which is the result used in Sec. IV. Equation (2) may be recast, on integration by parts, into the form

$$F_n(x)=\int_0^\infty d\beta\, e^{-\beta x}\frac{\partial}{\partial\beta}\prod_{i=1}^{n}\frac{1-e^{-\beta a_i}}{a_i}. \tag{A3}$$

From Eq. (3) we have immediately, as a simple check,

$$F_n(0)=\prod_{i=1}^{n}\frac{1-e^{-\beta a_i}}{a_i}\bigg|_{\beta=0}^{\beta=\infty}=\frac{1}{a_1a_2\cdots a_n}, \tag{A4}$$

a result used in Secs. II and III. Using mathematical induction, it is easy to provide a purely algebraic proof of (4). We leave this task to the interested reader.

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Renormalization of Regge Trajectories and Singularity Structure in Kikkawa-Sakita-Virasoro–Type Theories*

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An investigation is made of theories which satisfy the duality principle using the Veneziano amplitude as a Born term. In constructing the theory, it is found necessary to average over different ways of assigning the loop momenta to the points of the duality diagram. The Regge-pole terms in the asymptotic behavior are identified, and transcendental equations which express the full renormalization of the leading trajectory are recorded. (It is necessary to assume that the integrals can be so defined that this asymptotic behavior, found in the limit $\text{Re}s\to-\infty$, continues to be the dominant behavior as $\text{Re}s\to+\infty$.) The amplitude is shown to have the Landau-Cutkosky singularity structure corresponding to poles lying on the renormalized leading trajectory. In particular, if low-lying particles on this trajectory are the only stable particles in the theory, the real singularity structure required by unitarity is correctly obtained. It is then possible that the failure in a finite theory of exact factorization for all daughters would not spoil the theory.

I. INTRODUCTION

Recently Kikkawa, Sakita, and Virasoro (KSV)[1] have proposed a way of constructing a new form of perturbation theory, consistent with duality, in which the Veneziano amplitude[2] plays the role of a Born term. Such a series appears likely to be formally unitary and to correct the most glaring deficiency of the Veneziano model itself. However, KSV in a note added in proof, and also Bardakci, Halpern, and Shapiro (BHS)[3] have pointed out that in order to obtain full factorization of even the single-loop KSV expression in a way which is consistent with Veneziano-type functions associated with tree diagrams,[4] the integrand in the KSV integral must contain an infinite product which leads to an exponential divergence.

This disastrous conclusion is enforced by the requirement that factorization, and consequent unitarity-like discontinuity formulas around normal threshold singularities, is required for *all* poles contained in the Veneziano amplitude whatever their level in the daughter sequence. While this would be an agreeable property if it were obtainable, it is not clear that its failure robs the KSV approach of all its utility. Two lines of thought suggest that this is not necessarily the case. One is that the daughter properties of a Veneziano amplitude can be modified by the addition of nonleading terms. Bardakci and Mandelstam[5] have conjectured that these nonleading additions cannot be used in a way which leads to a simpler, and so probably less divergent, daughter sequence, but a proof has not, at present, been given that this is so. Secondly, the effect of unitarizing the theory will be to destroy the narrow-resonance approximation of the Veneziano amplitude. Resonance poles should move onto unphysical sheets, leaving only the stable-particle poles renormalized to locations which are still real. For simplicity, we shall always consider the model in which the only stable particle is the spin-0 member of the leading trajectory. If that leading trajectory factorizes properly, then the real normal thresholds corresponding to stable particles will have Cutkosky discontinuity formulas which correspond to physical unitarity. This will not be true for singularities involving daughter-trajectory particles, if the latter do not factorize

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† On leave of absence from the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England.
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properly, but if these singularities are translated onto unphysical sheets they may not spoil the physical unitarity of the theory. We return to a fuller discussion of this point in the conclusion.

