Nonrelativistic Strong-Coupling Model with a Recoiling Source*

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We have replaced the static extended source of traditional strong-coupling models with a point fermion of finite bare mass. We find self-consistent stationary-state solutions to the problem of the strongly coupled fermion and pion field in the neutral pseudoscalar theory. We find the usual set of rotational levels, $j=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, and, in addition, find a class of states which strongly suggest identification with the $N'(1470)$ and its possible rotational excitations. Our model provides a natural interpretation of the repulsive hard-core potential in nuclear forces, and also contains the mechanisms which will lead to negative-parity nucleon resonances and Regge recurrences when internal symmetries are included. Because of several important approximations, our numerical results are not yet to be taken seriously.

I. INTRODUCTION

THE strong-coupling theory has a long history dating back to some of the earliest attempts to form a field theory of the strong interactions. It has not been as productive of useful information about nucleon structure as one might have hoped, and most of the workers on this model have confined themselves to predicting the spectrum of baryon resonances using various forms of static sources in Hamiltonian models[1–4] or the properties of the noncompact strong-coupling groups.[5–7] Some attempts have also been made to explain nuclear forces[8] and pion-nucleon scattering,[9] again within the context of the static model.

The limitation to static models is a severe one and effectively denies one the opportunity to make unambiguous predictions of such experimentally interesting quantities as the electromagnetic form factors, photoproduction and pion-production amplitudes for resonances, and scattering cross sections (both elastic and inelastic).

In this paper, we propose a method for introducing a recoiling source into the strong-coupling theory. It must be emphasized at the outset that our first formulation of this model is rather primitive, and in its present form it is not greatly more satisfactory than the static model itself. But the principle involved is physically more realistic than the static-extended source, and if certain technical problems can be solved, this new model will have not only a richer structure in terms of resonances, but the capacity to predict unambiguously the quantities previously mentioned.

The basic innovation of this model is the interpretation of the source to which the pion field is strongly coupled. We introduce the source as a point fermion with a finite bare mass and a spin of $\frac{1}{2}$. We then look for solutions in which this fermion and its attendant pion field are in a self-consistent stationary state.

The self-consistency is obtained by treating the probability density of the fermion wave function as the source function which determines the pion field in the strong-coupling approximation. This pion field then serves as a potential well in which the fermion can be shown to have bound states. It is not obvious *a priori* that such a system will have self-consistent bound states, but we show in this paper that such states do exist.

We make several important approximations on the way to our results and these are discussed as they are introduced and in Sec. V of the paper. These approximations make our numerical results quite unreliable, but we have included some anyway to show the qualitative effects of variations of the fundamental parameters of the model.

We have only two free parameters in the model: the fermion bare mass m_0 and the bare pion-nucleon coupling constant g. Our aim is to predict the spectrum of baryon resonances with only these two parameters. Our preliminary results as presented here represent a qualitative success but not yet a quantitative one. This problem will be discussed in detail in Sec. V.

Our model has produced two new results which were not present in the old static strong-coupling theory. It provides a physically simple and natural explanation of the hard core in nuclear forces, and it provides an explanation of the existence of the class of resonances typified by the $N'(1470)$ or Roper resonance. The model also contains the potential (which is not ex-

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ploited in this paper) for including negative parity resonances and "Regge recurrences" in a natural way. All of these results are discussed in detail in the text.

The paper is organized as follows: Sec. II presents a brief review of the simplest nontrivial strong-coupling theory emphasizing the limitations imposed by a static extended source. In Sec. III, we present our proposal for introducing a recoiling source, and in Sec. IV, we find solutions to the model. In Sec. V, the virtues and limitations of the model are discussed and we suggest ways in which it might be improved.

II. STATIC STRONG-COUPLING THEORY

We begin our review of strong-coupling theory with the work of Pauli and Dancoff.[1] They considered the problem of a charge-symmetric interaction of a pseudoscalar field with a static extended source $[\rho(x)]$ which has both spin and isospin degrees of freedom.

In this review we shall treat the simpler problem of a neutral pseudoscalar field interacting with a neutral source which has spin $\frac{1}{2}$. The solutions to the two systems follow very similar lines, and by treating the simpler problem we economize greatly on notation.

We begin with the Hamiltonian

$$H=\frac{1}{2}\int d^3x[\pi^2(\mathbf{x})+|\nabla\phi(\mathbf{x})|^2+\mu^2\phi^2(\mathbf{x})] +\frac{g}{\mu}\int d^3x\,\rho(\mathbf{x})\boldsymbol{\sigma}\cdot\nabla\phi(\mathbf{x}). \quad (1)$$

The source density $\rho(x)$ is assumed to be spherically symmetric.

The essential step in reducing this problem to an easily soluble one in the limit where g is large is to define a set of three dynamical variables.

$$q_i=\int d^3x\,\phi(\mathbf{x})\nabla_i\rho(\mathbf{x}). \quad (2)$$

The q_i are operators which represent the strength of the overlap of the p-wave part of the pion field with the gradient of the source density. In terms of the q_i, the interaction part of the Hamiltonian becomes simply

$$H_I=-(g/\mu)\boldsymbol{\sigma}\cdot\mathbf{q}. \quad (3)$$

Since the q_i commute among themselves, the symmetry group of this Hamiltonian is $SU(2)\times T_3$ which is the simplest strong-coupling group considered by Goebel and collaborators.[5]

The next step is to get the rest of the Hamiltonian in terms of the q_i. This requires the splitting up of the pion field $\phi(x)$ into "bound" and "free" parts as follows:

$$\phi(\mathbf{x})=\mathbf{q}\cdot\nabla\xi(\mathbf{x})+\phi'(\mathbf{x}). \quad (4)$$

If we now insert (4) into (2), we find the following requirements on $\xi(x)$ and $\phi'(x)$:

$$\int d^3x\,\nabla_i\rho(\mathbf{x})\nabla_j\xi(\mathbf{x})=\delta_{ij}, \quad (5)$$

$$\int d^3x\,\nabla_i\rho(\mathbf{x})\phi'(\mathbf{x})=0. \quad (6)$$

We note that the only restrictions on $\xi(\mathbf{x})$ imposed by (5) are that it be spherically symmetric and normalized properly. Note also that (6) puts a restriction only on the p-wave part of $\phi'(\mathbf{x})$ and therefore all mesons in other angular momentum states relative to the source are considered free (i.e., noninteracting). This restriction to p-wave mesons is one of the primary deficiencies of the static model. It restricts the resonance spectrum to positive-parity states, and, in the charge-symmetric theory, those states in which the angular momentum equals the isospin. Even if higher symmetries such as $SU(3)$ are used, the resonance spectrum is still deficient in Regge recurrences and overstocked with unobserved multiplets.

Some attempts[3,5] have been made to remove the restriction to p-wave mesons, but all have remained within the context of the static model. In Sec. III we will propose a model which, at least in principle, is capable of providing both Regge recurrences and negative-parity states.

Now to get the pion-field Hamiltonian in terms of the q_i, we still need an expression for $\pi(\mathbf{x})$. This is also broken up into a bound and free part as follows:

$$\pi(\mathbf{x})=\boldsymbol{\pi}\cdot\nabla\rho(\mathbf{x})+\pi'(\mathbf{x}), \quad (7)$$

where the $\boldsymbol{\pi}_i$ are defined in such a way that

$$[\boldsymbol{\pi}_i,q_j]=-i\delta_{ij}. \quad (8)$$

Referring to Eq. (5) and using the fact that $\pi(x)$ and $\phi(x)$ satisfy the equal-time commutation rules,

$$[\pi(\mathbf{x}),\phi(\mathbf{x}')]=-i\delta^3(\mathbf{x}-\mathbf{x}'), \quad (9)$$

it is easy to show that the proper definition of $\boldsymbol{\pi}_i$ is

$$\boldsymbol{\pi}_i\equiv\int d^3x\,\pi(\mathbf{x})\nabla_i\xi(\mathbf{x}), \quad (10)$$

and it then follows that

$$\int d^3x\,\pi'(\mathbf{x})\nabla_i\xi(\mathbf{x})=0. \quad (11)$$

For completeness we include the commutation relations for the $\pi'(x)$, $\phi'(x)$:

$$[\pi'(\mathbf{x}),\phi'(\mathbf{x})]=-i[\delta^3(\mathbf{x}-\mathbf{x}')-\nabla\rho(\mathbf{x})\cdot\nabla'\xi(\mathbf{x}')]. \quad (12)$$

These commutation relations are nonlocal and lead to a rather complicated problem when one tries to calculate scattering of free mesons. This phenomenon (called

orthogonality scattering by Sachs[9]) will not concern us in this paper.

