Branch Cuts in the Balázs Method*

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The Balázs method of parametrizing distant singularities is discussed for the case when the near part of the left-hand cut is retained explicitly. A closed-form solution of the N/D equations is derived which allows the method to be applied in a systematic manner. The solution is approximate, but can be made extremely accurate if the cut retained is not too large. The method is applied to the case of Yukawa scattering where it is shown that the addition of the cut substantially reduces the ambiguities associated with a pure-pole approach and leads to solutions in good agreement with those obtained from the Schrödinger equation. The case $l=0$ requires separate treatment, and some modified treatments are discussed which improve the results.

I. INTRODUCTION

In using the N/D method,[1] it is often convenient to treat the near and distant parts of the left-hand cut in different ways. To calculate the near part of the cut, we can use the Froissart-Gribov expression for the partial-wave amplitude[2]

$$A_l(\nu)=\frac{1}{2\pi\nu}\int_4^\infty dt\, A_t(s,t)Q_l(1+t/2\nu)\,, \tag{1}$$

where

$$s=4(\nu+1)\,. \tag{2}$$

For simplicity, we will always write expressions for the equal-mass case, set $m=1$, and ignore u-channel effects.

The left-hand cut begins at $s=0$ ($\nu=-1$), and comes from the cut of $Q_l(1+t/2\nu)$. Using on the cut of Q_l

$$\mathrm{Im}Q_l(1+t/2\nu)=\tfrac{1}{2}\pi P_l(1+t/2\nu) \tag{3}$$

for $-1\leq 1+t/2\nu\leq +1$, we have

$$\mathrm{Im}A_l(\nu)=\frac{1}{4\nu}\int_4^{-4\nu} dt\, A_t(s,t)P_l(1+t/2\nu)\,, \tag{4}$$

where the region of integration is shown in Fig. 1. This expression holds out to the beginning of the double spectral function A_{tu} at $s=-32$ when A_t becomes complex. To obtain $A_t(s,t)$, we can expand the full scattering amplitude in terms of the t-channel partial waves

$$A(s,t)=\sum_{l'}(2l'+1)A_{l'}(t)P_{l'}(\cos\theta_t)\,, \tag{5}$$

$$A_t(s,t)=\sum_{l'}(2l'+1)\,\mathrm{Im}A_{l'}(t)P_{l'}(\cos\theta_t)\,, \tag{6}$$

where θ_t is the center-of-mass scattering angle for the t channel,

$$\cos\theta_t=1+2s/(t-4)\,. \tag{7}$$

The series for A_t in (6) converges for $-32<s<4$ and can therefore be used to continue A_t outside the physical region of the t channel to the region of integration in (4). In this way, then, $\mathrm{Im}A_l(\nu)$ can be calculated on the near part of the left-hand cut if the crossed-channel partial waves are known.

The method fails, however, for $s<-32$, since then the expansion for A_t must be used beyond its region of convergence.

Some time ago, a method was suggested by Balázs of estimating the contributions from these distant singularities.[3] The idea is that $A_l(s,t)$ can also be calculated in the region $0<s<4$ ($-1<\nu<0$) between the cuts; the series (6) converges there, and if desired, Regge pole theory can be used for large t. The amplitude itself, $A_l(\nu)$, can thus be obtained from (1). Then, if the distant parts of the left-hand cut are replaced by poles, their contributions can be determined by demanding that the *output* of the N/D calculation reproduce this amplitude as well as possible between the cuts.

Unfortunately, although the procedure is clearly defined, it is not easy to apply. The problem is that the matching conditions which determine the pole parameters are on the output amplitude, and the only way to satisfy them is by a trial and error method, solving the N/D integral equations numerically while varying the parameters until the conditions are satisfied. To circumvent this difficulty, what Balázs did in his actual calculations was to replace the entire left-hand cut by poles, thus reducing the equations to algebraic ones and making the problem numerically tractable.[3,4] However, using only a few poles, it appears to be quite

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] M. Froissart, in the Proceedings of the La Jolla Conference on Theoretical Physics, 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1961) [English transl.: Soviet Phys.—JETP **14**, 478 (1962)].
[3] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962); **129**, 872 (1963); **132**, 867 (1963); **134**, AB1(E) (1964).
[4] See also S. K. Bose and M. Der Sarkissian, Nuovo Cimento **30**, 878 (1963); V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963); P. Narayanaswamy and L. K. Pande, *ibid.*, **136**, B1760 (1964); M. Der Sarkissian, Nuovo Cimento **30**, 894 (1963); J. C. Pati and K. V. Vasavada, Phys. Rev. **144**, 1270 (1966); K. C. Gupta, R. P. Saxena, and V. S. Mathur, *ibid.*, **141**, 1479 (1966), for other applications of the pure pole method. In some of these, there are "short cuts" which are also replaced by poles and their residues calculated independently.

difficult to reproduce $A_l(\nu)$ over the entire gap; this leads to an ambiguity of the procedure, and the results can depend considerably on the details of how the matching is carried out.[5-8]

In this paper, we wish to reexamine the original proposal and look at the effect of keeping explicitly the near part of the left-hand cut.[9] One motivation for this is the hope that the matching difficulties referred to above will be alleviated by keeping the cut. Another is provided by the observation that the information contained in the cut and in the amplitude is not quite the same. For example, consider the effects of the t-channel process in the region $4<t<\bar{t}$. In (1), the amplitude is only partly determined by contributions from this region since the integral goes to infinity. In (4), on the other hand, the integral cuts off at $t=-4\nu$ ($u=0$) so that, for $s<4-\bar{t}$, A_l is completely determined. Thus, the low-mass t-channel singularities give more precise information about $\mathrm{Im}A_l$ than they do about the amplitude itself, and this additional information can be used if part of the cut is retained.

In Sec. II, an approximate closed form solution of the N/D equations is developed which can include an arbitrary finite section of the left-hand cut. The amplitude is given explicitly as a function of the distant-pole parameters so that the residues can be chosen *a priori* to make the output satisfy the matching conditions. In Sec. III, the method is applied to Yukawa scattering where we are basically interested in two questions:

(1) When part of the cut is kept, do the matching ambiguities disappear?

(2) Does the matching procedure really provide an effective method of parametrizing the contributions from the distant singularities?

For $l\neq 0$, the answer seems to be yes to both of these questions. For $l=0$, the straightforward method breaks down, and some alternative procedures are discussed which gives better results. In Sec. IV, our summary and final conclusions are given.

II. APPROXIMATE SOLUTIONS INCLUDING PART OF LEFT-HAND CUT

Here we give an approximate treatment of the N/D equations which has a closed-form solutions and retains

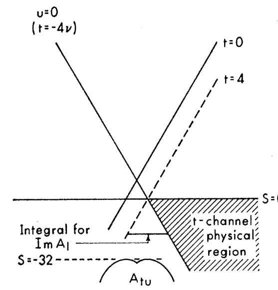

FIG. 1. Contributions of the t channel to the near left-hand cut.

a part of the left-hand cut explicitly. The treatment is made in the spirit of Pagels's method,[10] but the accuracy is improved by the fact that we will need our approximations only on a finite interval.

