that such pairs can only occur in diagrams for ${}^p_0\Gamma$ having no intermediate cuts $g_l(k'', q, \mathcal{E}'', \omega)$; thus, we need only consider the discontinuities in the function ${}^pI$ in these variables. The diagram appropriate for the evaluation of the discontinuity in $\mathcal{E}' - \mathcal{E}$ is shown in Fig. 3(b), while that for the variable $(\mathcal{E} + \mathcal{E}' + \omega)$ is found in Fig. 3(c). For $|\mathcal{E}|, |\mathcal{E}'| \lesssim \max(\beta^{-1}, \omega)$ and $|k|, |k'| \sim k_F$, one finds for the discontinuity in $\mathcal{E}' - \mathcal{E}$, to the desired order,[14, 15]

$$\Delta_1 I \equiv {}^pI^{\mathrm{II}}_{(2,2)k'k;q}(\mathcal{E}', \mathcal{E}, \omega) - {}^pI^{\mathrm{III}}_{(2,2)k'k;q}(\mathcal{E}', \mathcal{E}, \omega) = i\pi \Big[\sum_{\sigma_1} \int \frac{d\Omega_{k_1}}{4\pi} D(0) z_{k_1} z_{k_1+k'-k} |\Gamma_{k', k_1, k_1+k'-k, k}|^2$$

$$\times \frac{1}{k_F v_F} \delta(|\hat{k}_1 + \hat{k}' - \hat{k}| - 1)\Big] \int_{-\infty}^{\infty} d\mathcal{E}_1 \frac{\sinh[\beta(\mathcal{E}' - \mathcal{E})/2]}{\cosh(\beta\mathcal{E}_1/2)\cosh[\beta(\mathcal{E}_1 + \mathcal{E}' - \mathcal{E})/2]}; \tag{4.17}$$

for the discontinuity in $\mathcal{E} + \mathcal{E}' + \omega$, we have

$$\Delta_2 I \equiv {}^pI^{\mathrm{III}} - {}^pI^{\mathrm{IV}} = -\frac{i\pi}{2} \Big[\sum_{\sigma_1} \int \frac{d\Omega_{k_1}}{4\pi} D(0) z_{k_1} z_{k+k'-k_1} |\Gamma_{k', k, k_1, k+k'-k_1}|^2 \frac{1}{k_F v_F}$$

$$\times \delta(|\hat{k} + \hat{k}' - \hat{k}_1| - 1)\Big] \int_{-\infty}^{\infty} d\mathcal{E}_1 \frac{\sinh[\beta(\mathcal{E} + \mathcal{E}' + \omega)/2]}{\cosh(\beta\mathcal{E}_1/2)\cosh[\beta(\mathcal{E} + \mathcal{E}' + \omega - \mathcal{E}_1)/2]}. \tag{4.18}$$

In these two equations, the functions Γ are defined as after (4.10); in order to obtain this form, the fact that the main contribution in these expressions comes from $\mathcal{E}_1 \ll \mu$ has been used. The dependence of the functions Γ in these expressions on q, ω has been dropped, since it introduces terms of order s^2 in the final result.

The discontinuities $\Delta_j I$ only account for part of $\mathfrak{L}^0$, to simplify writing the rest, we define a function which is an average of ${}^p_0\Gamma_{(2,2)}$ over some of its branches; thus,

$${}^p_0\Gamma^a_{k'k;q}(\mathcal{E}', \mathcal{E}, \omega) = \tfrac{1}{2}[{}^p_0\Gamma^{\mathrm{II}}_{(2,2)k'k;q}(\mathcal{E}', \mathcal{E}, \omega) + {}^p_0\Gamma^{\mathrm{IV}}_{(2,2)k'k;q}(\mathcal{E}', \mathcal{E}, \omega)]. \tag{4.19}$$

We shall be interested in this function for $|\mathcal{E}|, |\mathcal{E}'| \lesssim \max(\beta^{-1}, \omega)$ and $|k|, |k'| \sim k_F$, in this region it can be characterized as a similar average I^a of ${}^pI_{(2,2)}$ plus the sum of all terms contributing to ${}^p_0\Gamma_{(2,2)}$ having at least one of the sections of the type $g_l (l = 1, 3)$. In terms of Γ^a and the discontinuities $\Delta_j I$ we can write the four branches of ${}^p_0\Gamma_{(2,2)}$ in the region mentioned as

$${}^p_0\Gamma^{\mathrm{I}}_{(2,2)} = \Gamma^a + \tfrac{1}{2}(\Delta_1 I - \Delta_2 I), \quad {}^p_0\Gamma^{\mathrm{II}}_{(2,2)} = \Gamma^a + \tfrac{1}{2}(\Delta_1 I + \Delta_2 I),$$

$${}^p_0\Gamma^{\mathrm{III}}_{(2,2)} = \Gamma^a - \tfrac{1}{2}(\Delta_1 I - \Delta_2 I), \quad {}^p_0\Gamma^{\mathrm{IV}}_{(2,2)} = \Gamma^a - \tfrac{1}{2}(\Delta_1 I + \Delta_2 I). \tag{4.20}$$

From (4.20) and (3.17), we now easily get

$$\mathfrak{L}^0_{k'k;q}(\mathcal{E}', \mathcal{E}, \omega) = \Gamma^a_{k'k;q}(\mathcal{E}', \mathcal{E}, \omega)\{\tanh[\tfrac{1}{2}\beta(\mathcal{E}' + \omega)] - \tanh(\tfrac{1}{2}\beta\mathcal{E}')\} + \tfrac{1}{2}\Delta_1 I\{2\coth[\tfrac{1}{2}\beta(\mathcal{E}' - \mathcal{E})] - \tanh(\tfrac{1}{2}\beta\mathcal{E}')$$

$$- \tanh[\tfrac{1}{2}\beta(\mathcal{E}' + \omega)]\} + \tfrac{1}{2}\Delta_2 I\{2\coth[\tfrac{1}{2}\beta(\mathcal{E} + \mathcal{E}' + \omega)] - \tanh(\tfrac{1}{2}\beta\mathcal{E}') - \tanh[\tfrac{1}{2}\beta(\mathcal{E}' + \omega)]\}. \tag{4.21}$$

