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Comments on Einstein Scalar Solutions

ALLEN I. JANIS*
University of Pittsburgh, Pittsburgh, Pennsylvania 15213
AND
DAVID C. ROBINSON† AND JEFFREY WINICOUR
Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio 45433
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Simple solutions of the Einstein scalar and Brans-Dicke field equations are exhibited, and the nature of the Killing horizons of some static solutions is discussed.

I. INTRODUCTION

RECENTLY there has been some discussion of the coupled gravitational and zero rest-mass scalar fields in connection with investigations of the Killing horizons of static asymptotically flat fields.[1–3] The purpose of this note is to exhibit some methods of obtaining simple solutions of the Einstein field equations when a massless scalar field is present and to indicate the corresponding solutions of the Brans-Dicke scalar-tensor theory. In particular, the spherically symmetric solution analyzed by Janis *et al.*[1] will be reconsidered in the light of some comments by Penney,[2] and the significance of the singular nature of the event horizon $g_{00}=0$ will be discussed.

II. SOLUTIONS

The Einstein field equations are given by

$$G_{\mu\nu}=-\kappa T_{\mu\nu}, \tag{1}$$

where $\kappa=8\pi G_0/c^4$, and G_0 is Newton's gravitational constant.[4] For rest-mass-zero scalar fields

$$T_{\mu\nu}=\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\psi_{,\rho}\psi^{;\rho}$$

and

$$\Box\psi=0. \tag{2}$$

When the gravitational equations (1) are satisfied, Eq. (2) is a consequence of the Bianchi identities. For static fields, the line element may be written

$$ds^2=e^{-2U}h_{ij}dx^idx^j-e^{2U}(dx^0)^2, \tag{3}$$

where U and h_{ij} are functions of x^i alone. The field equations (1) and (2) take the form[5]

$$\psi_{;i}{}^{i}=0, \tag{4a}$$
$$U_{;i}{}^{i}=0,$$
$$H_{ij}+2U_{,i}U_{,j}=-\kappa\psi_{,i}\psi_{,j}, \tag{4b}$$

where H_{ij} is the Ricci tensor for the auxiliary metric h_{ij}, and the covariant derivatives are taken with respect to this metric.

In the axially symmetric cases, Eqs. (4a) reduce to the two-dimensional Laplace equations when cylindrical coordinates are used, and Eq. (4b) is then simply integrated in terms of any two independent solutions, U and ψ, of these equations.[2,3]

Simple calculations, similar to those of Ref. 5, lead to the following results:

(i) If a vacuum solution of the Einstein equations is given by the metric of the line element

$$ds^2=e^{-2V}h_{ij}dx^idx^j-e^{2V}(dx^0)^2, \tag{5}$$

then a solution of the coupled Einstein scalar equations is given by the metric of the line element (3) and ψ, where

$$\psi=AU \quad\text{and}\quad U=V(1+\kappa A^2)^{-1/2},\quad A \text{ a constant}. \tag{6}$$

(ii) If a *static* solution of the Einstein scalar equations is given by the metric of the line element (3) and ψ, then a *static* solution of the coupled Einstein-Maxwell scalar field equations

$$G_{\mu\nu}=-\kappa(\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\psi_{,\rho}\psi^{,\rho}+F_{\mu\rho}F_{\nu}{}^{\rho}-\tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}),$$
$$\Box\psi=0,\quad F^{\mu\nu}{}_{;\nu}=0,\quad F_{[\mu\nu;\rho]}=0 \tag{7}$$

is given by

$$\psi,\quad F_{\rho\sigma}=(2/\kappa)^{1/2}e^{2W}(\delta_\rho{}^0U_{,\sigma}-\delta_\sigma{}^0U_{,\rho})$$

and the metric of the line element

$$ds^2=e^{-2W}h_{ij}dx^idx^j-e^{2W}(dx^0)^2,$$

where $W=-\ln\sinh U$.

Corresponding to each of these solutions of the Einstein field equations is a solution of the Brans-Dicke scalar-tensor theory.[6] For under the gauge transformations,

$$\bar{g}_{\mu\nu}=\lambda^{-1}g_{\mu\nu},\quad \bar{F}_{\mu\nu}=G_0{}^{1/2}F_{\mu\nu},$$
$$\lambda=e^{\psi/p},\qquad p=(2\omega+3/2\kappa)^{1/2}, \tag{8}$$

the field equations (7) become the Brans-Dicke field

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† National Academy of Sciences Postdoctoral Resident Research Associate.

1 A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Letters **20**, 878 (1968).

2 R. Penney, Phys. Rev. **174**, 1578 (1968).

3 R. Gautreau (unpublished).

4 Greek indices take the values 0, 1, 2, and 3 and Latin indices the values 1, 2, and 3.

5 J. Ehlers, Report to the International Conference on Relativistic Theories of Gravitation, London, 1965 (unpublished); J. Ehlers, Z. Physik **143**, 239 (1955).