We now substitute (4) and (7) into the free-pion Hamiltonian in (1). We find that if we add the further restriction on $\xi(\mathbf{x})$,

$$(-\nabla^2+\mu^2)\xi(\mathbf{x})=(1/N)\rho(\mathbf{x}), \tag{13}$$

the full Hamiltonian simplifies to

$$H=\tfrac{1}{2}R|\boldsymbol{\pi}|^2+(1/2N)|\mathbf{q}|^2-(g/\mu)\boldsymbol{\sigma}\cdot\mathbf{q} +\frac{1}{2}\int d^3x[\pi'^2(\mathbf{x})+|\nabla\phi'(\mathbf{x})|^2+\mu^2\phi'^2(\mathbf{x})] +\pi_i\int d^3x\,\pi'(\mathbf{x})\nabla_i\rho(\mathbf{x}), \tag{14}$$

where

$$R=\frac{1}{3}\int d^3x|\nabla\rho(\mathbf{x})|^2. \tag{15}$$

Now Eq. (13) implies that

$$\xi(\mathbf{x})=\frac{1}{4\pi N}\int d^3x'\,\rho(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \tag{16}$$

and using (15) and (5) we can derive the expression for N:

$$N=\frac{1}{3}\int d^3x\,[\rho^2(\mathbf{x})-\rho(\mathbf{x})N\xi(\mathbf{x})]. \tag{17}$$

The first three terms in H describe the dynamics of the $\boldsymbol{\pi}$ and $\mathbf{q}$ variables and resemble the Hamiltonian for a three-dimensional harmonic oscillator in which the amplitude of the vibration is coupled to a spin. The fourth term represents the "free" pions and its form is deceptively similar to that of the pion field when no source is present. However, one must keep in mind in solving the dynamics of the "free" pions that the field amplitudes satisfy nonlocal commutation rules.

The last term in (14) gives the coupling of the free and bound fields and it is independent of g. We would like to treat this term as a perturbation, but first we must extract from it the dependence on angular momentum. This is necessary because, as we will see, the level separation in the "rotational band" goes like g^{-2}. So if g is large the rotational states are relatively low-lying and close together. For these states the "pulling" of the levels caused by the coupling of the bound and free fields is quite significant.

In the neutral pseudoscalar theory the separation is very simply accomplished by noting that[10]

$$\mathbf{L}=-\int d^3x\,\pi(\mathbf{x})\mathbf{r}\times\nabla\phi(\mathbf{x}) =\mathbf{q}\times\boldsymbol{\pi}-\int d^3x\,\pi'(\mathbf{x})(\mathbf{r}\times\nabla)\phi'(\mathbf{x}). \tag{18}$$

[10] Contrary to the statement of Pauli and Dancoff following their Eq. (22′), the cross terms do vanish if $\rho(\mathbf{x})$ and $\boldsymbol{\xi}(\mathbf{x})$ are spherically symmetric.

Now using (18) we can write

$$\boldsymbol{\pi}=(1/q^2)\mathbf{q}(\mathbf{q}\cdot\boldsymbol{\pi})-(1/q^2)\mathbf{q}\times\mathbf{L}, \tag{19}$$

and if this is inserted into the last term of (14) we get

$$\frac{1}{q^2}q_i(\mathbf{q}\cdot\boldsymbol{\pi})\int d^3x\,\pi'(\mathbf{x})\nabla_i\rho(\mathbf{x}) -\frac{1}{q^2}(\mathbf{q}\times\mathbf{L})_i\int d^3x\,\pi'(\mathbf{x})\nabla_i\rho(\mathbf{x}). \tag{20}$$

We now define a new field variable

$$\pi''(\mathbf{x})=\pi'(\mathbf{x})-(1/q^2)(\mathbf{q}\times\mathbf{L})_i \times[\nabla_i\rho(\mathbf{x})-(1/T)\nabla_i\xi(\mathbf{x})], \tag{21}$$

where

$$T=\frac{1}{3}\int d^3x|\nabla\xi(\mathbf{x})|^2. \tag{22}$$

This definition is arranged so that (11) is still satisfied using $\pi''(\mathbf{x})$ instead of $\pi'(\mathbf{x})$. Solving (21) for $\pi'(\mathbf{x})$ and substituting into (14), we get finally

$$H=\frac{1}{2T}\frac{L^2}{q^2}+\frac{1}{2N}q^2-\frac{g}{\mu}\boldsymbol{\sigma}\cdot\mathbf{q}+\tfrac{1}{2}R|(1/q^2)\mathbf{q}(\mathbf{q}\cdot\boldsymbol{\pi})|^2 +\frac{1}{2}\int d^3x[\pi''^2(\mathbf{x})+|\nabla\phi'(\mathbf{x})|^2+\mu^2\phi'^2(\mathbf{x})] +\frac{1}{q^2}q_j(\mathbf{q}\cdot\boldsymbol{\pi})\int d^3x\,\pi'(\mathbf{x})\nabla_j\rho(\boldsymbol{\pi}). \tag{23}$$

In this Hamiltonian the first three terms are the ones we will consider in detail. The fourth term represents the kinetic energy stored in the radial oscillations of the q variable. The solution of the strong-coupling problem is one in which the magnitude of the vector $\mathbf{q}$ executes small zero-point oscillations about some equilibrium value q_0. The zero-point energy of these oscillations contributes some additive constant to the ground-state energy and the energies of the rotational states. As long as we do not consider excitations of higher vibrational states, this constant can be ignored.

The fifth term represents the free-meson energy. We will measure all energies relative to the free-meson vacuum so we drop this term. Finally, the last term represents the coupling of the radial oscillations of the bound field and the free field. This term can be assumed to be small. In addition to these terms there are others which are generated if we treat the transformation (21) consistently as a canonical transformation to new variables. Pauli and Dancoff discuss these other terms and show that they are small if g is sufficiently large. It is these neglected terms which lead to transitions between rotational states, so they must be taken into

account properly if decay widths are to be predicted. We do not discuss this problem in this paper.

We now restrict ourselves to the first three terms of (22) and assume that the third term is large. The problem is solved by diagonalizing this last term by means of the unitary transformation

$$U^\dagger \boldsymbol{\sigma}\cdot\mathbf{q} U = q\sigma_3. \tag{24}$$

It is straightforward to show that

$$U = e^{-\frac{1}{2}i\sigma_3\alpha} e^{-\frac{1}{2}i\sigma_2\beta}, \tag{25}$$

where α and β are, respectively, the azimuthal and polar angular coordinates of the vector $\mathbf{q}$. The form of U is quite familiar, and, in fact, if $U^\dagger$ operates on a state with spin up (along the space-fixed z axis), the effect is to rotate the spin vector so that it is oriented parallel to the vector $\mathbf{q}$.

The transformation U has no effect on the second term of H and the full dependence of the Hamiltonian on the angles α and β is contained in the term

$$(1/2Tq^2)U^\dagger L^2 U. \tag{26}$$

Now we know the expression for L^2 in terms of α and β and the above transformation can be worked out explicitly. The eigenstates of this transformed operator turn out to be the symmetric top wave functions $D_{m\pm 1/2}{}^{(j)}(\alpha,\beta,0)$, where the choice of zero for the third argument represents only a choice of an over-all phase. The transformed Hamiltonian has the form

$$\frac{1}{2Tq^2}U^\dagger L^2 U + \frac{1}{2N}q^2 - \frac{g}{\mu}\begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \tag{27}$$

and since g has been assumed positive, we see that the last term gives a strong binding in the state where $\boldsymbol{\sigma}\cdot\mathbf{q}=q$ and a strong repulsion when $\boldsymbol{\sigma}\cdot\mathbf{q}=-q$. Thus the correct eigenfunction to choose is $D_{m+1/2}{}^{(j)}(\alpha,\beta,0)$, and this represents a symmetric top with angular momentum projection $+\frac{1}{2}$ along its body-fixed symmetry axis.

In this state the potential energy becomes

$$(1/2N)q^2 - (g/\mu)q,$$

and we complete the square to get

$$(1/2N)q'^2 - g^2N/2\mu^2, \tag{28}$$

where $q'=q-gN/\mu$ represents fluctuations in the amplitude about the equilibrium value $q_0=gN/\mu$. The term in q'^2 can be transferred to the neglected part of H, and, in fact, combines with the fourth term in (24) to give the Hamiltonian for a one-dimensional oscillator.