The aim of this paper is to discuss some of the effects of imposing unitarity on the Veneziano formula by means of a KSV approach. We restrict ourselves to planar diagrams and, consequently, construct a theory which has only s and t channels. In particular, in such a theory we study how this renormalizes the particle and resonance poles. This renormalization manifests itself in two distinct ways. The first is by the displacement of Landau singularities and in particular the direct-channel poles. The second is through a modification of the asymptotic behavior of the amplitude corresponding to a renormalized Regge trajectory. KSV have already given a leading-order approximation discussion of the latter. In this paper, we give a complete calculation, using techniques developed to give a similarly complete calculation of the asymptotic behavior of ladder diagrams in conventional perturbation theory.[6,7] Of course, one requires that the two effects give the same answer, that is that the displaced direct-channel poles lie on the displaced trajectory. We show that the factorization conditions involved are always the same in the two cases, whatever daughter is considered, and that when these are satisfied the consistency condition is an identity.

Furthermore, for the case of the leading trajectory, these factorization conditions are shown to hold for virtually any expression constructed according to the general ideas of KSV, whether or not it contains terms corresponding to circling lines in duality diagrams. Presumably the factorization conditions for daughter trajectories will require increasing numbers of these lines, and if they are to hold for all daughters, one would expect to arrive by a somewhat different route at the disaster found by KSV and BHS. However, as we have argued above, it may be that a useful theory may be obtained without going to that limit.

Equations (5.13)–(5.17) give the transcendental equations which incorporate the unitarity corrections to the leading trajectory of the Veneziano model. Although these equations are in the form of series in the expansion parameter, even the lowest approximation corresponds to a partial infinite summation and incorporates important nonperturbative features. For example, it reproduces the Gribov-Pomeranchuk condensation of poles at Re$l=-\frac{1}{2}$ at the first elastic threshold.[7,8] However, the threshold is still at its unrenormalized position.

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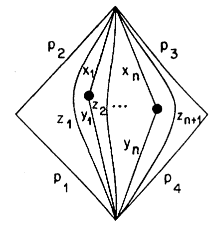

FIG. 1. Dual diagram showing the variables used.

II. MODEL

The integral associated with n-loop planar diagrams can be written in the form

$$I_n=(g^2)^{n+1}\prod_{i=1}^{n}\int_0^1 dx_i\int_0^1 dy_i\prod_{J=1}^{n+1}\int_0^1 dz_j z_j^{-\alpha_0(t)-1}$$

$$\times\exp\Big(\prod_{j=1}^{n+1} z_j f_n(x,y,z)s-d_n(x,y,z,t)\Big)g_n(x,y,z). \quad (2.1)$$

The variables x, y, and z are associated with lines of the dual diagram shown in Fig. 1; $s=(p_2+p_3)^2$, $t=(p_1+p_2)^2$; g^2 is the expansion parameter; α_0 is the linear trajectory of the original Veneziano amplitude. The exponent in the integrand is constructed according to the rules given by KSV. Its detailed form will depend on how many further lines are to be represented in the dual diagram Fig. 1. The variables associated with these lines are all functions of x, y, and z determined by the repeated application of the quadrilateral conditions, Eqs. (3.2) and (3.3) of KSV. We discuss the choice of these further variables in the next paragraph. At present we only indicate in (2.1) that whatever the choice, the coefficient of s in (2.1) will vanish when any one of the z_j vanishes. This was shown by KSV. The function g_n is the product of two terms. One is the $(\det A_n)^{-2}$ factor arising from performing the symmetric integration over the n-loop momenta. The other is whatever else is required, including a Jacobian factor. We leave the precise form unsettled, but will impose a simple requirement as the argument develops.