It is appropriate here to make an identification of the function Γ^a; this is an important step in showing how the reactive part of the transport equation is related to the zero-temperature Landau equation.[12,23,24] First let us emphasize that because of the tanh factors multiplying Γ^a in (4.21), we only have to determine it to zeroth order in s (as a consequence we need not consider terms with direct proportionality to n_i). With this in mind, let us investigate I^a. Assuming that the interaction matrix elements occurring in the irreducible scattering function are slowly varying functions of q (the problem of the long-range or Coulomb interaction must be handled separately; it is briefly discussed in Sec. VI), we find that the only possible significant dependence of I^a on q must come from the internal lines in diagrams for it. However, in such graphs no two lines are required to have a frequency (momentum) difference with value ω ($\vec{q}$), thus, there is no forced near coincidence of propagator poles in I^a and we conclude that the difference of this function from its value at $q, \omega = 0$ is at least of order s and hence may be ignored (an argument of this type for zero-temperature Green's functions is given by Nozières[6]). Since the only possible zeroth-order dependence on the temperature and impurity density in I^a must involve the quasiparticle width, it follows that the lack of coincidence of poles in I^a means that we can evaluate it in the zero-temperature zero-impurity density limit. Inasmuch as we have removed the singular behavior (branch cut) in the frequency variables by considering the average I^a, we can reasonably expect that this function is slowly varying with respect to $\mathcal{E}, \mathcal{E}'$, $|k'|$, and $|k'|$ for values near the Fermi surface. More precisely, consideration of the dependence on these variables would lead us to terms of higher order in s than are called for. The rest of the terms contributing to Γ^a all have diagrams with a number of pairs of lines with momentum, frequency differing by $\vec{q}, \omega$; however, these come only in the form $g_{1,3}(k'', q, \mathcal{E}'', \omega)$ and are always sandwiched between factors I which are, according to our assumptions, slowly varying functions of k'' near the Fermi surface. It is clear that all the zeroth-order dependence on s of these terms can only come from the region around the poles in $g_{1,3}$ near the real axis, i.e., for $\mathcal{E} \sim 0$ and $|k''| \sim k_F$. In this region, we can replace $g_{1,3}(k'', q, \mathcal{E}'', \omega)$ by

$$\sim (z_{k''})^2[\mathcal{E} + \omega - E_{k''+q/2} \pm i\gamma_{k''+q/2}(\mathcal{E}+\omega)]^{-1}[\mathcal{E} - E_{k''-q/2} \pm i\gamma_{k''-q/2}(\mathcal{E})]^{-1}, \tag{4.22}$$

where the plus (minus) sign refers to $g_1(g_3)$. Since both poles in (4.22) are on the same side of the real axis, we obtain upon integration over $|k''|$ essentially the momentum derivative of the slowly varying I factors; thus, there is no s dependence in zeroth order. Having considered all terms in Γ^a, we can now conclude that Γ^a can be taken as its zero temperature, wavelength, frequency, and impurity density limit. As in the case of I^a, for our purposes, we can take Γ^a to be its value at $\mathcal{E}, \mathcal{E}' = 0$; moreover, since by definition, Γ^a contains no sections g_2 (so that diagrams contributing to Γ^a are those which would contribute if $\omega \equiv 0$), we can identify it with the function $\Gamma_{k'k}{}^q(\mathcal{E}'\mathcal{E})$ defined by Landau[3,11,6,15] which in the zero-temperature case is the limit of Γ as $\omega/q \to 0$, $q \to 0$. This is the connection that allows us to identify the reactive part of our transport equation with the zero-temperature Landau equation. Just to complete this discussion, we note the symmetry of the function $\Gamma_{kk'}{}^a$ with respect to the interchange of k and k'.

We have discussed the s-dependent features of expressions involving Sec. II (with factors g_2); in Sec. V, we interest ourselves in the properties of some of the functions not involving sections of this type.

V. TERMS INDEPENDENT OF SMALL PARAMETERS (s); WARD IDENTITIES

For the same reasons cited in the case of Γ^a, the functions ${}_0\Lambda$, h^0, and Q do not depend on the quantities characterized by s, when evaluated to zeroth order in that parameter. (In particular, direct dependence on the impurity density n_i can be neglected.) The importance of these functions is that they provide the important s-independent quantities which enter into the theory. In this section, we display some identities involving these functions which will allow us to make a physical interpretation of the quasiparticle transport equation we obtain.

We begin our considerations with a discussion of the functions ${}_0^i\Lambda^\alpha(k, \mathcal{E})$. It is taken for granted that near the Fermi surface these functions are slowly varying in $|k|$ over a range of order s/v_F. The slowness of variation with respect to $\mathcal{E}$ (in the sense that this variation leads to no contribution of zeroth order in s) needs some elaboration. We give a plausibility argument as we did for Γ^a, i.e., we suggest that if the discontinuities in ${}_0\Lambda$ across the cuts $\operatorname{Im}\mathcal{E} = 0$ and $\operatorname{Im}(\mathcal{E} + \omega) = 0$ do not involve contributions of zeroth order, then for $|\mathcal{E}| \lesssim s$, we can safely take ${}_0\Lambda(\mathcal{E})$ to be its value at $\mathcal{E} = 0$. We pointed out in Sec. IV, that the techniques for evaluating discontinuities of Langer[20,22] must be applied with great caution when pairs of internal lines with frequency-momentum difference fixed at $\omega, \vec{q}$ are present. Such pairs are found in ${}_0\Lambda$ (as well as in ${}_0^p\Gamma$ which occurs in Q); however, by definition, they cannot be in the form g_2, but must be either g_1 or g_3. It is the essence of an argument of Eliashberg[36] that complications are introduced by a pair of lines of the type described above when factors g_2 are involved, but not otherwise. The Langer technique gives the correct result when only factors g_1 and g_3 occur. As was pointed out above, contributions to ${}_0\Lambda$ directly proportional to n_i need not be taken into account; thus, we need only consider the diagrams for ${}_0\Lambda$ which occur in the case of a pure system. The discontinuities in such diagrams in the variables $\mathcal{E}$ and $\mathcal{E}+\omega$ are determined by cutting, in the Langer method, three or a greater odd number of lines; it follows that the discontinuities are proportional to s when $|\mathcal{E}| \lesssim \max(\beta^{-1}, \omega)$ and thus, may be ignored.