We assume that $N(\nu)$ and $D(\nu)$ satisfy once-subtracted dispersion relations with a common subtraction point ν_0 so that[11] $D(\nu_0)=1$, $N(\nu_0)=A(\nu_0)$. $D(\nu)$ is assumed to have the right-hand cut of $A(\nu)$ and $N(\nu)$ to have a segment of the left-hand cut $\bar{L}$ plus poles at $\nu=-a_j^2$ as shown in Fig. 2. The equations for N and D are then[1]

$$N(\nu)=A(\nu_0)+\frac{(\nu-\nu_0)}{\pi}\int_{\bar{L}}\frac{\mathrm{Im}N(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}-(\nu-\nu_0)\sum_j\frac{\alpha_j}{(\nu+a_j^2)(\nu_0+a_j^2)},\quad(8)$$

$$D(\nu)=1-\frac{(\nu-\nu_0)}{\pi}\int_0^\infty\frac{d\nu''\rho(\nu'')N(\nu'')}{(\nu''-\nu)(\nu''-\nu_0)},\quad(9)$$

where

$$\rho(\nu)=[\nu/(\nu+1)]^{1/2}.\quad(10)$$

Substituting (8) into (9) gives

$$D(\nu)=1-\frac{(\nu-\nu_0)}{\pi}A(\nu_0)\int_0^\infty\frac{d\nu''\rho(\nu'')}{(\nu''-\nu)(\nu''-\nu_0)}+\frac{\nu-\nu_0}{\pi}\sum_j\frac{\alpha_j}{\nu_0+a_j^2}\int_0^\infty\frac{d\nu''\rho(\nu'')}{(\nu''-\nu)(\nu''+a_j^2)}+\frac{(\nu-\nu_0)}{\pi}\int_{\bar{L}}\frac{\mathrm{Im}N(\nu')d\nu'}{\nu'-\nu_0}\frac{1}{\pi}\int_0^\infty\frac{d\nu''\rho(\nu'')}{(\nu''-\nu)(\nu''-\nu')}.\quad(11)$$

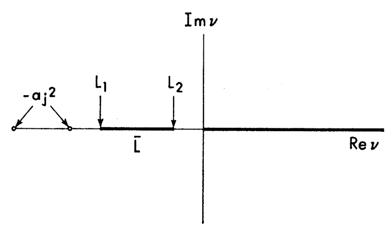

FIG. 2. Analytic structure of $A_l(\nu)$.

[5] M. L. Mehta and P. K. Srivastava, Phys. Rev. **137**, B423 (1965).

[6] M. R. Williamson and A. E. Everett, Phys. Rev. **147**, 1074 (1966).

[7] A. H. Bond, Phys. Rev. **147**, 1058 (1966).

[8] A. F. Antippa and A. E. Everett, Phys. Rev., **178**, 2443 (1969).

[9] Several calculations have been made keeping "short cuts" explicitly: J. C. Pati, Phys. Rev. **134**, B387 (1964); G. L. Kane, *ibid.*, **135**, B843 (1964); S. R. Choudhury and L. K. Pande, *ibid.* **135**, B1027 (1964); V. Singh and B. M. Udgaonkar, *ibid* **128**, 1820 (1962). In none of these papers, however, is there any analysis of the stability and matching problems encountered in the pure pole method.

[10] H. Pagels, Phys. Rev. **140**, B1599 (1965).

[11] In this section, the subscript l will be omitted throughout.

The integrals in ν'' can be carried out to give[12]

$$D(\nu)=1-A(\nu_0)[iG(\nu)-iG(\nu_0)]+(\nu-\nu_0)\times\sum_j\frac{\alpha_j[iG(\nu)-iG(-a_j^2)]}{(\nu_0+a_j^2)(\nu+a_j^2)}-(\nu-\nu_0)\times\frac{1}{\pi}\int_{\bar{L}}\frac{\mathrm{Im}N(\nu')[iG(\nu)-iG(\nu')]d\nu'}{(\nu'-\nu)(\nu'-\nu_0)},\quad(12)$$

where $G(\nu)$ is the function having only the right-hand cut of $\rho(\nu)$. Explicit forms for real ν are

$$\begin{aligned}\pi iG(\nu)&=\rho(\nu)\ln\frac{\rho(\nu)-1}{\rho(\nu)+1}\quad\text{for}\quad\nu\leqslant-1,\\&=-\bar{\rho}(\nu)[\pi-2\arctan\bar{\rho}(\nu)]\quad\text{for}\quad-1<\nu<0,\\&=\rho(\nu)\left(\ln\frac{1-\rho(\nu)}{1+\rho(\nu)}\pm i\pi\right)\quad\text{for}\quad\nu\geqslant0,\end{aligned}\quad(13)$$

where $\bar{\rho}(\nu)=[-\nu/(\nu+1)]^{1/2}$, and the $\pm i\pi$ refer to the top and bottom, respectively, of the cut for $\nu\geqslant0$. In the first term of the integral in (12), $iG(\nu)$ can be taken outside and the expression evaluated by using (8):

$$D(\nu)=1-iG(\nu)N(\nu)+iG(\nu_0)A(\nu_0)-(\nu-\nu_0)\times\sum\frac{\alpha_j iG(-a_j^2)}{(\nu+a_j^2)(\nu_0+a_j^2)}+\frac{(\nu-\nu_0)}{\pi}\times\int_{\bar{L}}\frac{iG(\nu')\,\mathrm{Im}N(\nu')d\nu'}{(\nu'-\nu_0)(\nu'-\nu)}.\quad(14)$$

This expression for $D(\nu)$ is exact. We note further that the integral in (14) could be evaluated exactly if $G(\nu)$ were meromorphic in ν instead of having the cut $0<\nu<\infty$. The approximation then consists of replacing the known function $G(\nu)$ by a meromorphic function $\tilde{G}(\nu)$ such that $G(\nu)\approx\tilde{G}(\nu)$ on $\bar{L}$. Since the approximation need hold only on a segment, this is easy to arrange as discussed in the Appendix. Let the poles of $\tilde{G}(\nu)$ be at $\nu=\nu_n$ and the corresponding residues of $i\tilde{G}(\nu_n)$ be β_n. Then a contour integral about the cut gives

$$\frac{1}{\pi}\int_L\frac{i\tilde{G}(\nu')\,\mathrm{Im}N(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}=\frac{i\tilde{G}(\nu)N(\nu)-i\tilde{G}(\nu_0)N(\nu_0)}{\nu-\nu_0}+\sum_n\frac{\beta_nN(\nu_n)}{(\nu-\nu_n)(\nu_0-\nu_n)}+\sum_j\frac{\alpha_ji\tilde{G}(-a_j^2)}{(\nu+a_j^2)(\nu_0+a_j^2)},\quad(15)$$