Figure 1 is the dual diagram associated with Fig. 2(a). In Fig. 2(b) we show some of the many diagrams

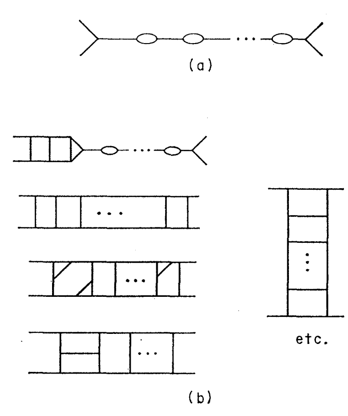

FIG. 2. (a) The diagram of which Fig. 1 is the dual diagram; (b) other diagrams related to (a) by duality.

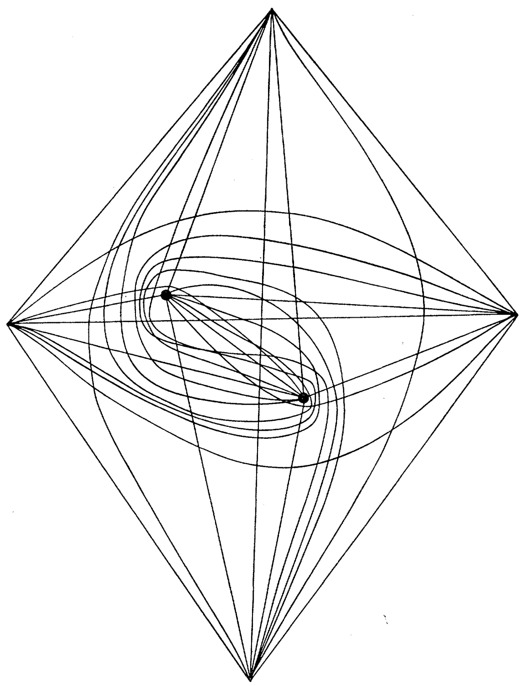

FIG. 3. Lines needed for the two-loop diagram.

related to Fig. 2(a) by duality in the way explained by KSV. The minimum set of further variables which must go into the construction of (2.1) is that which corresponds to all the lines needed for the dual diagrams of the set Fig. 2(b). When one attempts to construct such a set for diagrams with more than one loop, one immediately encounters a difficulty. It proves impossible to choose a set in such a way that each desired dual diagram is obtained once and once only. This is because the internal points of the dual diagram represent loop momenta and there is not a natural ordering of these loop momenta which holds universally for all the diagrams of Fig. 2. In fact one must be content with a sum over all the possible assignments of points to loop momenta so that every dual diagram is generated in $n!$ ways. For example, Fig. 3 shows some of the variables required for the two loop diagram. (Omitted from Fig. 3 are variables needed to correspond to diagrams with self-energy

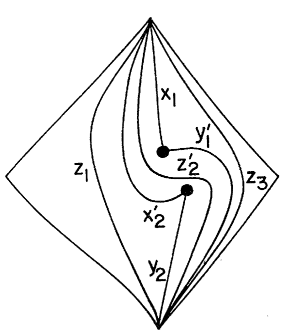

FIG. 4. A dual diagram contained in Fig. 3. The primed variables differ from those of Fig. 1.

insertions in the external lines. These variables in fact require a special discussion which is given in Sec. VI when wave-function renormalization is dealt with.) The dual diagram corresponding to Fig. 2(a) can be constructed in two ways, one corresponding to Fig. 1, the other to Fig. 4.

We are interested in the singularity structure of the integral (2.1), which will be discussed in later sections. We shall find that it has singularities occuring on the expected Landau curves and that these arise from points in the region of integration when the variables corresponding to the lines in the appropriate dual diagram vanish. Since there are $n!$ ways of constructing any given dual diagram, there are $n!$ distinct points in the region of integration which contribute to the given singularity. Each point, because of the symmetrical way of constructing I_n, yields the same contribution and if they were all added together we should find Cutkosky discontinuity formulas which differed from those required for unitarity by a factor of $n!$. It is therefore necessary that g_n should contain a factor $(n!)^{-1}$. It is clearly equivalent and much more convenient merely to evaluate the contribution at one of the points only and forget about the $(n!)^{-1}$. This we shall do in all that follows, as a calculational convenience, both for singularity structure and also for asymptotic behavior, to which we now turn our attention.