Some important properties of the functions ${}_0\Lambda$ are embodied in the Ward identities[11,15,6,37,38] which we review here. In a way similar to that used by Nozières and Luttinger[11] to reach their Eq. (4.32) and (4.33), we find

$$1-\frac{M_k(\mathcal{E}_n+\omega_m)-M_k(\mathcal{E}_n)}{\omega_m}=1+\sum_{n'} G_{k'}(\mathcal{E}_{n'}+\omega_m)G_{k'}(\mathcal{E}_{n'})\Gamma_{k'k;0}(\mathcal{E}_{n'},\mathcal{E}_n,\omega_m)=\Lambda^4{}_{0\omega_m}(k,\mathcal{E}_n); \tag{5.1}$$

thus, it follows that

$$1-\mathrm{Re}\frac{\partial M_k(\mathcal{E}+i\eta)}{\partial \mathcal{E}}=\lim_{\omega\to 0}\ \tfrac{1}{2}[{}^1\Lambda^4{}_{0\omega}(k,\mathcal{E})+{}^3\Lambda^4{}_{0\omega}(k,\mathcal{E})]. \tag{5.2}$$

In order to obtain a relationship between ${}_0\Lambda$ and Eq. (5.2), which for $|k|\sim k_F$ and $\mathcal{E}=E_k$ is the inverse of the renormalization constant z_k, we separate, as in Sec. III, terms with factors g_2 from the rest; thus for $|k|\sim k_F$ and $|\mathcal{E}|\lesssim s$ we have to zeroth order

$$\tfrac{1}{2}[{}^1\Lambda^4{}_{0\omega}(k,\mathcal{E})+{}^3\Lambda^4{}_{0\omega}(k,\mathcal{E})]={}^2_0\Lambda^4(k,\mathcal{E})$$

$$+\sum_{\sigma'}\int d\mathcal{E}'\int_{|k'|=k_F}\frac{d\Omega_{k'}}{4\pi}\ \bar{F}_{kk'}N'(\mathcal{E}',\omega)\frac{\omega}{\omega+i[\gamma(\mathcal{E}')+\gamma(\mathcal{E}'+\omega)]}\ {}^2\Lambda^4{}_{0\omega}(k',\mathcal{E}'). \tag{5.3}$$

In this equation,

$$N'(\mathcal{E},\omega)\equiv(1/2\omega)\{\tanh[\tfrac{1}{2}\beta(\mathcal{E}+\omega)]-\tanh(\tfrac{1}{2}\beta\mathcal{E})\}, \tag{5.4}$$

note that the integral of $N'(\mathcal{E},\omega)$ over $\mathcal{E}$ is unity and that in the limit $\omega\to 0$, N' is the negative derivative of the Fermi function;

$$\bar{F}_{kk'}\equiv z_k{}^2 D(0)\Gamma^a{}_{k'k}\Big|_{|k|,|k'|=k_F} \tag{5.5}$$

(we have used here the fact that z_k is independent of $\hat{k}$ in zeroth order). To be able to use (5.5) we have replaced $\tfrac{1}{2}({}_0\Gamma_{(2,1)}+{}_0\Gamma_{(2,3)})$ by $\tfrac{1}{2}({}^p_0\Gamma^{\mathrm{III}}_{(2,2)}+{}^p_0\Gamma^{\mathrm{I}}_{(2,2)})$ which, by the argument given earlier, is accurate to zeroth order in s. Similar reasoning allows us to use ${}^2_0\Lambda$ in (5.3).

An important feature of Eq. (5.3) is the fact that it is not dependent on the value of ω; to see this we study the function ${}^2\Lambda^4{}_{0\omega}(k,\mathcal{E})$. Note that the special case of $q=0$ of Eq. (3.16) is rotationally invariant and thus at the Fermi surface we need not consider the $\vec{k}$ dependence. ${}^2\Lambda^4{}_{0\omega}(\mathcal{E})$ is conveniently represented as the sum of two functions; one, χ_ω, is the part essentially independent of, $\mathcal{E}$, while the other $\chi'_\omega(\mathcal{E})$ has all the $\mathcal{E}$ dependence. These functions satisfy [see Eqs. (4.9), (4.10), (4.15), (4.17), (4.18), and (4.21)]

$$\chi_\omega={}^2_0\Lambda^4+\sum_{\sigma'}\int\frac{d\Omega_{k'}}{4\pi}\ \bar{F}_{kk'}\int d\mathcal{E}'\,N(\mathcal{E}',\omega)[\omega\,{}^2\Lambda_{0\omega}{}^4(\mathcal{E}')]/\{\omega+i[\gamma(\mathcal{E}')+\gamma(\mathcal{E}'+\omega)]\}, \tag{5.6}$$

$$\chi'_\omega(\mathcal{E})=iC^i{}_{k\mathcal{E}}\left[\frac{{}^2\Lambda^4{}_{0\omega}(\mathcal{E}')}{\omega+i[\gamma(\mathcal{E}')+\gamma(\mathcal{E}'+\omega)]}\right]+iC^p{}_{k\mathcal{E}\beta\omega}\left[\frac{{}^2\Lambda^4{}_{0\omega}(\mathcal{E}')}{\omega+i[\gamma(\mathcal{E}')+\gamma(\mathcal{E}'+\omega)]}\right]$$

$$+\frac{i[\gamma(\mathcal{E})+\gamma(\mathcal{E}+\omega)]}{\omega+i[\gamma(\mathcal{E})+\gamma(\mathcal{E}+\omega)]}\,{}^2\Lambda^4_{0\omega}(\mathcal{E}). \tag{5.7}$$