The eigenvalues of the rotational term are

$$(1/2Tq_0^2)[j(j+1)+\tfrac{1}{4}], \tag{29}$$

but the extra $\frac{1}{4}$ can be dropped since it is also a constant added to all energy levels. The final expression for the energy spectrum is then

$$E = E_0 - g^2N/2\mu^2 + (\mu^2/2g^2TN^2)[j(j+1)-\tfrac{3}{4}], \tag{30}$$

where we have kept the second term separate from E_0 for easy comparison with the results of Sec. IV, and the rotational energy has been set to zero for the ground state.

The above solution of the strong-coupling problem is quite straightforward and physically understandable, because we have chosen the simplest interesting system. If we try to solve the more complicated theories this same way, we find that the $SU(2)\times SU(2)$ theory is more complicated but still tractable and the $SU(2)\times SU(3)$ model is extremely cumbersome.[3] Fortunately, Cook and Sakita[5] have shown how to derive the eigenstates for any theory by using the elegant method of induced representations. Using this method we could have guessed immediately that our eigenfunctions would be the symmetric top wave functions, since these form the irreducible representations of the strong-coupling group $SU(2)\times T_3$, which is just the Galilean group in three dimensions. The eigenfunctions in more complicated theories turn out to be generalized symmetric top wave functions.[5]

This concludes our review of the static strong-coupling theory. The effects of the static, spherically symmetric source are seen to be the restriction to positive-parity states and the appearance of only one state for each value of the angular momentum. In the $SU(2)\times SU(2)$ model this restriction appears as the requirement that $I=J$ and in the $SU(2)\times SU(3)$ model, the allowed multiplets are those in which the component with hypercharge $+1$ also has $I=J$.

We now introduce a new formulation of the strong-coupling theory which will, in principle allow for a richer selection of resonances.

III. RECOILING SOURCE

A. Self-Consistent Equations

We begin with the Hamiltonian

$$H = \int d^3x\, \psi^\dagger(\mathbf{x})\left(m_0 + \frac{p^2}{2m_0}\right)\psi(\mathbf{x}) + \frac{1}{2}\int d^3x[\pi^2(\mathbf{x}) + |\nabla\phi(\mathbf{x})|^2 + \mu^2\phi^2(\mathbf{x})] - g\int d^3x\, \psi^\dagger(\mathbf{x})\boldsymbol{\sigma}\cdot\hat{r}\psi(\mathbf{x})\phi(\mathbf{x}). \tag{31}$$

We have added a term representing a nonrelativistic two-component fermion field $\psi(\mathbf{x})$, and we have also changed the interaction term by changing the $\boldsymbol{\sigma}\cdot\nabla$ of the Pauli-Dancoff model to $\boldsymbol{\sigma}\cdot\hat{r}$. In addition to being simpler to deal with, this latter form allows us to use a dimensionless coupling constant without introducing a mass.

The next step is to postulate that the eigenstates of the Hamiltonian can be given in terms of a set of basis states of the form

$$|\psi\rangle=|\psi_{lm}\rangle|\chi_n\rangle|\phi_{LM}\rangle, \tag{32}$$

where $|\psi_{lm}\rangle$ is a one-particle fermion state of definite orbital angular momentum about some arbitrarily chosen origin, and $|\chi_n\rangle$ is a two-component spinor. The definition of $|\psi_{lm}\rangle$ can be written explicitly as

$$\psi(\mathbf{x})|\psi_{lm}\rangle=\psi_{lm}(\mathbf{x})|0\rangle, \tag{33}$$

where $\psi_{lm}(x)$ is a one-particle wave function. This restriction to one-particle fermion states is essentially automatic in a nonrelativistic theory, but in a relativistic theory it amounts to the assumption that virtual fermion pair states have a small effect. This is just what is done for the relativistic hydrogen atom: The one-particle Dirac equation is used to get the energy levels, and then vacuum polarization effects are calculated in perturbation theory. Our hope is that this same procedure can be used consistently in our model. This is an assumption which has not yet been quantitatively justified.

The state vector $|\phi_{LM}\rangle$ in (32) represents a state of the pion field of definite angular momentum. This is not as yet a unique specification, and this definition will be made more specific below.

Now a general eigenstate of H will be constructed as a superposition of product states of the type (32). In general, an exact specification of any eigenstate of H will require an infinite number of the product states and in practice this series will be truncated for practical reasons. Presumably the greater the number of product states included in the sum the more accurately will be represented the actual spectrum of states of H.

At this point, before proceeding with the problem at hand, it will help to clarify the meaning of the subsequent calculations if we digress briefly to discuss a more familiar problem from the viewpoint of our model. We consider the problem of two nonidentical particles (taken to have equal masses for simplicity) which interact via a central force.

The Hamiltonian for this system is

$$H=p_1^2/2m+p_2^2/2m+V(|\mathbf{x}_1-\mathbf{x}_2|), \tag{34}$$

and, as is well known, one solves this by defining new momentum and position coordinates, one set referring to the momentum and position of the c.m. and the other to the momentum and position of an equivalent particle of reduced mass in the c.m. frame.

Let us suppose, however, that for some reason we are prevented from making this change of variables. We must then proceed with the techniques used by the many-body theorists when they deal with systems such as atoms or nuclei, and the standard approach is the independent-particle or Hartree type of calculation. It is instructive to examine the two-body problem using these many-body techniques since this is the closest analogy to the way we will proceed with our strong-coupling model.

The standard procedure in a Hartree calculation is to start with a state which is a product of single-particle wave functions, these wave functions having been determined in some convenient starting potential. To calculate the ground state of H, we might begin by writing

$$\Psi(\mathbf{x}_1,\mathbf{x}_2)=\psi_0(\mathbf{x}_1)\psi_0(\mathbf{x}_2), \tag{35}$$

where $\psi_0(\mathbf{x}_i)$ is the ground-state wave function of a particle of mass m moving in the potential well $V(|\mathbf{x}_i|)$. The eigenvalue equation is

$$H\Psi(\mathbf{x}_1,\mathbf{x}_2)=E\Psi(\mathbf{x}_1,\mathbf{x}_2), \tag{36}$$

and this can be written as two coupled equations by taking matrix elements with respect to $\psi_0(x_1)$ and $\psi_0(x_2)$, respectively, i.e.,

$$\begin{aligned}\langle\psi_0(1)|H|\psi_0(1)\rangle|\psi_0(2)\rangle&=E|\psi_0(2)\rangle,\\ \langle\psi_0(2)|H|\psi_0(2)\rangle|\psi_0(1)\rangle&=E|\psi_0(1)\rangle.\end{aligned} \tag{37}$$

Using (34) for H, these equations become

$$\begin{aligned}\left(\frac{p_2^2}{2m}+\int d^3x_1\,\psi_0^*(\mathbf{x}_1)V(|\mathbf{x}_1-\mathbf{x}_2|)\psi_0(\mathbf{x}_1)\right)\psi_0(\mathbf{x}_2)&\\ =(E-K_1)\psi_0(\mathbf{x}_2),&\\ \left(\frac{p_1^2}{2m}+\int d^3x_2\,\psi_0^*(\mathbf{x}_2)V(|\mathbf{x}_1-\mathbf{x}_2|)\psi_0(\mathbf{x}_2)\right)\psi_0(\mathbf{x}_1)&\\ =(E-K_2)\psi_0(\mathbf{x}_1),&\end{aligned} \tag{38}$$

where

$$K_j=\int d^3x_j\,\psi_0^*(\mathbf{x}_j)\frac{p_j^2}{2m}\psi_0(\mathbf{x}_j). \tag{39}$$

The solutions of these two equations using the starting wave functions in the potential integral leads to two new independent-particle wave functions. These new functions are put back into Eq. (38) and generate another new set. This process continues until the wave functions no longer change and we have a self-consistent set of solutions to (38).