III. ASYMPTOTIC BEHAVIOR

In order to investigate the asymptotic behavior of (2.1), we take its Mellin transform[6,7] with respect to $(-s)$. If the Mellin-transform variable is l, this yields

$$M_n(l)=\Gamma(-l)(g^2)^{n+1}\sum_{i=1}^{n}\int_0^1 dx_i\int_0^1 dy_i\prod_{j=1}^{n+1}\int_0^1 dz_j \times z_j^{l-\alpha_0-1}(f_n)^l g_n e^{-d_n}. \quad (3.1)$$

The expression (3.1) has poles when $l=\alpha_0-m$ $(m=0, 1, 2, \cdots)$ due to the divergence of the z_j integrations at $z_j=0$. These can be explicitly exhibited in the standard way by integrating by parts to yield

$$M_n(l)=\Gamma(-l)(g^2)^{n+1}\prod_{i=1}^{n}\int_0^1 dx_i\int_0^1 dy_i \times\prod_{j=1}^{n+1}\int_0^1 dz_j\left(\frac{z_j^{l-\alpha_0+m}}{(l-\alpha_0)\cdots(l-\alpha_0+m)}\right) \times(-)^{m+1}\frac{\partial^{m+1}}{\partial z_j^{m+1}}[f_n^l g_n e^{-d_n}]. \quad (3.2)$$

If we put $l=\alpha_0-m$ everywhere in (3.2) other than in the vanishing denominator factors, we obtain the leading-order approximation already discussed in the case $m=0$ by KSV. Summed over n, it yields a Regge pole. However, we wish to do better than that and sum up all contributions, not just the leading ones. Only then shall we get the correct trajectory. The technique is the exact analog of that employed in perturbation theory.[6,7] One expands each factor

$$z_j^{l-\alpha_0+m}=\sum_{r_j=0}^{\infty}\frac{(\ln z_j)^{r_j}(l-\alpha_0+m)^{r_j}}{r_j!},\qquad(3.3)$$

and collects terms according to the resulting net powers of $(l-\alpha_0+m)^{-1}$ displayed. Any term with an $r_j=0$ is such that the corresponding z_j integration can be performed explicitly. This replaces $\partial^{m+1}/\partial z_j^{m+1}$ by $\partial^m/\partial z_j^m$ evaluated at the limits $z_j=1$, 0. At $z_j=1$, variables dual to z_j become zero and their logarithms, which appear in the exponent, become infinite. There is then a vanishing contribution from $z_j=1$, and one is left with the contribution from $z_j=0$. Symbolically, we can represent the effect of integrating these terms with $r_j=0$ by the substitution

$$\int_0^1 dz_j(-)^{m+1}\frac{\partial^{m+1}}{\partial z_j^{m+1}}\to(-)^m\frac{\partial^m}{\partial z_j^m}\bigg|_{z_j=0}.\qquad(3.4)$$

The summation of multiple poles in (3.2) to give displaced poles corresponding to Regge poles depends upon factorization properties of these derivatives evaluated with $z_j=0$. We shall give a detailed discussion of the case $m=0$ in Sec. V. We do not attempt a general discussion of $m\neq 0$. Even in conventional perturbation theory, only special cases have been solved.[9] Our purpose in developing the general argument thus far is to be able to make a comparison with a different but related discussion in Sec. IV.