Here C^i and C^p are, respectively, impurity and interparticle scattering functionals; thus,

$$C^i_{k\mathcal{E}}[y_{k'\mathcal{E}'}] \equiv 2\pi n_i z_k^2 \int_{|k'|=k_F} \frac{d\Omega_{k'}}{4\pi} D(0)\ |t(k,k',\mathcal{E}+i\eta)|^2 (y_{k'\mathcal{E}} - y_{k\mathcal{E}})\ , \tag{5.8}$$

and
$$C^p_{k\mathcal{E}\beta\omega}[y_{k'\mathcal{E}'}] \equiv \frac{\pi}{4}\sum_{\sigma_1\sigma_2}\int_{|k_i|=k_F} \frac{d\Omega_{k_1}}{4\pi}\frac{d\Omega_{k_2}}{4\pi}\frac{1}{k_F v_k}\, z_k^4 D^2(0) \int d\mathcal{E}_1 d\mathcal{E}_2 |\Gamma_{k_1,\,k+k_2-k_1,\,k_2,\,k}|^2 \delta$$
$$\times\,(|\hat{k}+\hat{k}_2-\hat{k}_1|-1)\left\{ y_{k_1\mathcal{E}_1}\frac{N'(\mathcal{E}_1,\omega)}{N'(\mathcal{E},\omega)}\left(\frac{\cosh[\frac{1}{2}\beta(\mathcal{E}_1+\omega)]}{\cosh[\frac{1}{2}\beta(\mathcal{E}+\omega)]}+\frac{\cosh(\frac{1}{2}\beta\mathcal{E}_1)}{\cosh(\frac{1}{2}\beta\mathcal{E})}\right)\frac{1}{\cosh(\frac{1}{2}\beta\mathcal{E}_2)\cosh[\frac{1}{2}\beta(\mathcal{E}+\mathcal{E}_2-\mathcal{E}_1)]}\right.$$
$$-\ \tfrac{1}{2}y_{k_2\mathcal{E}_2}\frac{N'(\mathcal{E}_2,\omega)}{N'(\mathcal{E},\omega)}\left(\frac{\cosh[\frac{1}{2}\beta(\mathcal{E}_2+\omega)]}{\cosh(\frac{1}{2}\beta\mathcal{E})}+\frac{\cosh(\frac{1}{2}\beta\mathcal{E}_2)}{\cosh[\frac{1}{2}\beta(\mathcal{E}+\omega)]}\right)\cdot\frac{1}{\cosh[\frac{1}{2}\beta\mathcal{E}_1]\cosh[\frac{1}{2}\beta(\mathcal{E}+\mathcal{E}_2+\omega-\mathcal{E}_1)]}$$
$$\left.-\ \tfrac{1}{2}y_{k\mathcal{E}}\left(\frac{\cosh(\frac{1}{2}\beta\mathcal{E})}{\cosh[\frac{1}{2}\beta(\mathcal{E}+\mathcal{E}_2-\mathcal{E}_1)]}+\frac{\cosh[\frac{1}{2}\beta(\mathcal{E}+\omega)]}{\cosh[\frac{1}{2}\beta(\mathcal{E}+\omega+\mathcal{E}_2-\mathcal{E}_1)]}\right)\frac{1}{\cosh(\frac{1}{2}\beta\mathcal{E}_1)\cosh(\frac{1}{2}\beta\mathcal{E}_2)}\right\}. \tag{5.9}$$

For future reference, we define a functional $\tilde{C}^p$ closely related to C^p, i.e.,

$$\tilde{C}^p_{k\mathcal{E}\beta\omega}[Y_{k'\mathcal{E}'} \equiv N'(\mathcal{E}',\omega)y_{k'\mathcal{E}'}] = N'(\mathcal{E},\omega)C^p_{k\mathcal{E}\beta\omega}[y_{k'\mathcal{E}'}]\ . \tag{5.10}$$

Note the important fact[39] that for a rotationally invariant function $y_{\mathcal{E}}$,

$$\int d\mathcal{E}\, N'(\mathcal{E},\omega)\, C^p_{k\mathcal{E}\beta\omega}[y_{\mathcal{E}}] = 0\ ; \tag{5.11}$$

clearly, also $C^i_{k\mathcal{E}}[y_{\mathcal{E}}]$ vanishes. From (5.7), (5.11), and the properties of χ_ω and $\chi'_\omega(\mathcal{E})$, it easily follows that

$$\chi_\omega = \int d\mathcal{E}\, N'(\mathcal{E},\omega)\frac{\omega}{\omega+i[\gamma(\mathcal{E})+\gamma(\mathcal{E}+\omega]}\,{}^2\Lambda^4_{0\omega}(\mathcal{E})\ . \tag{5.12}$$

Putting this into (5.6), we get

$$\chi_\omega = {}_0^2\Lambda^4 + \sum\nolimits_{\sigma'}\int\frac{d\Omega_{k'}}{4\pi}\,\bar{F}_{kk'}\chi_\omega\ ; \tag{5.13}$$

the solution of this equation is clearly independent of ω. From (5.3), (5.12), and (5.13), we can identify χ_ω with $\frac{1}{2}({}^1\Lambda^4_{0\omega}+{}^3\Lambda^4_{0\omega})$ which is thus also ω-independent. Note also, from the remarks following (5.2),

that $$\chi_\omega = 1/z_k. \tag{5.14}$$

Equations (5.13) and (5.14) will prove extremely useful in obtaining a quasiparticle transport equation.