The final result if only (35) is used is at best a crude approximation to the real ground state. This can be demonstrated by choosing a soluble potential, doing the problem both ways, and comparing the answers. A particularly simple potential is the harmonic oscillator, which gives

$$H=p_1^2/2m+p_2^2/2m+\tfrac{1}{2}k|\mathbf{x}_1-\mathbf{x}_2|^2. \tag{40}$$

We know the correct answer for the ground state in terms of c.m. and relative coordinates:

$$\Psi(\mathbf{r},\mathbf{R})=Ce^{i\mathbf{P}\cdot\mathbf{R}}e^{-\frac{1}{2}\beta r^2}, \tag{41}$$

where $\mathbf{P}$ is the momentum of the c.m., and C is the

normalization. For the special case $\mathbf{P}=0$, we have

$$\Psi(r)=Ce^{-\frac{1}{2}\beta r^2}, \tag{42}$$

where[11]

$$\beta=(\tfrac{1}{2}mk)^{1/2}.$$

The energy of the ground state is

$$E_{\text{exact}}=\tfrac{3}{2}(2k/m)^{1/2}=(3/\sqrt{2})\omega. \tag{43}$$

If we now follow the procedure indicated in (38), we obtain the following equations:

$$\begin{aligned}(p_2^2/2m+\tfrac{1}{2}kx_2^2)\psi_0(\mathbf{x}_2)&=(E-K_1-V_1)\psi_0(\mathbf{x}_2),\\ (p_1^2/2m+\tfrac{1}{2}kx_1^2)\psi_0(\mathbf{x}_1)&=(E-K_2-V_2)\psi_0(\mathbf{x}_1),\end{aligned} \tag{44}$$

where

$$V_i=\int d^3x\,\psi_0^*(\mathbf{x}_i)(\tfrac{1}{2}kx_i^2)\psi_0(\mathbf{x}_i). \tag{45}$$

The problem has separated into two uncoupled problems because of the simple form of the potential and the approximate ground state. Since

$$V(|\mathbf{x}_1-\mathbf{x}_2|)=\tfrac{1}{2}k(x_1^2+x_2^2-2\mathbf{x}_1\cdot\mathbf{x}_2), \tag{46}$$

the self-consistency search does not have to be done. The symmetry of the problem tells us that $K_1=K_2$ and $V_1=V_2$, and the fact that the matrix element of $\mathbf{x}_1\cdot\mathbf{x}_2$ vanishes for our simple product of S-wave states then ensures that

$$K_1+V_1=K_2+V_2=\tfrac{1}{2}E.$$

The separation of this particular problem makes the calculations simple but does not change qualitatively the nature of the result.

Now the total energy of our approximate ground state is

$$E_{\text{approx}}=2[\tfrac{3}{2}(k/m)^{1/2}]=3\omega. \tag{47}$$

Comparing this with (43), we see that we have overestimated the ground-state energy by a factor of $\sqrt{2}$ or about 40%. Our approximate wave function is

$$\Psi_{\text{approx}}(\mathbf{x}_1,\mathbf{x}_2)=C\exp[-(\beta/\sqrt{2})r_1^2]\exp[-(\beta/\sqrt{2})r_2^2]. \tag{48}$$

It is instructive to put (48) in terms of the relative and c.m. coordinates $(\mathbf{r},\mathbf{R})$:

$$\Psi_{\text{approx}}(\mathbf{r},\mathbf{R})=C\exp(-\sqrt{2}\beta R^2)\exp[-(\beta/2\sqrt{2})r^2]. \tag{49}$$

From Eq. (49), we can see explicitly how the translational invariance of the wave function is broken, and also the reason why the energy is overestimated: We have effectively placed the whole system in an external oscillator well centered at the origin. Notice also that the wave function falls off less rapidly in r than the correct one.

If a better approximation to the ground state is desired, the next step is to include ground-state correlations or virtual excitations to excited states. This is done in our simple model by considering the set of states

$$\psi_0(x_i)\psi_0(x_j),\quad \psi_0(x_i)\psi_{1m}(x_j),\quad \psi_{1m}(x_i)\psi_{1m'}(x_j), \tag{50}$$

where m and m' take on the values ±1 and 0. This is a total of 16 states, and, in general, each of the above products should be determined by a self-consistent calculation. Once this is done, matrix elements of the Hamiltonian can be taken between various product states and the Hamiltonian diagonalized to give the proper superpositions for the stationary states of H.

If this is done for our oscillator example, the new ground-state energy is

$$E_C=[4-(7/4)^{1/2}](k/m)^{1/2}=2.68\omega, \tag{51}$$

which is to be compared with the value 3.00ω for the uncorrelated approximation, and the correct value of 2.13ω [Eq. (43)]. The corrected wave function is

$$\Psi_C(\mathbf{x}_1,\mathbf{x}_2)=C\exp[-(\beta/\sqrt{2})(r_1^2+r_2^2)]\times(1+0.62\beta\mathbf{x}_1\cdot\mathbf{x}_2). \tag{52}$$

In terms of $\mathbf{r}$ and $\mathbf{R}$, this becomes

$$\Psi_C(\mathbf{r},\mathbf{R})=C\exp(-\sqrt{2}\beta R^2)\exp[-(\beta/2\sqrt{2})r^2]\times(1+0.62\beta R^2-0.15\beta r^2). \tag{53}$$

We see that the effect of including the lowest virtual excitations is to make the wave function flatter as a function of R^2 (i.e., the whole system is now in a shallower well) and to make the falloff more rapid as a function of r^2 (also in the correct direction).

We note in passing that the above procedure leads to too many states, i.e., the spurious states well known to nuclear theorists.[12] Techniques exist for treating these states properly in nuclear physics, but the author is at present unqualified to say anything more about them. In particular, it is not at all clear how the nuclear-physics techniques might be adapted to our model, which is quite different from the usual many-particle model of the nucleus. Fortunately we avoid this problem in this initial exposition of the model by not considering correlations.

B. Fermion Wave Functions

We now return to our model of the nucleon. We write our approximate ground state as [cf. Eq. (32)]

$$|\Psi_{\frac{1}{2}m}\rangle=|\psi_{00}\rangle|\Phi_{\frac{1}{2}m}\rangle, \tag{54}$$

where

$$|\Phi_{\frac{1}{2}m}\rangle=\sum_n C(\tfrac{1}{2}L\tfrac{1}{2};n,m-n)\chi_n|\Phi_{L,m-n}\rangle. \tag{55}$$

[11] Note that $\frac{1}{2}m$ is the reduced mass.

[12] See, e.g., G. E. Brown, *Unified Theory of Nuclear Models* (John Wiley & Sons, Inc., New York, 1967). We also acknowledge the helpful comments of J. D. Bjorken and M. Einhorn on this point.

We can use either $L=0$ or $L=1$ to make the total $J=\frac{1}{2}$. We consider the eigenvalue problem

$$H|\Psi_{\frac{1}{2}m}\rangle=E|\Psi_{\frac{1}{2}m}\rangle, \tag{56}$$

and by analogy to Eq. (37), we take expectation values as follows:

$$\langle\psi_{00}|H_{mm'}|\psi_{00}\rangle|\Phi_{\frac{1}{2}m'}\rangle=E|\Phi_{\frac{1}{2}m}\rangle, \tag{57a}$$

$$\langle\Phi_{\frac{1}{2}m'}|H_{m'm}|\Phi_{\frac{1}{2}m}\rangle|\psi_{00}\rangle=E|\psi_{00}\rangle. \tag{57b}$$

Inserting (31) into the expectation value in (57a), we get

$$\langle\psi_{00}|H|\psi_{00}\rangle$$
$$=m_0+K_0+\frac{1}{2}\int d^3x[\pi^2(\mathbf{x})+|\nabla\phi(\mathbf{x})|^2+\mu^2\phi^2(\mathbf{x})]$$
$$-g\boldsymbol{\sigma}\cdot\int d^3x\,\psi_{00}^*(\mathbf{x})\hat{r}\psi_{00}(\mathbf{x})\phi(\mathbf{x}), \tag{58}$$

which is strongly reminiscent of Eq. (1). The identification is exact if we set

$$(1/\mu)\nabla\rho(x)=\psi_{00}(\mathbf{x})\hat{r}\psi_{00}(\mathbf{x}). \tag{59}$$

It is clear now that we can solve Eq. (57a) by the same techniques as were used in the static strong-coupling problem.[1] We write

$$\begin{aligned}\phi(\mathbf{x})&=\mathbf{q}\cdot\hat{r}\,\xi(\mathbf{x})+\phi'(\mathbf{x}),\\ \pi(\mathbf{x})&=\boldsymbol{\pi}\cdot\hat{r}\rho(\mathbf{x})+\pi'(\mathbf{x}),\end{aligned} \tag{60}$$

where

$$q_i=\int d^3x\,\hat{r}_i\rho(\mathbf{x})\phi(\mathbf{x}),\quad \pi_i=\int d^3x\,\hat{r}_i\xi(\mathbf{x})\pi(\mathbf{x}). \tag{61}$$

We also demand that

$$\int d^3x\,\hat{r}_i\rho(\mathbf{x})\hat{r}_j\xi(\mathbf{x})=\delta_{ij} \tag{62}$$

and

$$\int d^3x\,\hat{r}_i\xi(\mathbf{x})\pi'(\mathbf{x})=\int d^3x\,\hat{r}_i\rho(\mathbf{x})\phi'(\mathbf{x})=0. \tag{63}$$

Finally, we define

$$\rho(\mathbf{x})\equiv|\psi_{00}(\mathbf{x})|^2 \tag{64}$$

and find $\xi(\mathbf{x})$ from

$$(-\nabla^2+\mu^2)\hat{r}\,\xi(\mathbf{x})=\hat{r}\rho(\mathbf{x})/N. \tag{65}$$