and finally,

$$D(\nu)\approx1-N(\nu)\Delta G(\nu)+A(\nu_0)\Delta G(\nu_0)+(\nu-\nu_0)\times\sum_n\frac{\beta_nN(\nu_n)}{(\nu_0-\nu_n)(\nu-\nu_n)}-(\nu-\nu_0)\sum_j\frac{\bar{\alpha}_j\Delta G(-a_j^2)}{\nu+a_j^2},\quad(16)$$

where

$$\Delta G(\nu)=iG(\nu)-i\tilde{G}(\nu),\quad(17)$$

$$\bar{\alpha}_j=\alpha_j/(\nu_0+a_j^2).\quad(18)$$

In (8), we have $\mathrm{Im}N(\nu)=D(\nu)\,\mathrm{Im}A(\nu)$, and using (16) for D with $\Delta G=0$ on the cut gives

$$N(\nu)\approx A(\nu_0)+(\nu-\nu_0)\Bigg[[1+A(\nu_0)\Delta G(\nu_0)]I_0(\nu)+\sum_n\frac{\beta_nN(\nu_n)I_n(\nu)}{\nu_0-\nu_n}-\sum_j\bar{\alpha}_j\left(\frac{1}{\nu+a_j^2}+\Delta G(-a_j^2)I_j(\nu)\right)\Bigg],\quad(19)$$

where

$$I_0(\nu)=\frac{1}{\pi}\int_{\bar{L}}\frac{\mathrm{Im}A(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)},\quad(20)$$

$$I_n(\nu)=\frac{1}{\pi}\int_L\frac{\mathrm{Im}A(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_n)},\quad(21)$$

$$I_j(\nu)=\frac{1}{\pi}\int_{\bar{L}}\frac{\mathrm{Im}A(\nu')d\nu'}{(\nu'-\nu)(\nu'+a_j^2)}.\quad(22)$$

The constants $N(\nu_n)$ can be determined by setting $\nu=\nu_n$ in (19), and then $N(\nu)$ and $D(\nu)$ are given explicitly in terms of the pole parameters a_j^2 and α_j. The functions I_0, I_n, and I_j can be evaluated numerically for any given $\mathrm{Im}A(\nu)$, or, in most cases, determined analytically by means of the formula

$$\int_{z_1}^{z_2}d_zf(z)=\sum\mathrm{res}\left(f(z)\ln\frac{z-z_2}{z-z_1}\right),\quad(23)$$

which is valid whenever $f(z)$ is meromorphic in z and $f(z)\to0$ ($|z|\to\infty$); the sum is over the residues of the poles of $f(z)\ln[(z-z_2)/(z-z_1)]$.

As can be seen in (16), $D(\nu)$ also carries the left-hand cut of $N(\nu)$. The discontinuity, however, is proportional to $\Delta G(\nu)$ which can be made quite small for reasonable cut segments. Some examples are discussed in the Appendix.

III. YUKAWA SCATTERING

Yukawa scattering, with contributions on the entire left-hand cut calculated from Born terms, has been previously discussed by other authors.[13,14] Our method. of course, treats the distant parts of the cut in an entirely different way, and it is of interest to see how well the method works in this rather well understood situation. We review here only the results necessary

[12] J. Dilley, J. Math. Phys. 8, 2022 (1967).

[13] M. Luming, Phys. Rev. **136**, B1120 (1964).

[14] P. D. B. Collins and R. C. Johnson, Phys. Rev. **169**, 1222 (1968).

for our purpose. The potential is

$$V(\nu) = -ge^{-r}/r \tag{24}$$

and has the analytic structure shown in Fig. 3 (ν is the square of the momentum, with units $\hbar=2m=1$; the potential has been taken with unit range). The left-hand cuts are generated by successive Born terms so that the cut is given exactly by the first Born approximation $\mathrm{Im}A_l^{\mathrm{I}}$ for $-1<\nu\leqslant-\frac{1}{4}$, by $\mathrm{Im}A_l^{\mathrm{I}}+\mathrm{Im}A_l^{\mathrm{II}}$ for $-9/4<\nu\leqslant-1$, etc. On the other hand, all orders of Born approximation contribute to the amplitude in the gap $-\frac{1}{4}<\nu<0$. For reference, we list here the first and second approximations:

$$A_l^{\mathrm{I}}(\nu) = (g/2\nu)Q_l(1+\tfrac{1}{2}\nu), \tag{25}$$

$$\mathrm{Im}A_l^{\mathrm{I}}(\nu) = (\pi g/4\nu)P_l(1+\tfrac{1}{2}\nu) \quad (\nu\leqslant-\tfrac{1}{4}), \tag{26}$$

$$A_l^{\mathrm{II}}(\nu) = \frac{g^2}{4\nu\sqrt{(-\nu)}}\int_4^\infty \frac{dt\, Q_l(1+t/2\nu)}{[t(t-t_0)]^{1/2}}, \tag{27}$$

$$\mathrm{Im}A_l^{\mathrm{II}}(\nu) = \frac{\pi g^2}{8\nu\sqrt{(-\nu)}}\int_4^{-4\nu} \frac{dt\, P_l(1+t/2\nu)}{[t(t-t_0)]^{1/2}} \quad (\nu\leqslant-1), \tag{28}$$

where

$$t_0 = 4(1+\tfrac{1}{4}\nu). \tag{29}$$

The N/D equations, including some segment $\bar{L}$ of the left-hand cut (see Fig. 3), can be treated as in Sec. II except that no subtraction is made for N, and $G(\nu)$ is replaced by $\sqrt{\nu}$. For convenience, the subtraction point ν_0 of D is taken at the beginning of the cut, $\nu_0=-\frac{1}{4}$, and $\tilde{G}(\nu_0)$ chosen such that $\tilde{G}(\nu_0)=\sqrt{\nu_0}$, so that $\Delta G(\nu_0)=0$. The choice of $\tilde{G}(\nu)$ and the resulting accuracy of the solutions is discussed in the Appendix.

The basic idea, as indicated in the Introduction, is to try to make the N/D output reproduce the amplitude, in this case given by the Born terms, as closely as possible in the region $-0.25<\nu\leq 0$. Our procedure is to use two poles with the residues chosen so that the N/D output agrees with $A_l(\nu)$ at two points, $\nu=-0.2$ and 0. The pole positions are then varied so as to optimize the agreement throughout the region, with the restriction that the first pole must be somewhere near the point L_1 which is the end of the cut being explicitly treated.[15]

$l\neq 0$

The cases with $l\neq 0$ are all characterized by the fact that the first Born term A_l^{I} gives a reasonable approximation to the actual amplitude. The threshold behavior is correct, $A_l^{\mathrm{I}}(\nu)\to\nu^l$ $(\nu\to 0)$, and the first Born term also dominates near the left-hand cut, so that it gives at least a qualitatively correct picture

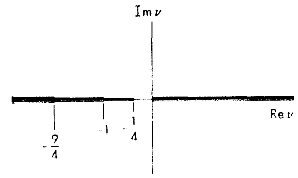

FIG. 3. Analytic structure for Yukawa scattering.

throughout the gap. The behavior of the first two Born approximations are indicated in Table I for the cases $g=1$ and $g=3$, $l=1$. For $g=1$, even the contribution of the second Born term is very small and the amplitude is very close to the correct one. For $g=3$, the relative importance of higher-order terms is considerably larger, and the accuracy of the amplitudes A_1^{I} or $A_1^{\mathrm{I}}+A_1^{\mathrm{II}}$ correspondingly less.