Another Ward identity which we need is much simpler and is important in determining the relationship of the induced current density to the quasiparticle distribution; it reads[11]

$$\frac{k_\alpha}{m} + \mathrm{Re}\,\frac{\partial M_k(\mathcal{E}+i\eta)}{\partial k_\alpha} = \tfrac{1}{2}[{}_0^1\Lambda^\alpha(k,\mathcal{E}) + {}_0^3\Lambda^\alpha(k,\mathcal{E})]\ . \tag{5.15}$$

The definition of $\vec{v}_k$, the quasiparticle velocity,

$$\vec{v}_k = \frac{\partial E_k}{\partial \vec{k}} = \frac{\vec{k}}{m} + \mathrm{Re}\frac{\partial M_k(\mathcal{E}+i\eta)}{\partial \vec{k}}\Bigg|_{E_k} + \mathrm{Re}\frac{\partial M_k(\mathcal{E})}{\partial \mathcal{E}}\Bigg|_{E_k}\vec{v}_k \tag{5.16}$$

leads to the result

$$v_k^{\alpha} = \tfrac{1}{2} z_k [{}^1_0\Lambda^{\alpha}(k,\mathcal{E}) + {}^3_0\Lambda^{\alpha}(k,\mathcal{E})]\Big|_{\mathcal{E}\sim 0} . \tag{5.17}$$

Since we are interested in (5.17) only to zeroth order in s, we can ignore impurity effects and can write for $|k| \sim k_F$

$$v_k^{\alpha} = z_k {}^2_0\Lambda^{\alpha}(k,\mathcal{E})\Big|_{\mathcal{E}\sim 0} = z_k \int \frac{d\mathcal{E}'}{4\pi i} \sum_{k'} \frac{k'^{\alpha}}{m} Q_{q\omega}(k',k,\mathcal{E}',\mathcal{E})\Big|_{\mathcal{E}\sim 0} ; \tag{5.18}$$

the first equality follows from the discussion at the beginning of this section and the second follows from Eq. (3.15).

As a final result of this section, we derive a useful identity for the zeroth-order part of h_k^o. From Eq. (2.13), it is evident that for $\omega_m \neq 0$, $h_k(0,\omega_m) = 0$. Analytically continuing in the upper half-plane, we find, as well, that $h_k(0,\omega)$ vanishes; thus, from (3.13), it follows that

$$h_k^o = -\int \frac{d\mathcal{E}}{4\pi i} g_2(k,0,\mathcal{E},\omega)\, 2\omega N'(\mathcal{E},\omega)\, {}^2\Lambda^4_{0\omega}(k,\mathcal{E})$$
$$- \int \frac{d\mathcal{E}}{4\pi i} \sum_{\sigma'} \int \frac{d\Omega_{k'}}{4\pi} D(0)\, \overline{Q}(k,k',\mathcal{E},\mathcal{E}')\big|_{\mathcal{E}'\sim 0} (z_{k'})^2 \chi_{\omega} \tag{5.19}$$

Here we have used Eq. (5.12), and we take [see Eq. (3.14)]

$$\overline{Q}(k,k',\mathcal{E},\mathcal{E}') \equiv Q(k,k',\mathcal{E},\mathcal{E}') - 4\pi i\, \delta(\mathcal{E}-\mathcal{E}')\delta_{kk'} . \tag{5.20}$$

Normally, we use h_k^o in a sum over k with a function B_k which is not a rapidly varying function of $|k|$ at the Fermi surface; from (4.6), it follows that such a sum is given by

$$\sum_k B_k h_k^o = -\sum_{\sigma} \int_{|k|=k_F} \frac{d\Omega_k}{4\pi} D(0)(BQ)_k z_k^2 \chi_{\omega} , \tag{5.21}$$

where $$(BQ)_k = \int \frac{d\mathcal{E}'}{4\pi i} \sum_{k'} B_{k'} Q(k',k,\mathcal{E}',\mathcal{E})\big|_{\mathcal{E}\sim 0}. \tag{5.22}$$

We have now built up the machinery for the derivation of a quasiparticle transport equation, which is the subject of Sec. VI.

VI. QUASIPARTICLE DISTRIBUTION FUNCTION AND TRANSPORT EQUATION

In this section, we shall show that a quasiparticle distribution can be defined which depends on spin σ, direction on the Fermi surface $\hat{k}$, and frequency $\mathcal{E}$. The *partial* connection of this function to the bare-particle distribution function will be discussed and its relationship to the current and particle densities will be explicitly determined. Finally, we shall obtain a transport equation for this quasiparticle distribution function. The reactive part of this equation is of the Landau form[1]; the absorptive part is the sum of an impurity scattering term of an expected form,[7–10] and an interparticle scattering term.[2]

To begin our considerations, let us study Eq. (3.13) for $h_k(q,\omega)$; using Eqs. (4.6), (5.19), and (5.20), we obtain for the impurity averaged bare-particle distribution function to lowest order in s,

$$\frac{f_{q\omega}(k)}{\phi_{q\omega}} = h_k(q,\omega) = \int\frac{d\mathcal{E}}{4\pi i}\, 2N'(\mathcal{E},\omega)\omega[g_2(k,q,\mathcal{E},\omega)\, {}^2\Lambda^4_{q\omega}(k,\mathcal{E})$$
$$-g_2(k,0,\mathcal{E},\omega)\, {}^2\Lambda^4_{0\omega}(k,\mathcal{E})] + \int\frac{d\mathcal{E}}{4\pi i}\sum_{\sigma'}\int_{|k'|=k_F}\frac{d\Omega_{k'}}{4\pi}D(0)\overline{Q}(k,k',\mathcal{E},0)$$
$$\times\int d\mathcal{E}'\, N'(\mathcal{E}',\omega)(z_{k'})^2\left\{\frac{\omega}{\omega-\vec{v}_{k'}\cdot\vec{q}+i[\gamma_k(\mathcal{E}'+\omega)+\gamma_k(\mathcal{E}')]}\, {}^2\Lambda^4_{q\omega}(k',\mathcal{E}')-\chi_\omega\right\} . \tag{6.1}$$

[In this equation, we have used the fact that the integral over $\mathcal{E}$ of $N'(\mathcal{E},\omega)$ is unity.] Just as in the case of $h_k{}^o$, discussed in Sec. V, we avail ourselves of the fact that f is normally used in a sum over k with a smooth function B_k; thus, we have, for the induced quantity associated with B_k,

$$\sum_k B_k f_{q\omega}(k) = \sum_\sigma \int_{|k|=k_F}\frac{d\Omega_k}{4\pi}D(0)\, z_k(BQ)_k\int d\mathcal{E}\,\overline{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}). \tag{6.2}$$