The Hamiltonian analogous to (14) is now

$$H=m_0+K_0+\tfrac{1}{2}R|\boldsymbol{\pi}|^2+(1/2N)|\mathbf{q}|^2-g\boldsymbol{\sigma}\cdot\mathbf{q}$$
$$+\frac{1}{2}\int d^3x[\pi'^2(\mathbf{x})+|\nabla\phi'(\mathbf{x})|^2+\mu^2\phi'^2(\mathbf{x})]$$
$$+\pi_i\int d^3x\,\pi'(\mathbf{x})\hat{r}_i\rho(\mathbf{x}), \tag{66}$$

where the constants R and N are now given by

$$R=\frac{1}{3}\int d^3x\,\rho^2(\mathbf{x}), \tag{67}$$

$$N=\frac{1}{3}\int d^3x\,\rho(\mathbf{x})\eta(\mathbf{x}), \tag{68}$$

and

$$\eta(\mathbf{x})=N\xi(\mathbf{x})=\frac{1}{4\pi}\int d^3x'(\hat{r}\cdot\hat{r}')\rho(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}'-\mathbf{x}|}}{|\mathbf{x}'-\mathbf{x}|}. \tag{69}$$

The transformation analogous to (21) is

$$\pi''(\mathbf{x})=\pi'(\mathbf{x})-(1/q^2)(\mathbf{q}\times\mathbf{L})\cdot\mathbf{r}[\rho(\mathbf{x})-(1/T)\xi(\mathbf{x})], \tag{70}$$

where

$$T=\frac{1}{3}\int d^3x\,\xi^2(\mathbf{x}). \tag{71}$$

As pointed out by Pauli and Dancoff, this is just the first-order change in $\pi'(\mathbf{x})$ caused by the unitary transformation

$$\pi''(\mathbf{x})=e^{+iV}\pi'(\mathbf{x})e^{-iV}, \tag{72}$$

where

$$V=\frac{1}{Tq^2}(\mathbf{q}\times\mathbf{L})_j\int d^3x\,\phi'(x)\hat{r}_j\xi(x). \tag{73}$$

It is assumed that all matrix elements of V are small (of order g^{-2}) so that the shift in $\pi'(\mathbf{x})$ is the only important term to order g^{-2}. Clearly, the transformation (72) will also change other terms in the Hamiltonian, but these are already smaller than the main ones by a factor of g^{-2}.

The Hamiltonian we will use to solve (57a) is therefore very similar to Eq. (23):

$$H=L^2/2Tq^2+(1/2N)|\mathbf{q}|^2-g\boldsymbol{\sigma}\cdot\mathbf{q}+m_0+K_0$$
$$+\tfrac{1}{2}R|(1/q^2)\mathbf{q}(\mathbf{q}\cdot\boldsymbol{\pi})|^2$$
$$+\frac{1}{2}\int d^3x[\pi''^2(\mathbf{x})+|\nabla\phi'(\mathbf{x})|^2+\mu^2\phi'^2(\mathbf{x})]$$
$$+\frac{1}{q^2}q_j(\mathbf{q}\cdot\boldsymbol{\pi})\int d^3x\,\pi'(\mathbf{x})\hat{r}_j\rho(\mathbf{x}). \tag{74}$$

Everything now proceeds just as in Sec. II, and we see that the proper definition of the states $|\phi_{LM}\rangle$ is that they are states in which q has a definite magnitude and in which $L^2=|\mathbf{q}\times\boldsymbol{\pi}|^2$ and $L_z=(\mathbf{q}\times\boldsymbol{\pi})_z$ have definite eigenvalues. It is straightforward to show that the result of applying the transformation U given by (25)

to the state (55) is[13]

$$\begin{pmatrix} D_{m+1/2}{}^{(1/2)}(\alpha,\beta,0) \\ D_{m-1/2}{}^{(1/2)}(\alpha,\beta,0) \end{pmatrix} \quad \text{for} \quad L=1$$

and (75)

$$\begin{pmatrix} D_{m+1/2}{}^{(1/2)}(\alpha,\beta,0) \\ -D_{m-1/2}{}^{(1/2)}(\alpha,\beta,0) \end{pmatrix} \quad \text{for} \quad L=0.$$

In the limit of strong coupling, the top element of each of these column vectors represents the bound state and the top elements are identical. Thus the wave functions are independent of whether we use $L=0$ or $L=1$, but the energies differ because the eigenvalue of $U^\dagger L^2 U$ is different in the two cases. The lowest energy corresponds to choosing $L=0$, and the energy difference between the two states is proportional to g^{-2} just as the rotational state separations.

One must resist the temptation to see the above doubling of the states as a parity degeneracy. In fact both the $L=0$ and $L=1$ states have the *same* parity because $\mathbf{q}$ is an axial vector. This must be true since $\phi(\mathbf{x})$ must be a pseudoscalar field and therefore must satisfy

$$\phi(\mathbf{x})=-\phi(-\mathbf{x}).$$

Referring to (60), we see that this can be true only if

$$P\mathbf{q}P^{-1}=+\mathbf{q}. \tag{76}$$

But we have written $\mathbf{q}$ in terms of its magnitude q and the spherical angles α, β. For (76) to be satisfied these angles (which are still operators) must behave differently under parity than ordinary spherical angles. In fact, we have

$$P\alpha P^{-1}=\alpha \quad \text{and} \quad P\beta P^{-1}=\beta, \tag{77}$$

which shows immediately that no matter what the angular momentum L of a state, its wave function $Y_{LM}(\alpha,\beta)$ always has a parity of $+1$. Thus the negative-parity states must come from negative-parity wave functions for the core fermion.

Since the two states (75) have the same parity, we can form a linear combination of them to get a state with zero for the lower element and $D_{m+1/2}{}^{(1/2)}(\alpha,\beta,0)$ for the upper element. This is our bound-state wave function.

We have now solved Eq. (57a) and see that the proper normalized eigenstate to use for $|\Phi_{jm}\rangle$ is

$$\langle\alpha\beta|\Phi_{jm}\rangle=[(2j+1)/4\pi]^{1/2}D_{m+1/2}{}^{(j)}(\alpha,\beta,0). \tag{78}$$

In addition, the vector $\mathbf{q}$ has the average magnitude $q_0=gN$. This tells us in turn how to evaluate the expectation value occurring in Eq. (57b). The eigenstate (78) is an eigenstate of the transformed Hamiltonian $U^\dagger H U$ [see Eq. (27)]. Thus the expectation value we want is

$$\langle\Phi_{\frac{1}{2}m}|H|\Phi_{\frac{1}{2}m}\rangle=\langle D_{m+1/2}{}^{(1/2)}|U^\dagger H U|D_{m+1/2}{}^{(1/2)}\rangle, \tag{79}$$

where we keep only the 1-1 matrix element of the 2×2 matrix $U^\dagger H U$.

Equation (57b) now can be written

$$[m_0+p^2/2m_0+E_\phi-g\xi(\mathbf{x}) \times\langle D_{m+1/2}{}^{(1/2)}|(U^\dagger\boldsymbol{\sigma}\cdot\hat{r}U)_{11}\mathbf{q}\cdot\hat{r}|D_{m+1/2}{}^{(1/2)}\rangle]\psi_{00}(x) =E\psi_{00}(x), \tag{80}$$

where we have used (60) for $\phi(\mathbf{x})$ and the fact that the expectation value of $\phi'(\mathbf{x})$ is zero in a state with no free mesons. Also, in (80) we have defined

$$E_\phi=\langle D_{m+1/2}{}^{(1/2)}|H_{0\phi}|D_{m+1/2}{}^{(1/2)}\rangle =\langle D_{m+1/2}{}^{(1/2)}|U^\dagger[\tfrac{1}{2}R|\boldsymbol{\pi}|^2 +(1/2N)|\mathbf{q}|^2]U|D_{m+1/2}{}^{(1/2)}\rangle. \tag{81}$$

Now it is straightforward to show that the 1-1 component of $U^\dagger\boldsymbol{\sigma}\cdot\hat{r}U$ is precisely $\hat{q}\cdot\hat{r}$, so that the last term on the left-hand side of (80) becomes

$$-gq\xi(\mathbf{x})\hat{r}_i\hat{r}_j\langle D_{m+1/2}{}^{(1/2)}|\hat{q}_i\hat{q}_j|D_{m+1/2}{}^{(1/2)}\rangle.$$

We treat $\hat{q}_i\hat{q}_j$ as a tensor operator with both a scalar and tensor (spin-2) component, but only the scalar component has an expectation value in the state specified. In this state, we have

$$\langle D_{m+1/2}{}^{(1/2)}|\hat{q}_i\hat{q}_j|D_{m+1/2}{}^{(1/2)}\rangle=\tfrac{1}{3}\delta_{ij}. \tag{82}$$

Thus Eq. (57b) becomes

$$(-\nabla^2/2m_0-\tfrac{1}{3}gq\xi(\mathbf{x}))\psi_{00}(x) =(E-E_\phi-m_0)\psi_{00}(\mathbf{x}), \tag{83}$$

which is the Schrödinger equation for a particle of mass m_0 in an attractive potential. The potential is the function $\xi(\mathbf{x})$ which is connected to $\rho(\mathbf{x})$ (i.e., the probability density of the fermion) by Eq. (69).