6 C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961); R. H. Dicke, *ibid.* **125**, 2163 (1962).

equations

$$\bar{\Box}\lambda=0, \tag{9a}$$

$$\bar{G}_{\mu\nu}=-\frac{8\pi}{\lambda c^4}\bar{T}_{\mu\nu}-\frac{\omega}{\lambda^2}(\lambda_{,\mu}\lambda_{,\nu}-\tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\lambda_{,\rho}\lambda_{,\sigma})-(1/\lambda)(\lambda_{;\mu\nu}-\bar{g}_{\mu\nu}\bar{\Box}\lambda). \tag{9b}$$

The energy-momentum tensor of the electromagnetic field $\bar{T}_{\mu\nu}$ is given by

$$\bar{T}_{\mu\nu}=\bar{F}_{\mu\rho}\bar{F}_{\nu}{}^{\rho}-\tfrac{1}{4}\bar{g}_{\mu\nu}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}.$$

Consequently, if $g_{\mu\nu}$, ψ, and $F_{\mu\nu}$ are solutions of the Einstein field equations (7), $\bar{g}_{\mu\nu}$, $\bar{F}_{\mu\nu}$, and λ satisfy the Brans-Dicke equations. In general, the trace $\bar{T}$ of the energy-momentum tensor of the nongravitational fields acts as the source of the scalar field, and Eq. (9a) has the form

$$\bar{\Box}\lambda=\bar{T}\times 8\pi/(3+2\omega)c^4. \tag{10}$$

However, the trace of the energy-momentum tensor of the electromagnetic field is zero, and, in this case, the scalar field equation (9a) is source free.

III. SPHERICALLY SYMMETRIC SOLUTIONS

By applying the result (i) above to the Schwarzschild solution of the vacuum Einstein equations

$$ds^2=(R+m/R-m)[dR^2+(R^2-m^2)(d\theta^2+\sin^2\theta d\varphi^2)]-(R-m/R+m)dt^2, \tag{11}$$

the Einstein scalar solution, discussed by Janis *et al.* in Ref. 1, is obtained:

$$ds^2=(R+m/R-m)^{1/\mu}[dR^2+(R^2-m^2)(d\theta^2+\sin^2\theta d\varphi^2)]-(R-m/R+m)^{1/\mu}dt^2,$$
$$\psi=(A/2\mu)\ln(R-m/R+m) \tag{12}$$

and

$$\mu=\tfrac{1}{2}(4+2\kappa A^2)^{1/2}\geqslant 1.$$

The corresponding Brans-Dicke solution is given by

$$ds^2=(R+m/R-m)^{(A+2p)/2\mu p}\times[dR^2+(R^2-m^2)(d\theta^2+\sin^2\theta d\varphi^2)]-(R+m/R-m)^{(A-2p)/2\mu p}dt^2, \tag{13}$$
$$\lambda=(R-m/R+m)^{A/2\mu p}.$$

By introducing the constants B, C, D, and the coordinate r, defined by

$$R=\tfrac{1}{2}m(r/B+B/r)$$

and

$$\frac{C}{D}=\frac{A}{\mu p},\quad \frac{A}{\mu p}-\frac{2}{\mu}=-\frac{2}{D},$$

this solution may be put in the form

$$ds^2=\frac{m^2}{4B^2}\left(\frac{r+B}{r-B}\right)^{2(C+1-D)/D}\left(1+\frac{B}{r}\right)^4\times[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)]-\left(\frac{r+B}{r-B}\right)^{-2/D}dt^2, \tag{14}$$

$$\lambda=(r-B/r+B)^{C/D},$$

where

$$D^2=(C+1)^2-C(1-\tfrac{1}{2}\omega C)\geqslant 0.$$

This is just that one of the four spherically symmetric static solutions, found by Brans,[7] which is asymptotically flat.

Although the event horizon $g_{00}=0$ is a regular hypersurface for the Schwarzschild metric, the curvature scalar is singular there for the other two metrics. In the case of the Einstein scalar fields, the event horizon is a singular point. For the Brans-Dicke fields, however, the nature of the event horizon depends on the values taken by the parameters C and ω.

The value of the coupling constant ω has been set by Dicke in his interpretation of the precession of the perihelion of Mercury at $\omega\approx 6$, and the range of values that C may take is determined by the asymptotic form of the solutions for λ and g_{00}. For physically realistic fields, the trace of the energy-momentum tensor of the matter source $\bar{T}$, the energy density ϵ, and the pressure p of matter satisfy the inequalities

$$\bar{T}=3p-\epsilon\leqslant 0,\quad \epsilon\geqslant 0,\quad p\geqslant 0.$$

In these cases, it follows directly from Eqs. (9) and (10) and the values of g_{00} and λ given by Eq. (14) that C must be negative. The event horizon $g_{00}=0$ is then a singular point.