Here, we have used Eqs. (4.6), (5.22), and the definition

$$\overline{n}_{q\omega}(\hat{k},\sigma,\mathcal{E})/\phi_{q\omega} \equiv N'(\mathcal{E},\omega)\overline{\nu}_{q\omega}(\hat{k},\sigma,\mathcal{E})$$
$$= N'(\mathcal{E},\omega)\left\{\frac{\omega}{\omega-\vec{v}_k\cdot\vec{q}+i[\gamma_k(\mathcal{E}+\omega)+\gamma_k(\mathcal{E})]}\, z_k{}^2\Lambda^4_{q\omega}(k,\mathcal{E})\Big|_{|k|=k_F} - z_k\chi_\omega\right\}; \tag{6.3}$$

we shall identify $\overline{n}_{q\omega}(\hat{k},\sigma,\mathcal{E})$ with the deviation of the quasiparticle distribution function from local quasiparticle equilibrium. From our identification of Γ^a with Γ^q (see Sec. IV), we are led to describe the total induced quasiparticle distribution function by

$$n_{q\omega}(\hat{k},\sigma,\mathcal{E}) = \overline{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) - N'(\mathcal{E},\omega)\int d\mathcal{E}'\sum_{\sigma'}\int\frac{d\Omega_{k'}}{4\pi}\overline{F}_{kk'}\,\overline{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}'). \tag{6.4}$$

If we define a function $F_{kk'}$ on the Fermi surface so that it satisfies

$$F_{kk'} = \overline{F}_{kk'} + \sum_{\sigma''}\int_{|k''|=k_F}\frac{d\Omega_{k''}}{4\pi}F_{kk''}\overline{F}_{k''k} \tag{6.5}$$

(note that since $\overline{F}$ is symmetric with respect to interchange, so is F), we can easily write the inverse of the relationship (6.4), namely,

$$\overline{n}_{q\omega}(k,\sigma,\mathcal{E}) = n_{q\omega}(\hat{k},\sigma,\mathcal{E}) + N'(\mathcal{E},\omega)\int d\mathcal{E}'\sum_{\sigma'}\int\frac{d\Omega_{k'}}{4\pi}F_{kk'}n_{q\omega}(\hat{k}',\sigma',\mathcal{E}') . \tag{6.6}$$

(For a discussion of the physical significance of the difference between n and $\overline{n}$ see, for instance, Refs. 6, 7, and 9.)

To justify our identification of $\overline{n}$, and thus, n, we now show that the special cases of (6.2) which give the induced current and particle densities can be written in the standard form for quasiparticles.[1,6,7] For the particle density we have

$$\rho_{q\omega} = \sum_k f_{q\omega}(k) = \sum_\sigma\int_{|k|=k_F}\frac{d\Omega_k}{4\pi}D(0)z_k(1Q)_k\int d\mathcal{E}\,\overline{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) ; \tag{6.7}$$

noting from (5.22) and (3.15) that $(1Q)_k$ is just ${}^2_0\Lambda^4(k,0)$, we find by reference to (5.13), (5.14), and (6.4), that

$$\rho_{q\omega} = \sum_\sigma \int_{|k|=k_F} \frac{d\Omega_k}{4\pi} D(0) \int d\mathcal{E}\, n_{q\omega}(\hat{k},\sigma,\mathcal{E}) \tag{6.8}$$

[we have used again the fact that $\int d\mathcal{E} N'(\mathcal{E},\omega)=1$]. The current density is given by

$$j_{q\omega}{}^{\alpha} = \sum_k \frac{k_\alpha}{m} f_{q\omega}(k) = \sum_\sigma \int_{|k|=k_F} \frac{d\Omega_k}{4\pi} D(0) z_k \left(\frac{k_\alpha}{m} Q\right)_k \int d\mathcal{E}\, \bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}); \tag{6.9}$$

this result together with (5.18) leads to the relation

$$j_{q\omega}{}^{\alpha} = \sum_\sigma \int \frac{d\Omega_k}{4\pi} D(0) v_k{}^{\alpha} \int d\mathcal{E}\, \bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}). \tag{6.10}$$

[In a translationally invariant system, another Ward identity, not discussed here, can be used to give the current in terms of n, thus

$$j_{q\omega}{}^{\alpha} = \sum_\sigma \int \frac{d\Omega_k}{4\pi} D(0) \frac{k_\alpha}{m} \int d\mathcal{E}\, n_{q\omega}(\hat{k},\sigma,\mathcal{E}) \;.] \tag{6.11}$$

If in (6.8), (6.10), and (6.11), we think of the $\mathcal{E}$ integration as an integration over the quasiparticle energy, then we see that indeed these equations are in the standard quasiparticle form.

Now that we have defined the quasiparticle functions n and $\bar{n}$, we are in a position to point out that it is not possible, in any obvious way, to determine f completely in terms of n or $\bar{n}$; that is, of course, because the first term in (6.1) does not have an integration over the variable $|k|$ which is found in the factors g_2. This feature is not unique to the impure case we are studying, but clearly is also relevant for a pure system. In contrast with this result is the work on a pure system of Résibois[16,17] and of Watabe and Dagonnier[18] and that on an impure system of Sigel and Argyres.[25,26] In these studies, a quasiparticle distribution function defined by the invertible relation $n_{q\omega}(k)=\sum_{k'} M_{kk'} f_{q\omega}(k')$ was found which made the transport equation derived for f readily interpretable in terms of the Landau theory. We stress that these calculations were done to finite order in the interparticle (and the impurity) interaction. We may speculate as to the possible reasons for the differences in results: (a) The results of Résibois, etc., may not hold up to all orders in the interaction strengths. (b) There may be more than one way to define a quasiparticle distribution function having appropriate features. In this connection note that in the theory of Eliashberg for a pure system, and our work for an impure system, an important variable in describing n and $\bar{n}$ is the frequency $\mathcal{E}$; a variable of this type never occurs in the Résibois theory, instead the appropriate quasiparticle distribution function depends on the magnitude of the momentum. This difference could indicate that these two types of quasiparticle distribution functions differ in an essential way although, as we shall see, they satisfy similar equations.