The best fits to the amplitudes are given in Table I for three cases: (a) no cut (two poles only), (b) a cut $-0.5\leqslant\nu\leqslant-0.25$ (the cut is included to where $\mathrm{Im}A_l^{\mathrm{I}}(\nu)$ changes sign), plus two poles, and (c) a cut $-1\leqslant\nu\leqslant-0.25$ (the cut is taken to the point where the second Born term starts to contribute), plus two poles.

As expected, the pure pole solutions are unable to reproduce the amplitudes with any great accuracy, particularly in the region near the left-hand cut. In this situation, of course, what constitutes a "best fit" becomes somewhat a matter of taste, and we can only say that the ones given seem as good as any. This type of ambiguity is the one referred to in the introduction.

The solutions with cuts do much better, and give good fits to the amplitude in all cases except $g=3$ in first Born approximation. Here, it is seen that there is a small discrepancy with the cut solution $-0.5\leqslant\nu\leqslant-0.25$ (the discrepancy is real and cannot be removed by adjusting the pole positions). Furthermore, the discrepancy increases with the length of the included cut. This phenomena seems very strange at first sight, but it must be remembered that the first Born cut and unitarity are not equivalent to the first Born amplitude [for example, the entire first Born cut plus unitarity implies $A_1(0)\neq 0$, whereas $A_1^{\mathrm{I}}(\nu)$ vanishes at threshold]. The correct statement is that all of the Born cuts plus unitarity is equivalent to the sum of the Born amplitudes in regions where the Born series converges.[16] Thus, for cases like $g=3$ where there are noticeable corrections from higher-order Born terms to the amplitude, the left-hand cut and unitarity can become inconsistent with the approximate amplitude being used. In this situation, as more of the left-hand cut is retained, the poles have to generate ficticious forces in such a way as to cancel effectively some of the cut contribution, and if too much of the cut is used, the poles may be unable to bring about agreement. In

[15] This procedure differs somewhat from that proposed by Balázs who gave an *a priori* prescription for determining the pole positions from the kernel of the integral equation for N.

[16] For the case $g=3$, there is an s-wave bound state, and the Born series actually does not converge above the bound-state energy. However, for $l\neq 0$, the first few Born terms still give an approximate description of the actual amplitude.

TABLE I. Amplitude matching for $l=1$.

					$g=1,\ l=1$						
ν	−0.2499	−0.23	−0.2	−0.175	−0.15	−0.125	−0.1	−0.075	−0.05	−0.025	0
$A_1{}^{\mathrm{I}}(\nu)$	−5.8326	−1.0489	−0.5177	−0.3371	−0.2300	−0.1589	−0.1083	−0.0705	−0.0415	−0.0185	0
a Pure pole solution; poles at $\nu=-0.28,\ -1.5$	−1.7829	0.9734	−0.5177	−0.3391	−0.2308	−0.1587	−0.1076	−0.0698	−0.0408	−0.0182	0
b Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -1.7$	−5.8326	−1.0489	−0.5177	−0.3371	−0.2300	−0.1589	−0.1083	−0.0706	−0.0415	−0.0185	0
c Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.5,\ -5.2$	−5.8325	−1.0488	−0.5177	−0.3371	−0.2301	−0.1590	−0.1084	−0.0707	−0.0416	−0.0186	0
$A_1{}^{\mathrm{I}}(\nu)+A_1{}^{\mathrm{II}}(\nu)$	−5.8389	−1.0546	−0.5227	−0.3414	−0.2338	−0.1620	−0.1108	−0.0724	−0.0428	−0.0192	0
a Pure pole solution; poles at $\nu=-0.28,\ -1.7$	−1.7901	−0.9795	−0.5227	−0.3433	−0.2343	−0.1615	−0.1097	−0.0713	−0.0418	−0.0186	0
b Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -2.2$	−5.8388	−1.0546	−0.5227	−0.3414	−0.2337	−0.1620	−0.1107	−0.0724	−0.0427	−0.0191	0
c Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.5,\ -5.2$	−5.8390	−1.0547	−0.5227	−0.3414	−0.2337	−0.1620	−0.1107	−0.0724	−0.0427	−0.0192	0
					$g=3,\ l=1$						
ν	−0.2499	−0.23	−0.2	−0.175	−0.15	−0.125	−0.1	−0.075	−0.05	−0.025	0
$A_1{}^{\mathrm{I}}(\nu)$	−17.4979	−3.1467	−1.5531	−1.0112	−0.6901	−0.4766	−0.3248	−0.2116	−0.1244	−0.0555	0
a Pure pole solution; poles at $\nu=-0.28,\ -1.3$	−5.3610	−2.9238	−1.5531	−1.0168	−0.6918	−0.4757	−0.3227	−0.2095	−0.1229	−0.0548	0
b Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -1.25$	−17.4998	−3.1475	−1.5531	−1.0109	−0.6897	−0.4765	−0.3248	−0.2118	−0.1248	−0.0559	0
c Cut $-1\leqslant\nu\leqslant-0.25$ poles at $\nu=-1.5,\ -2.5$	−17.4964	−3.1458	−1.5531	−1.0119	−0.6912	−0.4782	−0.3265	−0.2133	−0.1259	−0.0565	0
$A_1{}^{\mathrm{I}}(\nu)+A_1{}^{\mathrm{II}}(\nu)$	−17.5543	−3.1985	−1.5980	−1.0505	−0.7238	−0.5049	−0.3475	−0.2288	−0.1361	−0.0615	0
a Pure pole solution; poles at $\nu=-0.28,\ -2$	−5.4043	−2.9719	−1.5980	−1.0565	−0.7258	−0.5038	−0.3449	−0.2260	−0.1338	−0.0602	0
b Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -2.4$	−17.5545	−3.1985	−1.5980	−1.0505	−0.7238	−0.5049	−0.3476	−0.2289	−0.1362	−0.0616	0
c Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.25,\ -2.5$	−17.5551	−3.1988	−1.5980	−1.0504	−0.7236	−0.5047	−0.3474	−0.2288	−0.1361	−0.0616	0