We have now defined the self-consistency problem which must be solved. We can put the coupled equations into dimensionless form by setting

$$x=\mu r, \quad \gamma=m_0/\mu, \quad \epsilon=(1/\mu)(E-E_\phi-m_0). \tag{84}$$

Then the Schrödinger equation for the fermion wave function is

$$[-(1/2\gamma)(d^2/dx^2)-\tfrac{1}{3}g^2\eta(\mathbf{x})]u(x)=\epsilon u(x), \tag{85a}$$

and $\eta(x)$ is determined from

$$\eta(x)=-\frac{1}{4\pi}\int_0^\infty dx'\,u^2(x')j_1(ix_<)h_1{}^{(1)}(ix_>), \tag{85b}$$

which is the dimensionless form of (69) when $\rho(\mathbf{x})$ is spherically symmetric. In the above equations we have set

$$\psi_{00}(\mathbf{x})=[1/\sqrt{(4\pi)}][u(r)/r]. \tag{86}$$

Self-consistent solutions to Eqs. (85) have been

[13] We use the notation of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

found, and we discuss them in detail in the next section.

C. Field Wave Functions

The solutions we want for the field are the eigenstates of the Hamiltonian given by the first six terms of (74). The seventh term is dropped because we will measure our energies relative to the "free-meson" vacuum, and the last term is the coupling between the free fields and the radial oscillations of the bound field. This last term will contribute to the decays of the baryon states, and in second order will cause a mass shift. We assume the second-order effects to be small and we will check this approximation presently.

As we have seen in Eq. (78), the angular wave function for the q variables is a $D_{m+1/2}{}^{(j)}(\alpha\beta 0)$, and the eigenvalue equation is

$$[U^\dagger L^2 U]_{11} D_{m+1/2}{}^{(j)}(\alpha\beta 0) = (j+\tfrac{1}{2})^2 D_{m+1/2}{}^{(j)}(\alpha\beta 0). \quad (87)$$

The energy contributed by the first term, therefore, is

$$E_{\rm rot} = (j+\tfrac{1}{2})^2/2Tg^2N^2, \quad (88)$$

where T is given by (71) and N by (68). We have used

$$q^2 = g^2N^2, \quad (89)$$

which means that q has been set equal to its equilibrium value. This approximation is checked below and will turn out to be not entirely justified.

After the transformation U [Eq. (25)] the second, third, and sixth terms of Eq. (74) become

$$\tfrac{1}{2}R[-d^2/dq^2 - (2/q)(d/dq)] + (1/2N)q^2 - gq. \quad (90)$$

The eigenfunctions and eigenvalues of this part of the Hamiltonian are easily determined. We write

$$\psi(q) = u(q)/q,$$

and complete the square on the last two terms. The eigenvalue equation, then, is

$$[-\tfrac{1}{2}R(d^2/dq'^2) + (1/2N)q'^2]u(q') = (E_q + \tfrac{1}{2}g^2N)u(q'), \quad (91)$$

where $q' = q - gN$.

This is the equation for a one-dimensional harmonic oscillator. In the lowest vibrational state we have

$$E_q + \tfrac{1}{2}g^2N = \tfrac{1}{2}(R/N)^{1/2}, \quad (92)$$

so that the energy contributed to the system by the radial oscillations is

$$E_q = \tfrac{1}{2}(R/N)^{1/2} - \tfrac{1}{2}g^2N, \quad (93)$$

with the first term representing the zero-point radial oscillations and the second the field binding energy [cf. Eq. (30)].

The radial eigenfunction for the ground vibrational state is

$$\psi_0(q') = (\beta/\pi)^{1/4}(e^{-\frac{1}{2}\beta q'^2}/q), \quad (94)$$

where

$$\beta = (RN)^{-1/2}.$$

We can use this to estimate the error we make by neglecting this vibration-rotation coupling in Eq. (88). The first nonvanishing correction to (88) is proportional to $\langle q'^2\rangle$ in the state $\psi_0(q')$:

$$\frac{\Delta E_{\rm rot}}{E_{\rm rot}} = \frac{3\langle q'^2\rangle}{g^2N^2} = \frac{3}{2g^2N^2}(RN)^{1/2}. \quad (95)$$

Finally, we note that Eq. (92) gives us the energy separation between the vibrational states (again neglecting rotation-vibration coupling):

$$\Delta E_{\rm vib} = (R/N)^{1/2}. \quad (96)$$

We will put numbers into these formulas when we present our solutions below.

Our final expression for the energy is now obtained using (88) and (93):

$$E = m_0 + K_0 + \frac{(j+\frac{1}{2})^2}{2g^2N^2T} - \tfrac{1}{2}g^2N + \frac{1}{2}\left(\frac{R}{N}\right)^{1/2}, \quad (97)$$

where m_0 is the bare mass of the elementary fermion, and K_0 is its kinetic energy determined by

$$K_0 = \int d^3x\, \psi_{00}{}^*(\mathbf{x})\left(\frac{-\nabla^2}{2m_0}\right)\psi_{00}(\mathbf{x}). \quad (98)$$

Finally, we remind the reader that $\psi_{00}(\mathbf{x})$ is determined self-consistently by solving the set of coupled Eqs. (85).

IV. SELF-CONSISTENT SOLUTIONS

We will now consider the solutions in three groups: (a) the ground state, (b) rotational states, and (c) S-wave core excitations.

A. Ground State

In the ground state, $j=\frac{1}{2}$ so Eq. (97) becomes

$$E = m_0 + K_0 + 1/2g^2N^2T - \tfrac{1}{2}g^2N + \tfrac{1}{2}(R/N)^{1/2}. \quad (99)$$

We have solved the self-consistency problem for various values of m_0 and g as follows:

(i) Start with a square-well potential $\eta_0(x)$ as shown in Fig. 1. Equation (85a) is solved in this potential with m_0 and g specified. This produces an eigenvalue (ϵ) for the ground state in the square well and a wave function $u(x)$ like the one shown in Fig. 1.

(ii) The $u(x)$ determined from step (i) is inserted into (85b) and a new $\eta(x)$ is generated.

(iii) Equation (85a) is now solved again using the $\eta(x)$ generated in step (ii) and a new eigenvalue ϵ results.

(iv) This procedure is repeated until the eigenvalue ϵ stops changing. A typical final result is shown in Fig. 2.

We have experimented with a number of different starting potentials in step (i) and find that, as long as

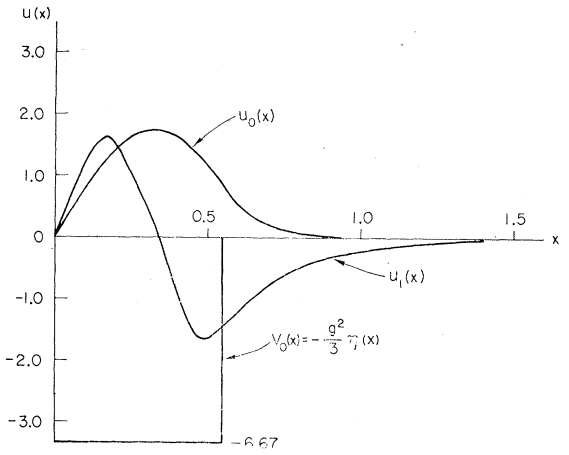

FIG. 1. Square-well potential and solutions for the ground and first excited S-wave states used to start search for self-consistent solutions. The x coordinate in all the figures is measured in pion Compton wavelengths, and the potential is in units of the pion mass.

the starting potential has a bound state, the procedure converges to the same final eigenvalue and wave function no matter what starting potential we use. The rate of convergence will vary, but the final answer is always the same.

This fact is not really surprising. The wave function $u(x)$ is constrained to be zero at both $x=0$ and $x=\infty$ and it has no nodes. Therefore, no matter what the starting potential is, the shape of the initial $u(x)$ will not differ very radically from that of Fig. 1.