To get the transport equation satisfied by n (or $\bar{n}$), we multiply Eq. (3.16) by the factor ωz_k, and then, doing a little algebra, write it in terms of $\bar{\nu}$ [as defined in (6.3)], thus,

$$\begin{aligned}(\omega - \vec{v}_k\cdot\vec{q})\bar{\nu}_{q\omega}(\hat{k},\sigma,\mathcal{E}) - \omega\sum_{\sigma'}\int \frac{d\Omega_{k'}}{4\pi} F_{kk'} \int d\mathcal{E}' N'(\mathcal{E}',\omega)\bar{\nu}_{q\omega}(\hat{k}',\sigma',\mathcal{E}') - \vec{v}_k\cdot\vec{q} z_k \chi_\omega]\\ = iC^{i}_{k\mathcal{E}}[\bar{\nu}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')+z_k\chi_\omega] + iC^{p}_{k\mathcal{E}\beta\omega}[\bar{\nu}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')+z_k\chi_\omega] \;.\end{aligned} \tag{6.12}$$

Here, we have used (4.21), (5.5), (5.8), (5.9), and (5.13). It is not difficult to see[39] that when applied to a constant the functional C^p vanishes; this is also obviously true of C^i, thus,

$$C^{p}_{k\mathcal{E}\beta\omega}[z_k\chi_\omega] = C^{i}_{k\mathcal{E}}[z_k\chi_\omega] = 0 \;. \tag{6.13}$$

Multiplying (6.12) by $\phi_{q\omega}N'(\mathcal{E},\omega)$, we obtain the transport equation for n

$$\omega n_{q\omega}(\hat{k},\sigma,\mathcal{E}) - \vec{v}_k\cdot\vec{q}\,\bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) - \vec{v}_k\cdot\vec{q}\,\phi_{q\omega}N'(\mathcal{E},\omega)$$

$$= i\,\{C^{i}_{k\mathcal{E}}[\bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')] + \tilde{C}^{p}_{k\mathcal{E}\beta\omega}[\bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')]\}\,, \tag{6.14}$$

where we have used (6.4), (5.10), and (5.14). From (5.8), we can easily see that the impurity scattering functional can be written

$$C^{i}_{k\mathcal{E}}[\bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')] = 2\pi n_i \int \frac{d\Omega_{k'}}{4\pi} D(0) \int d\mathcal{E}' z_k z_{k'} \,|t(k,k',\mathcal{E}+i\eta)|^2$$

$$\times\, \delta(\mathcal{E}-\mathcal{E}')[\bar{n}_{q\omega}(\hat{k}',\sigma,\mathcal{E}') - \bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E})]\,, \tag{6.15}$$

which is the impurity scattering term as found in Refs. 23 and 24.

The discussion of $\tilde{C}^p$ is a little more involved; we can discuss this functional in two regions which, since they overlap, cover together the whole range for which the theory is valid. The first of these is characterized by the inequality $\omega\beta \ll 1$. Included in this region are the cases in which interparticle collisions are important (i.e., $\gamma^p \gtrsim \omega$) and as well, some of the cases in which they are not (i.e., $\gamma^p \sim \sim \beta^{-2}\mu^{-1} \ll \omega \ll \beta^{-1}$). Defining the equilibrium Fermi function

$$N^0(\mathcal{E}) = (e^{\beta\mathcal{E}} + 1)^{-1}\,, \tag{6.16}$$

we can write in this limit ($\omega\beta \ll 1$)

$$n_{q\omega}(\hat{k},\sigma,\mathcal{E}) = \bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) + \frac{\partial}{\partial\mathcal{E}} N^0(\mathcal{E}) \int d\mathcal{E}' \sum_{\sigma'} \int \frac{d\Omega_{k'}}{4\pi} \bar{F}_{kk'} \bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')\,, \tag{6.17}$$

this follows since $N'(\mathcal{E},\omega) \to -\partial N^0(\mathcal{E})/\partial\mathcal{E}$. Note also

$$\bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) \to -\frac{\partial}{\partial\mathcal{E}} N^0(\mathcal{E}) \bar{\nu}_{q\omega}(\hat{k},\sigma,\mathcal{E})\phi_{q\omega}\,.$$

The interparticle scattering functional in this regime no longer depends directly on ω and can be written after some trivial manipulation

$$\tilde{C}^{p}_{k\mathcal{E}\beta}[\bar{n}_{q\omega}(k',\sigma',\mathcal{E}')] = -\tfrac{1}{2} \sum_{\sigma_1\sigma_2\sigma_3} \int_{|k_i|=k_F} \frac{d\Omega_{k_1}}{4\pi}\frac{d\Omega_{k_2}}{4\pi}\frac{d\Omega_{k_3}}{4\pi} D^3(0) \int d\mathcal{E}_1 d\mathcal{E}_2 d\mathcal{E}_3 z_{k_1} z_{k_2} z_{k_3} z_k$$

$$\times\, |\Gamma_{k_1,k_2,k_3,k}|^2 (2\pi)^4 \delta(\vec{k}+\vec{k}_3-\vec{k}_1-\vec{k}_2)\,\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma}\,\delta(\mathcal{E}+\mathcal{E}_3-\mathcal{E}_1-\mathcal{E}_2)$$

$$\times\, L\{\bar{N}_{q\omega}(\hat{k},\sigma,\mathcal{E})\,\bar{N}_{q\omega}(\hat{k}_3,\sigma_3,\mathcal{E}_3)[1-\bar{N}_{q\omega}(k_1,\sigma_1,\mathcal{E}_1)][1-\bar{N}_{q\omega}(\hat{k}_2,\sigma_2,\mathcal{E}_2)]$$