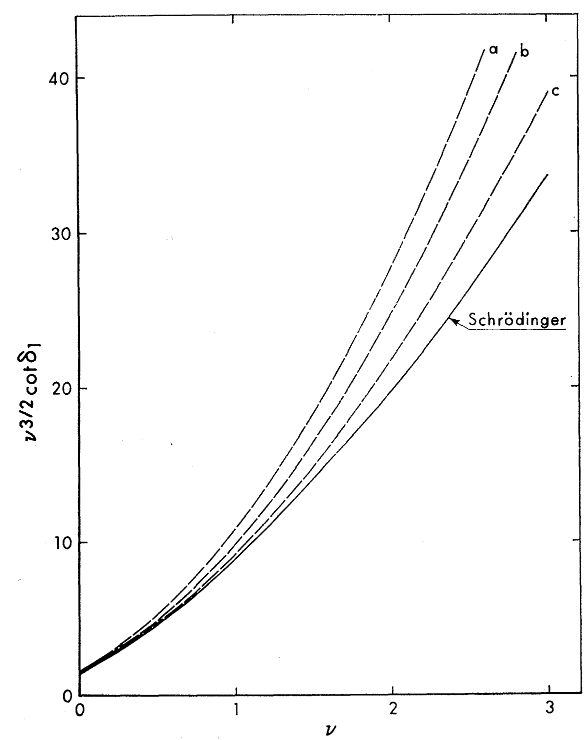

FIG. 4. Effective-range plots $g=1$, $l=1$, and first Born inputs; the labels correspond to those of Table I.

the present instance, the addition of the second Born contribution makes it possible for both cut solutions to be brought into agreement with the amplitude.

The outputs to the various calculations are shown in Figs. 4–7 where they are compared with the solution of the Schrödinger equation. For the case $g=1$, the cut solutions are clearly better than the pure pole solution, with the longer cut $-1\leqslant\nu\leqslant-0.25$ giving the best results. For the case $g=3$, the situation is a bit more confused because of the comparative poorness of the amplitude being matched. The pure pole solutions are surprisingly good, but the significance of this is diminished by the ambiguity of choosing a best fit. In both cases, the results improve when the amplitude used for matching is improved by the addition of the second Born contribution, a situation indicating that the use of the amplitude in this way to parametrize the contributions of the distant singularities has some validity.

Except for the case $g=3$ in first Born approximation, the cut solutions can reproduce the amplitude being used with remarkable accuracy. The pure pole solutions, on the other hand, always have an ambiguity associated with the way in which the "best fit" is chosen. One way to measure this ambiguity is to recalculate the amplitude with a "local fit" by requiring the N/D output match $A_1(\nu)$ and $(d/d\nu)A_1(\nu)$

TABLE II. Amplitude matching for $l=0$.

	$g=1,\ l=0$									
ν	−0.2499	−0.249	−0.245	−0.24	−0.23	−0.22	−0.2	−0.15	−0.1	−0.05
$A_0{}^{\text{I}}(\nu)$	7.8272	5.5436	3.9919	3.3530	2.7454	2.4094	2.0118	1.5272	1.2771	1.1157
a_{I} Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.65,\ -1.1$	7.8272	5.5436	3.9919	3.3530	2.7455	2.4099	2.0136	1.5364	1.3021	1.1744
b_{I} Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=1.25,\ -2$	7.8272	5.5436	3.9918	3.3530	2.7454	2.4097	2.0131	1.5345	1.2984	1.1679
$A_0{}^{\text{I}}(\nu)+A_0{}^{\text{II}}(\nu)$	8.0949	5.8115	4.2605	3.6226	3.0169	2.6831	2.2904	1.8202	1.5912	1.4655
a_{II} Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -1.2$	8.0949	5.8115	4.2605	3.6226	3.0169	2.6832	2.2908	1.8250	1.6075	1.5107
b_{II} Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.25,\ -2$	8.0949	5.8115	4.2605	3.6225	3.0169	2.6831	2.2908	1.8254	1.6084	1.5123
	$g=3,\ l=0$									
ν	−0.2499	−0.249	−0.245	−0.24	−0.23	−0.22	−0.2	−0.15	−0.1	−0.05
$A_0{}^{\text{I}}(\nu)$	23.4815	16.6309	11.9756	10.0590	8.2361	7.2282	6.035	4.582	3.381	3.347
a_{I} Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.65,\ -3$	23.4815	16.6309	11.9755	10.0591	8.2376	7.2327	6.051	4.659	4.049	3.898
b_{I} Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.25,\ -2$	23.4815	16.6309	11.9753	10.0585	8.2359	7.2295	6.044	4.636	3.999	3.801
$A_0{}^{\text{I}}(\nu)+A_0{}^{\text{II}}(\nu)$	25.8912	19.0421	14.3934	12.4853	10.6802	9.6911	8.543	7.219	6.658	6.495
a_{II} Cut $-0.5\leqslant\nu\leqslant-0.25$; poles at $\nu=-0.6,\ -1.1$	25.8912	19.0421	14.3936	12.4868	10.6878	9.7099	8.597	7.501	7.526	9.259
b_{II} Cut $-1\leqslant\nu\leqslant-0.25$; poles at $\nu=-1.5,\ -4$	25.8912	19.0421	14.3932	12.4853	10.6824	9.6984	8.567	7.381	7.212	8.362
b_{II}' Cut $-1<\nu<-0.25$; poles at $\nu=-1.1,\ -1.5$	25.8912	19.0421	14.3933	12.4856	10.6839	9.7018	8.577	7.423	7.329	8.704

at some point ν_1, and then vary ν_1 to see how the phase shifts change. The results for the first pure pole solution in Table I ($g=1$ in first Born approximation), where ν_1 has been varied between -0.2 and -0.05, are shown in Fig. 8, and indicate the typical matching

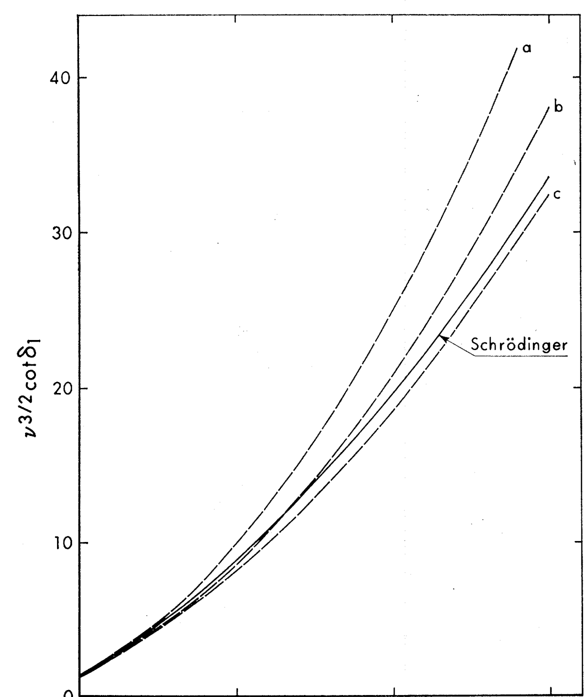

fig. 5

FIG. 5. Effective-range plots $g=1$, $l=1$, and second Born inputs; the labels correspond to those of Table I.