We present some of the results of these calculations in Table I. The five contributions to the ground-state energy are listed across the table in the order in which they appear in Eq. (97). We can now comment on the qualitative effects of variations in the basic parameters g and m_0.

As g is increased for a given m_0 the main effects are to increase the binding energy and the radial vibration energy. The kinetic energy of the fermion increases and the rotational energy decreases, but these are rather slowly varying as functions of g. The fact that

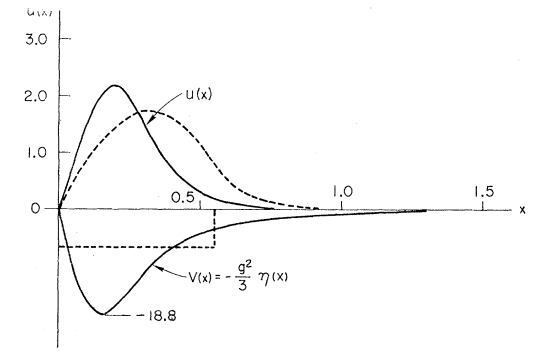

FIG. 2. Self-consistent solution for the ground state when $g=25$ and $m_0=6$. The starting potential and wave function are shown by the dashed lines for comparison.

the rotational energy is relatively insensitive to g makes it impossible in our simple version of the model to make the rotational separations small enough to fit the observed Δ and $N_{5/2}^*$ energies. In Sec. IVB we will give one mechanism which will lower these energies and at this point we can mention another.

With the data of Table I we can check the approximation indicated in Eq. (95). Using values of R, N, and g for a typical case (e.g., $g=25$, $\gamma=6$), we find that

$$\Delta E/E_{\text{rot}} \approx 1,$$

which, of course, violates our assumption that the centrifugal barrier has a negligible effect on the equilibrium value of q. In fact, we have estimated that the centrifugal barrier can increase q_0 by as much as 40 or 50%.

In this paper, we will not pursue this matter further because to do it properly would require an expansion of the self-consistency problem to include a self-consistent determination of q_0. This adds considerably to the complexity of the calculations and will be necessary to obtain reliable numbers. We will present this expanded calculation in a subsequent paper.

The effects of changing m_0 are less dramatic than those of g. Decreasing m_0 (which for a given g tends to spread out the fermion wave function) has the effect of reducing in magnitude all of the terms in (97). The only useful generalizations we can make are that decreasing m_0 decreases the energy of the ground state, and also raises the energy of the N' state relative to the N state (see Sec. IVC). However, both of these effects can also be achieved by increasing g.

Because of the approximations we have made which make our numbers only qualitatively significant, we have not made an extensive search for the best set of values for m_0 and g. A representative set (which fixes the mass of the nucleon at its known value) is $g=25$, $\gamma_0=6$, and we use this set for our illustrations in the following sections.[14] In Fig. 3 we show the fermion probability density $\rho(x)$ and the pion field strength $\xi(x)$ for these values of g and γ.

B. Rotational Excitations

These states are characterized by the angular field wave functions

$$U^\dagger|\chi_n\rangle|\phi_{LM}\rangle = |D_{m+1/2}{}^{(j)}\rangle, \tag{100}$$

where

$$j=\tfrac{3}{2},\tfrac{5}{2},\cdots.$$

This wave functions is, of course, to be multiplied by a wave function for the radial q oscillations and a wave function of the core fermion which must be determined self-consistently. We must now make an approximation for this core wave function.

[14] Note that since we are overestimating the energies of the states by neglecting correlations, we will be overestimating the value of g and/or underestimating the value of m_0 needed to fit the ground state to the nucleon mass.

TABLE I. Detailed breakdown of contributions to ground-state energy and energies of excited nucleon states. The values of $m_0=6$ and $g=25$ are used in the text as a representative choice because the mass of the nucleon is well fitted. All energies are in units of the pion mass, and the last row under the last five columns gives the known (or tentative) masses of the states considered.

m_0	g	K_0	Rot.	Bind.	Vib.	N	$\Delta_{3/2}$	$N^*_{5/2}$	N'	$\Delta'_{3/2}$
4	22	0.57	1.59	− 3.27	3.27	6.17	10.94	18.89	7.06	13.69
6	22	0.77	1.90	− 4.77	4.56	8.46	14.16	23.66	8.68	13.24
8	22	0.98	2.23	− 6.31	5.88	10.78	17.47	28.62	10.64	15.47
10	22	1.17	2.59	− 7.76	7.12	13.12	20.89	29.84	12.61	16.99
12	22	1.34	2.92	− 9.14	8.29	15.41	24.17	38.77	14.76	19.44
4	28	1.19	1.13	− 7.79	4.60	3.15	6.54	12.19	5.60	8.51
6	25	1.17	1.66	− 7.63	5.55	6.74	11.72	20.02	8.28	11.58
6.5	27	1.61	1.61	−10.97	6.73	5.49	10.32	18.37	8.16	11.07
7	27	1.71	1.70	−11.77	7.18	5.83	10.93	19.43	8.60	11.54
						6.72	8.97	11.90	10.65	12.20

Referring to Eq. (82), we recall that only the scalar part of $\hat{q}_i\hat{q}_j$ contributed in the state $j=\frac{1}{2}$. But for $j\geq\frac{3}{2}$ the tensor part also has an expectation value, and this expectation value is not spherically symmetric. As a consequence, Eq. (83) contains an extra potential proportional to the $l=2$ spherical harmonic. This extra term means that the fermion wave function can no longer be purely S-wave and must contain a mixture of $l=2, 4, \cdots$, etc.

At the present stage of our model this is an unsolved problem. We have not yet determined a good way to treat nonspherically symmetric source functions, so that we must neglect the tensor potential. This makes the equation for $\psi_{00}(x)$ independent of j and identical to Eq. (83). In this case we can use the energy formula (97) for the rotational states with only j changing, and the energy separations of the first two rotational excited states from the ground state are

$$\Delta E_{\frac{3}{2}\frac{1}{2}}=3/2g^2N^2T,$$
$$\Delta E_{\frac{5}{2}\frac{1}{2}}=8/2g^2N^2T. \tag{101}$$

All we can say about our neglect of the higher angular components of the fermion wave functions is that their inclusion should lower the above energies relative to the ground state. Thus, the formula (101) will overestimate the rotational level separations, but at this point we cannot say quantitatively by how much the relative separations will be changed.

Equations (101) give the same rotational spectrum as the old static strong-coupling theory. The recent evidence for a $T=\frac{5}{2}$ nucleon isobar[15] has renewed interest in this formula, and we note for completeness the old result that the energy of the $\frac{5}{2}$ level is predicted by (101) to be about 1740 MeV if the Δ mass is fitted to 1235 MeV. It is "seen" at about 1650 MeV, which is not inconsistent with our previous remark that Eqs. (101) will somewhat overestimate the level separations.

We have now discussed two different and probably quite significant corrections whose combined effect should be to reduce the rotational-level separations appreciably. We are unable in the present form of the model to bring these rotational levels down enough so as to make comparison with experiment very meaningful. The rotational band (just the lowest three levels) for the values of g and m_0 chosen to be representative is shown in Fig. 4, and it is clear that our model grossly exaggerates the rotational splittings.

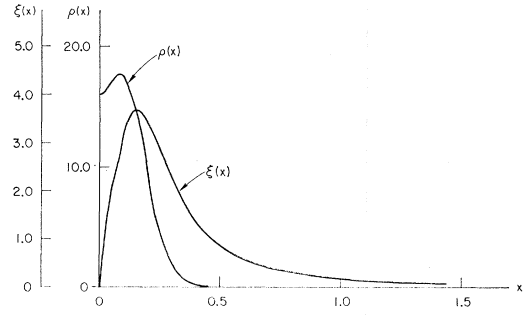

FIG. 3. Fermion probability density $[\rho(x)]$ and pion field amplitude $[\xi(x)]$ for $g=25$ and $m_0=6$. Note that $\xi(x)$ approaches a multiple of (e^{-x}/x) as x gets large.

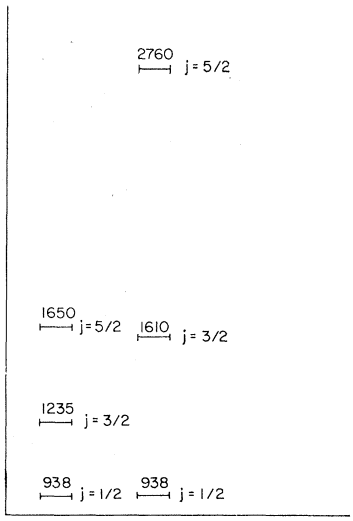

FIG. 4. Experimental values of the first three states in the lowest rotational band are shown on the left and our values on the right. We are off by a factor of about 2 when $g=25$ and $m_0=6$. We have used the energy value for $j=\frac{5}{2}$ of Ref. 15.