$$-\,[1-\bar{N}_{q\omega}(\hat{k},\sigma,\mathcal{E})][1-\bar{N}_{q\omega}(\hat{k}_3,\sigma_3,\mathcal{E}_3)]\bar{N}_{q\omega}(\hat{k}_1,\sigma_1,\mathcal{E}_1)\bar{N}_{q\omega}(\hat{k}_2,\sigma_2,\mathcal{E}_2)\}\,, \tag{6.18}$$

where $\bar{N}_{q\omega}(\hat{k},\sigma,\mathcal{E}) = N^0(\mathcal{E}) + \bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E})\,,$ (6.19)

and L is a linearization operator with respect to $\bar{n}$. Note that (6.18) is in the standard form of interfermion scattering. For $\omega\beta \ll 1$, then, the transport equation can be written

$$\omega n_{q\omega}(\hat{k},\sigma,\mathcal{E}) - \vec{v}_k\cdot\vec{q}\,\bar{n}_{q\omega}(\hat{k},\sigma,\mathcal{E}) + \vec{v}_k\cdot\vec{q}\,\phi_{q\omega}\frac{\partial}{\partial\mathcal{E}}N^0(\mathcal{E})$$

$$= iC^{i}_{k\mathcal{E}}[\bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')] + i\tilde{C}^{p}_{k\mathcal{E}\beta}[\bar{n}_{q\omega}(\hat{k}',\sigma',\mathcal{E}')]\,. \tag{6.20}$$

The second region in which a simple discussion of the interparticle scattering functional can be made is characterized by the inequality $\gamma^p \ll \max(\omega, qv_F, \gamma^i)$. Note that this region includes all β with $\beta\omega \gtrsim 1$ and in particular zero temperature; also note that it overlaps the region previously discussed (i.e., $\omega\beta \ll 1$), thus, the two cases cover all temperatures with $\beta^{-1}\mu^{-1} \ll 1$. To see the behavior in the regime under consideration, we show that it is consistent to assume that $\bar{\nu}_{q\omega}(\hat{k}, \sigma, \mathcal{E})$ is independent of $\mathcal{E}$. From (5.9) and (5.10) for a function $\bar{\nu}$ independent of $\mathcal{E}$, it is not hard to see that the integrals over frequency are of the same form as in (4.10), and thus, as in (4.11) and (4.12) we have

$$\tilde{C}^{p}[N'(\mathcal{E}', \omega)\bar{\nu}] \sim (\beta^{-2} + \omega^{2})\mu^{-1} N'(\mathcal{E}, \omega)\bar{\nu} \ll \max(\omega, qv_F, \gamma^{i}) N'(\mathcal{E}, \omega)\bar{\nu} ;$$

the interparticle scattering functional can thus be ignored and we are left with

$$\omega n_{q\omega}(\hat{k}, \sigma, \mathcal{E}) - \vec{v}_k \cdot \vec{q}\, \bar{n}_{q\omega}(\hat{k}, \sigma, \mathcal{E}) - \vec{v}_k \cdot \vec{q}\, \phi_{q\omega} N'(\mathcal{E}, \omega) = iC^{i}_{k\mathcal{E}}[\bar{n}_{q\omega}(\hat{k}', \sigma', \mathcal{E}')] . \tag{6.21}$$

We can remove the common factor $N'(\mathcal{E}, \omega)$ from (6.21) to get an equation satisfied by $\bar{\nu}_{q\omega}(\hat{k}, \sigma, \mathcal{E})$, which, it is not difficult to see, has a solution independent of $\mathcal{E}$ (consistent with the assumption). We can thus use (6.21) to determine n and $\bar{n}$ or equivalently the relation for $\bar{\nu}_{q\omega}(\hat{k}, \sigma)$ (we drop the superfluous label $\mathcal{E}$)

$$\omega\left[\bar{\nu}_{q\omega}(\hat{k}, \sigma)\phi_{q\omega} - \sum_{\sigma'}\int \frac{d\Omega_{k'}}{4\pi} F_{kk'}\bar{\nu}_{q\omega}(\hat{k}', \sigma')\phi_{q\omega}\right] - \vec{q} \cdot \vec{v}_k \bar{\nu}_{q\omega}(\hat{k}, \sigma)\phi_{q\omega} - \vec{v}_k \cdot \vec{q}\phi_{q\omega}]$$

$$= iC_k^{i}[\bar{\nu}_{q\omega}(\hat{k}', \sigma')\phi_{q\omega}] \tag{6.22}$$

in the region under discussion. If, as well, we write Eqs. (6.8), (6.10), and (6.11) in terms of $\bar{\nu}_{q\omega}(\hat{k}, \sigma)\phi_{q\omega}$ by carrying out the $\mathcal{E}$ integrations, we recover the results of Betbeder-Matibet and Nozières[23,24] which are obtained by the use of zero-temperature techniques. [A cautionary note: If one is interested in the damping of the undriven modes of the system, then Eq. (6.14) should be used especially if $\gamma^{p}/\gamma^{i} \gtrsim 1$.]

To summarize our results, we have found a transport equation which for temperatures high enough for interparticle scattering to be important has a nondissipative part of the usual Landau form and a dissipative part which is the sum of the standard impurity and interparticle scattering terms. For lower temperatures, the equation we get is equivalent to the zero-temperature results which do not include interparticle scattering.

VII. COULOMB FORCES

We discuss, finally, the modifications of the theory necessary for the case of the long-range Coulomb forces. As this problem has been adequately handled for Green's functions by Nozières and Luttinger[11] and in the book by Nozières,[6] we just briefly state the major changes. The transport equation obtained in this case is in the same form as (6.14) [or in the various limits as (6.20), (6.21), or (6.22)] except that $\phi_{q\omega}$, the external potential, is replaced by the mean potential in the medium $\bar{\phi}_{q\omega} = \phi_{q\omega}/\epsilon(q, \omega)$, where $\epsilon(q, \omega)$ is the impurity averaged longitudinal dielectric constant. The rest of the quantities involved in the transport equation are calculated by using only proper diagrams [i.e., those not having factors $v(q) = 4\pi e^2/q^2$]. The screening of the long-range part of the impurity potential (if present) is included in the t matrix; this result can be seen in detail in the work of Langer.[19]

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