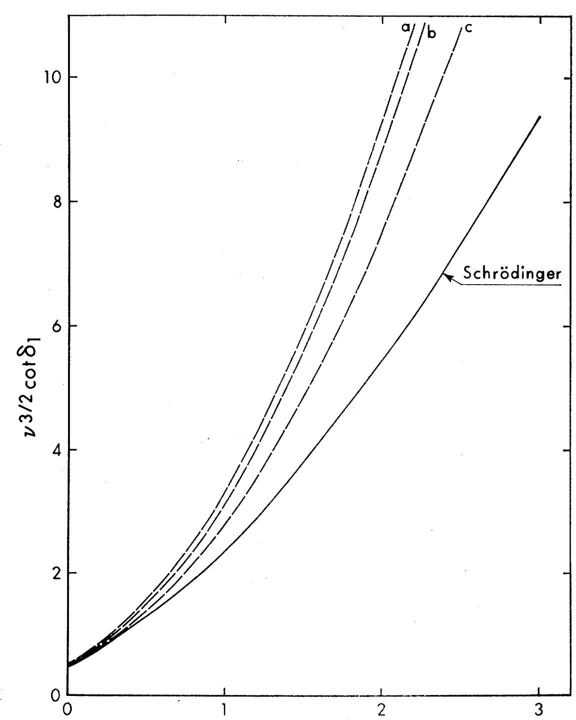

FIG. 6. Effective-range plots $g=3$, $l=1$, and first Born inputs; the labels correspond to those of Table I.

TABLE III. Fits to the function $iG(\nu)$.

					Relativistic						
					Cut $-2\leqslant\nu\leqslant-1$						
ν	−2	−1.9	−1.8	−1.7	−1.6	−1.5	−1.4	−1.3	−1.2	−1.1	−1
$iG(\nu)$	−0.79352	−0.78126	−0.76845	−0.75503	−0.74093	−0.72608	−0.71038	−0.69374	−0.67601	−0.65704	−0.63662
$\nu_n=2$; I, $\Delta G(\nu)$	0	0.00019	0.00029	0.00032	0.00029	0.00020	−0.00007	−0.00006	−0.00016	−0.00017	0
$\nu_1=0.4$, $\nu_2=5$; II, $\Delta G(\nu)$	0	0.00001	0.00001	0	0	0	0	0	0	0	0
					Cut $-4\leqslant\nu\leqslant-1$						
ν	−4	−3.7	−3.4	−3.1	−2.8	−2.5	−2.2	−1.9	−1.6	−1.3	−1
$iG(\nu)$	−0.9681	−0.9477	−0.9258	−0.9020	−0.8763	−0.8479	−0.8165	−0.7813	−0.7409	−0.6937	−0.6366
$\nu_n=2.8$; I, $\Delta G(\nu)$	0	0.0010	0.0017	0.0020	0.0019	0.0013	0.0004	−0.0008	−0.0019	−0.0021	0
$\nu_1=0.05$, $\nu_2=5$; II, $\Delta G(\nu)$	0	0.0002	0.0002	0.0002	0.0001	0.0	−0.0001	−0.0001	−0.0001	−0.0003	0
					Cut $-9\leqslant\nu\leqslant-1$						
ν	−9	−8.2	−7.4	−6.6	−5.8	−5	−4.2	−3.4	−2.6	−1.8	−1
$iG(\nu)$	−1.1903	−1.1640	−1.1352	−1.1034	−1.0678	−1.0275	−0.9810	−0.9258	−0.8577	−0.7685	−0.6366
$\nu_n=4.1$; I, $\Delta G(\nu)$	0	0.0034	0.0060	0.0078	0.0084	0.0076	0.0052	0.0010	−0.0046	−0.0089	0
$\nu_1=0.05$, $\nu_2=8$; II, $\Delta G(\nu)$	0	0.0007	0.0010	0.0010	0.0006	0	−0.0006	−0.0009	−0.0003	0.0019	0
					Nonrelativistic						
					Cut $-1\leqslant\nu\leqslant-0.25$						
ν	−1	−0.925	−0.85	−0.775	−0.7	−0.625	−0.55	−0.475	−0.4	−0.325	−0.25
$iG(\nu)$	−1	−0.961769	−0.921954	−0.880341	−0.836660	−0.790569	−0.741620	−0.689202	−0.632456	−0.570088	−0.5
$\nu_1=0.19$, $\nu_2=1.85$, ΔG	0	0.000001	0.000001	0	−0.000002	−0.000003	−0.000003	0	0.000001	−0.000005	0

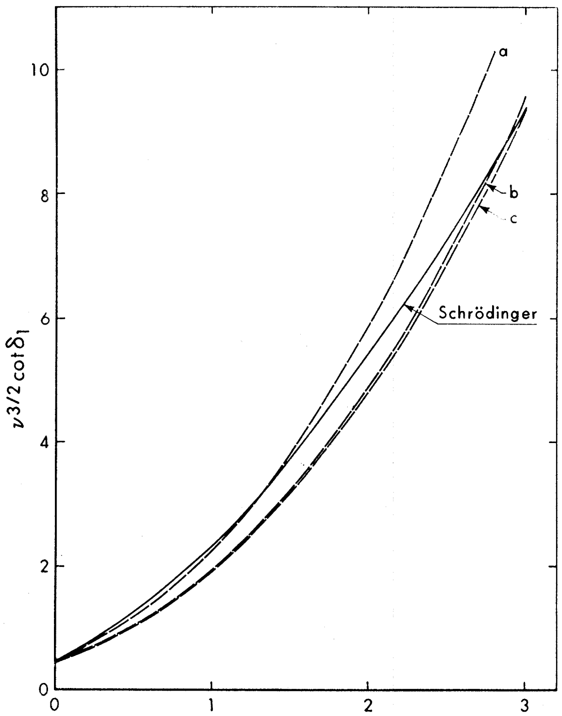

FIG. 7. Effective-range plots $g=3$, $l=1$, and second Born inputs; the labels correspond to those of Table I.