It should be noted that axially symmetric solutions of the Einstein scalar fields have been investigated by Gautreau and Penney.[2,3] The former considered solutions of the type given by Eqs. (5) and (6) and found that these solutions also have singular event horizons $g_{00}=0$. In contrast to the static vacuum case,[8] it was found that solutions with singular point horizons, which act as multipole sources of the field, may exist when the scalar field is present.

In previous considerations of the spherically symmetric Einstein scalar solution,[1] an analysis of the behavior of the luminosity distance

$$r_L=(R+m)^{1+1/\mu}(R-m)^{1-1/\mu}, \tag{15}$$

in the limit of vanishing coupling constant κ, led to the suggestion that the physical solution corresponding to a spherically symmetric point mass might differ from the usual Schwarzschild solution by having a singular point event horizon $g_{00}=0$. Penney has suggested that if nonspherically symmetric solutions of the scalar wave equation are considered, one is led to the usual Schwarzschild solution in the limit of vanishing coupling constant, and, consequently, analyses of this

[7] C. Brans, Phys. Rev. **125**, 2194 (1961).
[8] J. Winicour, A. I. Janis, and E. T. Newman, Phys. Rev. **176**, 1507 (1968).

type do not indicate a physical singularity at the Schwarzschild radius. However, the example he gives in support of this contention contains a calculational error, and, in fact, his solution is singular at the horizon $g_{00}=0$.[9]

[9] In Ref. 2, Eq. (19) should be replaced by

$$R_\mu{}^\mu=-\frac{\kappa a^2[R(R-2m)+m^2\sin^2\theta]}{R^2[R(R-2m)+m^2\cos^2\theta]^2}\exp\left\{\frac{\kappa a^2 R(R-2m)\sin^2\theta}{2[R(R-2m)+m^2\cos^2\theta]^2}\right\}.$$

The curvature scalar is then singular when $R(R-2m)+m^2\cos^2\theta=0$, for all values of κ.

A more serious objection has been raised by Misner,[10] who has pointed out that it is not possible to provide a material source, under normal conditions of hydrostatic support, for the solution (12), beyond a minimum distance

$$r_L=(m^2/4\mu^2)(\mu+1)^{2+2/\mu}(\mu-1)^{2-2/\mu}.$$

Consequently, the singularity at $g_{00}=0$ would seem to be of no physical significance in this case.

[10] C. Misner (to be published).

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Inconsistency of Asymptotic $SU(2)\times SU(2)$ for the Proper Amplitudes with Local Chiral Algebra*

R. Acharya
Institut für Theoretische Physik, Universität Bern, Switzerland

AND

H. H. Aly†
Department of Mathematics, American University of Beirut, Beirut, Lebanon

AND

P. Narayanaswamy†
Department of Physics, American University of Beirut, Beirut, Lebanon

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A recent proposal of asymptotic symmetry for the proper amplitudes is tested with reference to the Adler-Weisberger sum rule and is found to lead to an inconsistency, thereby demonstrating its incompatibility with local chiral current algebra.

THERE has recently been a very interesting suggestion by Barry, Gounaris, and Sakurai,[1] who seek a modification of the Adler-Fubini sum rule[2,3] by postulating specific asymptotic requirements on the proper amplitudes rather than on amplitudes involving matrix elements of currents. In view of Gerstein's proof[4] of the inconsistency of this approach with local current algebra involving currents of *unequal* masses, it is interesting to ask if the inconsistency persists for equal masses. In this paper, we propose to answer this in the affirmative within the framework of asymptotic $SU(2)\times SU(2)$ symmetry. We first present a derivation of the modified sum rule resulting from asymptotic $SU(2)\times SU(2)$ of the proper amplitude and explicitly demonstrate that it violates the well-established Adler-Weisberger[5] (AW) sum rule. Finally, we give a simple and direct argument based on Regge theory as to why the asymptotic symmetry for the proper amplitudes will be inconsistent with the usual form of asymptotic chiral symmetry.

Let the amplitudes for the forward weak vector and weak axial-vector scattering off nucleons be defined by

$$M_{\mu\nu}{}^{(V)i,j}(\nu,q^2)=i\int d^4x\, e^{-iq\cdot x}\theta(x_0)\times\langle N(p)|[V_\mu{}^i(x),V_\nu{}^j(0)]|N(p)\rangle,$$
$$M_{\mu\nu}{}^{(A)i,j}(\nu,q^2)=i\int d^4x\, e^{-iq\cdot x}\theta(x_0)\times\langle N(p)|[A_\mu{}^i(x),A_\nu{}^j(0)]|N(p)\rangle, \quad (1)$$

where i and j denote isospin and $\nu=-p\cdot q/m$, m being the nucleon mass. Following Fayyazuddin and Hussain,[6] let us write the invariant decomposition of the spin-

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† Present address: Physics faculty, Southern Illinois University, Edwardsville, Illinois.
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