[15] A. Benvenuti, E. Marquit, and F. Oppenheimer, Phys. Rev. Letters **22**, 970 (1969). Other references are given in this paper. For contrary indications see J. S. Danburg *et al.*, *ibid.* **23**, 41 (1969).

We can draw some comfort from another set of numbers in Table I. It is clear that for the ranges of m_0 and g which we are considering, the excitation energy for the next radial vibration state is generally quite large compared to the first rotational excitation. The same centrifugal-barrier effect which should lower the rotational levels should raise the vibrational levels, i.e., the "potential well" in which the radial q oscillations take place is made narrower and pushed out to larger values of q by the centrifugal barrier.

Of course, the radial q excitations give another class of states in the model, and we might predict another nucleonlike isobar somewhere above 2 BeV. But the radial q oscillations will have large amplitudes in such a resonance and one might expect it to decay very rapidly, i.e., the approximation of small width becomes unreliable. At this stage of our model we cannot predict whether such a state would be observable or not.

C. Core Excitations

Up to this point our results have not been very different from those obtainable from a static extended-source model. Our model has a potential for a much more complete description of the ground state and rotational levels, but this potential is yet to be exploited. However, it is in the excitations of the core fermion that this model makes its new contribution.

Since we have postponed consideration of nonspherical source functions, the only core excitations we can consider are those with $l=0$. In our self-consistency problem this means searching for self-consistent fermion wave functions which have a single node. Such solutions exist and an example is shown in Figs. 5 and 6.

It is our proposal that this S-wave excitation be identified with the $N_{11}'(1470)$, i.e., the well-known Roper[16] resonance. We are motivated in this by the identity of the quantum numbers with those of the nucleon, and by the experimentally observed fact that the N_{11}' does not decay strongly into the π-N channel. If the N_{11}' is really an S-wave core excitation, and the decay proceeds via emission of a p-wave pion (as the strong-coupling approximation would demand) then we have a selection rule which forbids the decay, i.e., no 0-0 transitions.[17]

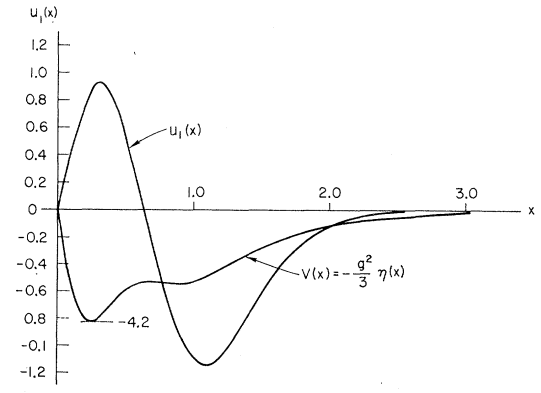

FIG. 5. Self-consistent solution for the first excited core state (N' resonance), with $g=25$ and $m_0=6$. Note that the wave function and potential extend considerably further out than in the ground state.

[16] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

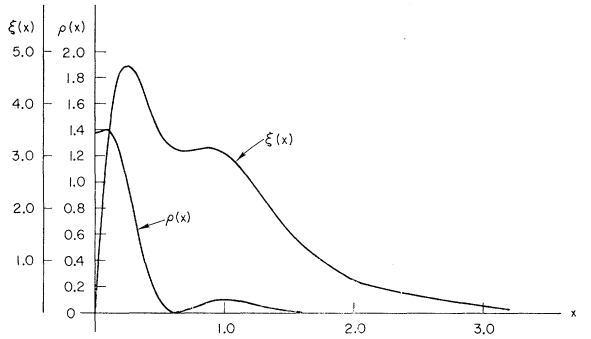

FIG. 6. Probability density of the fermion and pion field intensity for N' resonance when $g=25$ and $m_0=6$. We expect that the N' will be much larger than the nucleon (see Ref. 17).

Of course we also expect that the N_{11}' will have rotational excitations, but we find again that our approximations are too unreliable to allow a prediction of their energies. In particular, the fermion wave function is more spread out in the N' state and the neglect of rotation-vibration coupling is even less valid than in the ground state. We note, however, that a "probable" resonance, which could be called Δ', with the quantum numbers of the Δ is claimed by Lovelace[18] to occur at about 1688 MeV. Our model will have to be improved before we will be able to verify this energy value with any confidence, although our numbers do indicate that the rotational separations of the primed states are smaller than those of the unprimed.

Our chosen set of parameters gives 1140 MeV for the energy of the N' and 1600 MeV for the Δ'. These values are included for completeness only and are quite unreliable.

The possible existence of a second S-wave excitation has not been considered. However, we see no reason to believe that it and even higher radial excitations will not exist.

V. SUMMARY AND DISCUSSION

It has been our intention in this paper to present the basic structure of a strong-coupling model which in-

[17] Another interesting prediction of our model of the Roper resonance is its rather large radius (see Figs. 5 and 6). In diffraction production of the N' the diffraction peak is known to be extremely steep, suggesting a large radius, although this process is complicated by the possible presence of kinematical "Deck-type" effects. [See, e.g., G. Bellettini *et al.*, Phys. Letters 18, 167 (1965).] We wish to thank Dr. Jon Pumplin for reminding us of this result.

[18] C. Lovelace, in *The Fourth International Conference on Elementary Particles, Heidelberg, 1967*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1967).

cludes a point elementary fermion instead of a static, extended-source function. In order to make the paper of reasonable length and make the presentation clear, we have made a number of rather significant approximations. These can be summarized as follows:

(a) Nonrelativistic fermion kinematics. Referring to Table I, we note that the expectation value of the kinetic energy of the fermion is generally a small fraction of its mass. If we take this as an estimate of the validity of our nonrelativistic approximation, we are encouraged to believe that the approximation is not too bad. But this could be misleading since if $\gamma=E/m_0=7/6$, for example, then $\beta=v/c\approx 0.5$. This makes us believe that relativistic kinematic corrections will be at least of the order of 20% and that vacuum polarization effects could also be non-negligible.

(b) "Independent-particle" states. We have seen that the use of product wave functions destroys translation invariance and must overestimate the energies of the states. To include ground-state correlations requires a much more extensive calculation and also a technique to handle $l\neq 0$ fermion states. This remains for future work.

(c) Neglect of configuration mixing in states with $j\geq\frac{3}{2}$. This has been discussed in Sec. IV B.

(d) Neglect of vibration-rotation coupling in the radial q oscillations. As we have noted in Sec. IV A this requires a more elaborate self-consistency problem and will be considered in the next paper.

(e) Assumption of small decay widths and consequently small second-order level shifts. We have checked this by calculating the width of the Δ in the case when $\gamma_0=6$ and $g=25$. The width is very large, but we can account for most of this by noting that the phase space for the decay goes as the cube of the energy of the emitted pion. Since we have overestimated the energy separation by so much, this phase-space factor multiplies the error enormously. We also find that the width is proportional to $(gN)^{-2}$ as advertised, and that the same effect which we expect to reduce the rotational band separations will also reduce the widths.

(f) Neglect of nonlinear pion-field interactions (e.g., $\lambda\phi^4$).

The above list of approximations leads us to distrust any numerical prediction so far obtained. But it should be noted that none of them shows any prospect of qualitatively changing the spectrum of states (with the possible exception of a strong $\lambda\phi^4$ term in the Hamiltonian). We can then count the following as qualitative successes of the model:

(1) Prediction of the Δ and $N_{5/2}{}^*$ resonances. These are of course old predictions, but recent results[15] have generated new interest in them. The problems that a $T=\frac{5}{2}$ nucleon isobar causes for some other models are quite serious.

(2) Physical interpretation of the hard core in nuclear forces. Hard cores in potentials have been traditionally the result of the Pauli exclusion principle. Our model sustains this tradition, and attributes the hard core to the resistance of two core fermion wave functions to overlap.

(3) Prediction of the Roper resonance. The position and decay properties of the $N'(1470)$ are qualitatively accounted for in this model. As a bonus (or penalty) we also get a Δ' which may (or may not) actually exist.

(4) Negative-parity resonances and "Regge recurrences." These occur naturally in our model and their properties will be predictable once the technical problem of the nonspherically symmetric source can be surmounted.

These qualitative successes, we feel, justify the further calculational effort which will be necessary to get numbers in which we might have some confidence.

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