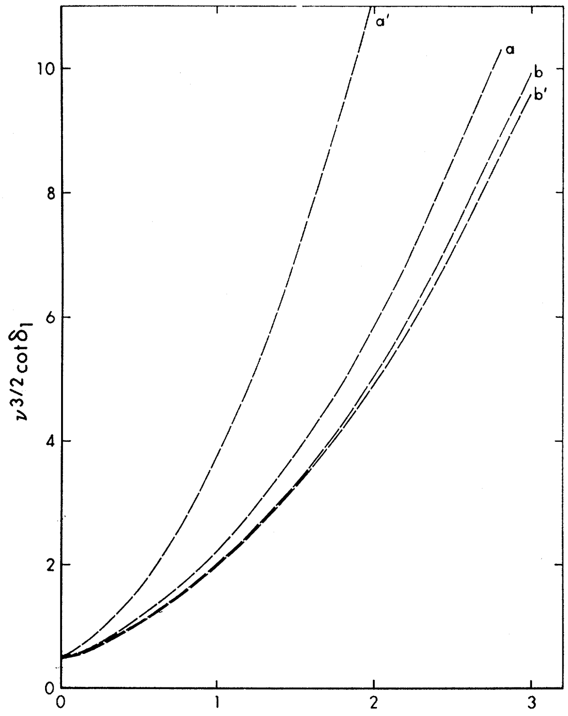

FIG. 9. Pole stability of solutions, $g=3$, $l=1$, second Born input. The curves are (a) no cut, poles at $\nu=-0.3$, -0.85; (a') no cut, poles at $\nu=-0.28$, -2; (b) cut $-0.5\leqslant\nu\leqslant-0.25$, poles at $\nu=-0.55$, -2.24; (b') cut $-0.5\leqslant\nu\leqslant-0.25$, poles at $\nu=-0.6$, -2.4.

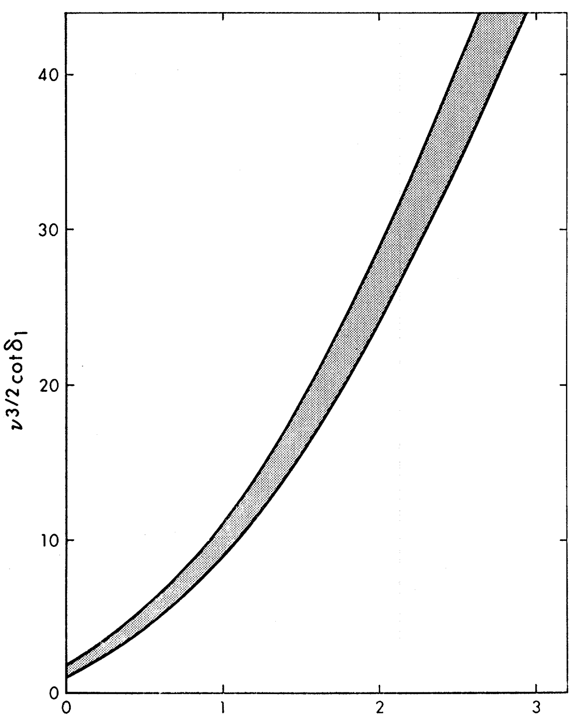

FIG. 8. Matching stability of pure pole solutions $g=1$, $l=1$, and first Born input.

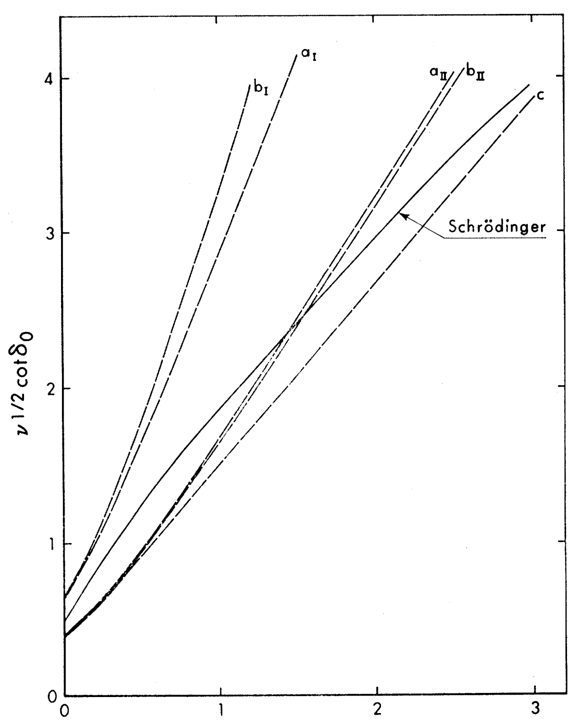

FIG. 10. Effective-range plots $g=1$, $l=0$; the labels correspond to those of Table II.

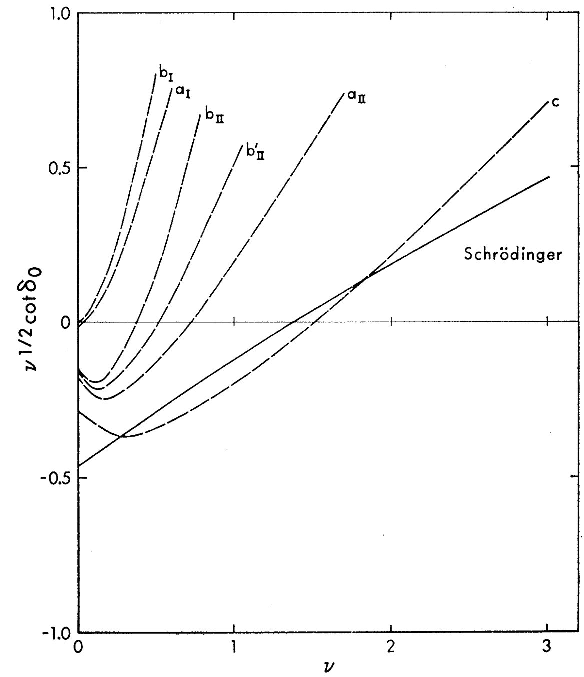

FIG. 11. Effective-range plots $g=3$, $l=0$; the labels correspond to those of Table II.

stability of the pole solutions. Except for the single case referred to above, the matching stability of all the cut solutions was completely negligible.

The cut solutions are also rather stable with respect to changes of the pole positions. That is, if the position of the first (nearer) pole is changed slightly and the other pole position then adjusted as before to give a best fit to the amplitude, the output changes very little. The results of such a variation are shown in Fig. 9 for $g=3$ (second Born input), the case where the poles give their greatest contribution. A change of only 0.02 in the position of the first pole for the pure pole solution causes a marked change in the results, whereas

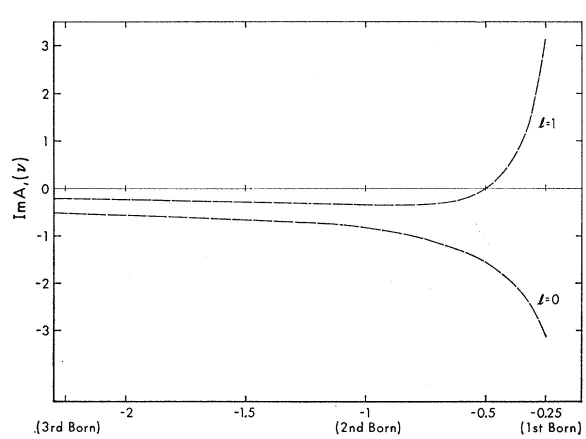

FIG. 12. Cut discontinuities $g=1$.

a change of 0.05 for the cut solution has a much smaller effect; a change of 0.25 for the cut solution $-1\leqslant\nu\leqslant-0.25$ (not shown) has a similar effect.

$l=0$

For $l=0$, the situation changes drastically because of large contributions from the higher-order Born terms (see Table II). Even for $g=1$, the amplitude is inadequate, whereas its use for the $g=3$ case is obvious nonsense because of the bound state.

Two different ways of circumventing this difficulty were tried. The first method utilizes the fact that the relative contribution of the higher-order terms decreases greatly on the peak of the amplitude near the cut so that the amplitude may be accurate there even if it is poor elsewhere. The method works rather well for $g=1$; the matching of the solutions to the Born amplitudes is shown in Table II, and the phase shifts in Fig. 10. The procedure used was to fix the residues by matching the N/D solutions to the amplitude and its first derivative very near the cut, then to adjust the pole positions so as to follow the amplitude as far down the peak as possible. This can be done down to about $\nu=-0.22$ without any significant difficulties or ambiguities arising, and the second Born outputs give quite good results. For $g=3$, however, the amplitudes can be matched only in a very small region, and choosing the best solution becomes more difficult. For example, two different second Born solutions for the cut $-1\leqslant\nu\leqslant-0.25$ are given in Table III; it is very difficult to choose between these solutions on the basis of their behavior near the cut, yet the outputs are quite different (see Fig. 11). In addition to this ambiguity, it can be seen that none of the solutions gives particularly good results.

The alternative approach utilizes the idea that the contributions from distant singularities should be nearly independent of l.[17] The cut discontinuities for $g=1$ are shown in Fig. 12, and the curves for $g=3$ have a similar form. It can be seen that though the behavior of the $l=0$ and $l=1$ cuts is quite different near the branch point, it is at least qualitatively the same far to the left. The idea is then to use as much of the left-hand cut as possible, to determine the pole residues from an $l=1$ calculation, and then to use the same residues for $l=0$. This must be done in a somewhat indirect manner, however, because of the difficulty of matching even the $l=1$ amplitudes when a very long segment of the cut is kept. What we do, therefore, is use the cut all the way to $\nu=-2.25$ (where the third Born cut begins) and then fix the residues so as to roughly reproduce the $l=1$ phase shifts as calculated before from the cut $-1\leqslant\nu\leqslant-0.25$.[18] Then, using the same residues, we can calculate for $l=0$. The results are

[17] L. A. P. Balázs, Phys. Rev. **162**, 1482 (1967).

[18] The actual procedure was to set $A_1(0)=0$ and also require the phase shifts to agree at $\nu=2$.

shown in Figs. 10 and 11 (curve c) and are seen to be quite satisfactory even for the very difficult case $g=3$. Furthermore, these results are insensitive to the pole positions; the curves shown are for poles at $\nu=-3, -10$, but moving the poles to $\nu=-2.5, -5$ gives virtually the same results.

IV. SUMMARY AND CONCLUSIONS

In actual practice, the use of the Balázs method has been greatly restricted by numerical difficulties. When part of the cut is retained to treat correctly the long-range forces, the resulting integral equations make the matching procedure extremely tedious and time-consuming. If, on the other hand, the cut is discarded and a pure pole approach used, important ambiguities in the solutions can arise. In this paper, we have given a closed-form solution of the N/D equations which greatly simplifies the calculations and makes it possible to use the more general approach in a systematic way. The closed-form solutions are approximate, but can be made very accurate without any difficulty if the cut being used is not too long.

A study of Yukawa scattering indicates that the straightforward Balázs method can be made reliable for $l\neq 0$ by keeping a part of the left-hand cut. Ambiguities in the matching procedure decrease substantially, and the output agrees well with Schrödinger equation solutions. The accuracy of the amplitude being used for matching seems to play an important role in the degree of success of the method. An improvement of this amplitude not only leads to better output, but also makes the matching procedure easier and allows more of the left-hand cut to be included.

For $l=0$, the method becomes much harder to apply because of the difficulty in calculating a sufficiently good amplitude to reproduce. Two methods were suggested for dealing with this problem. The first method fits only the peak near the left-hand cut where the accuracy of the amplitude is highest, and seems to work well for weak coupling. The second method uses the $l=1$ results and the idea that the effects of distant singularities are independent of l. This method gave good results even for the bound-state case, but requires the knowledge of a fairly large segment of the left-hand cut.

APPENDIX

The accuracy of the method depends on $\Delta G=iG-i\tilde{G}$ being small on the cut $\bar{L}$. This condition is easy to arrange as long as the cut segment is not too large. We have made no effort to find an optimal method of constructing the approximating function $\tilde{G}$, but merely give two examples:

$$\text{(I)}\quad i\tilde{G}(\nu)=iG(L_2)+c_1(\nu-L_2)/(\nu-\nu_n)\,,$$

$$\text{(II)}\quad i\tilde{G}(\nu)=c_2+c_3(\nu-L_2)/(\nu-\nu_1)+c_4(\nu-L_1)/(\nu-\nu_2)\,.$$

The one-pole form (I) obviously agrees with $iG(\nu)$ at L_2, and c_1 can be taken such that they agree at the other endpoint L_1. The pole position ν_n is then varied to minimize ΔG in the interior. In the second form (II) the constants c_2, c_3, and c_4 can be taken to make $\tilde{G}$ agree exactly with G at the end points of the cut and the midpoint $\frac{1}{2}(L_1+L_2)$. The degree to which G is approximated by these forms is given in Table III for different cut segments.

In actual calculations, of course, it is always possible to check the accuracy of the method by seeing how well the solutions obtained satisfy the exact integral equations. This was done for some of the Yukawa scattering calculations, and it was found that the solutions satisfied the integral equations very well. The most critical situation here is the region $-0.25\leqslant\nu\leqslant 0$ where the matching is being attempted to very high accuracy and there is the danger of small systematic errors. In particular, one wants to know that discrepancies such as those observed for $g=3$, $l=1$ in the first Born approximation are real, and not the product of deficiencies of the numerics. For this purpose, a high-accuracy two-pole form

$$i\tilde{G}(\nu)=\frac{iG(L_1)(\nu-L_2)}{L_1-L_2}+\frac{iG(L_2)(\nu-L_1)}{L_2-L_1}+\frac{(\nu-L_2)(\nu-L_1)}{L_1-L_2}\left(\frac{b_1}{\nu-\nu_1}+\frac{b_2}{\nu-\nu_2}\right)$$

was used. Here, $\tilde{G}(\nu)$ agrees with $G(\nu)=\sqrt{\nu}$ automatically at $\nu=L_1$, L_2; b_1, and b_2 were chosen to give agreement at two interior points, then ν_1 and ν_2 adjusted to give the best over-all agreement. The agreement obtained for the cut $-1\leqslant\nu\leqslant-0.25$ with this form is shown in Table III. Substitution of the solutions into the integral equations indicated that errors in the amplitudes given in Table I are probably never greater than one in the last decimal place.