A word of caution must finally be sounded on the results of the discussion presented here. The Mellin-transform method is only able to handle the limit $-s\to\infty$, and it correctly obtains the behavior in that case. In the case of conventional perturbation theory, analyticity and the fact that one can obtain bounds on the integrals which show that they cannot exceed power-law behavior for $|s|\to\infty$ in any direction, together, then, assure that the result holds for limits taken in any direction in the complex plane. In the case we are now discussing the second of these conditions can not be shown in general and so we can not generally exclude the presence of entire functions which would have exponentially vanishing behavior as $\text{Re}s\to-\infty$, but bad behavior as $\text{Re}s\to+\infty$. In fact, it is an important constraint to be satisfied on the detailed form of (2.1) that it is free from this undesirable behavior. We are at present unable to make a useful contribution towards determining how to do this and must proceed under the tacit assumption that it can be done. The Regge-pole properties that we obtain will then be those which hold in any sensible theory that can be constructed. It seems wholly reasonable to suppose that such a theory can be found.

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IV. DIRECT-CHANNEL POLES

The amplitude (2.1) has multiple poles in t corresponding to the divergencies of the z_i integrations at $z_i=0$. Graphically these correspond to the multiple poles in diagrams like Fig. 2(a) and the first diagram of Fig. 2(b). When these are summed over n, we expect them to turn into displaced simple poles as in conventional renormalization theory. This is now investigated.

In order to exhibit the angular momentum content of the poles, we first expand the part of the exponent in (2.1) which depends on s:

$$e^{f_n s}=\sum_{p=0}^{\infty}\frac{f_n^p s^p}{p!}.\qquad(4.1)$$

Integration by parts then exhibits the poles

$$I_n=\sum_{p=0}^{\infty}\frac{(g^2)^{n+1}s^p}{p!}\prod_{i=1}^{n}\int_0^1 dx_i\int_0^1 dy_i \times\prod_{j=1}^{n+1}\frac{z_j^{-\alpha_0+m+p}}{(\alpha_0-p)\cdots(\alpha_0-p-m)}\frac{\partial^{m+1}}{\partial z_i^{m+1}} \times[f_n^p g_n e^{-d_n}].\qquad(4.2)$$

The leading-pole behavior is given by putting $\alpha_0=p+m$ everywhere in (4.2) except in the denominators which vanish. If one wants to do better than a leading-order approximation, one must expand the $z_j^{-\alpha_0+m+p}$ factors in powers of $\ln z_j$, and integrate when possible, exactly as described in the analogous manipulations of Sec. III. It is clear that the factorization conditions required in this case are exactly the same as those required in Sec. III with l taken equal to the integer p.

Thus we see that there is complete consistency between the Regge poles obtained by an investigation of high-energy behavior and the direct-channel poles obtained by renormalization. The factorization conditions required are equivalent and the direct-channel poles are indeed the poles lying on the Regge trajectories.[10] In particular, the sequence of poles with $m=0$, $p=0$, 1, 2, $\cdots$, lie on the leading trajectory.

While the equivalence is to be expected on the basis of using the Sommerfeld-Watson transform in a well-behaved theory, it has seemed worthwhile to check it explicitly in this case. For the leading trajectory we shall also be able to show that it holds for all Landau

[10] In comparing (3.2) and (4.2) recall that $\Gamma(-l)=(\sin\pi l\cdot\Gamma(l+1))^{-1}$.

singularities, not just the direct-channel poles; this will require a generalization of the method used above.

V. FACTORIZATION AND LEADING TRAJECTORY

While the factorization conditions needed in the arguments of Secs. III and IV are difficult to discuss in general, it is possible to establish them rather easily for the case of the leading trajectory. This we now proceed to do.

We require that when $z_j=0$, the expression

$$f_n{}^l g e^{-d} \tag{5.1}$$

factorizes into a product of two terms, one of which depends only on variables associated with lines in the dual diagram lying to the left of the line corresponding to z_j, the other depending only on variables to the right. For the term e^{-d} this factorization is immediate. It follows from the fact that when we have $z_j=0$, duality forces the variables corresponding to lines crossing the z_j line to go to 1. Their logarithms, which appear in the exponent, then vanish and the terms which remain in d correspond to lines in two subdiagrams joined together only by the z_j line. Then e^{-d} factors into the product of the two e^{-d} factors corresponding to these subdiagrams. Similarly, the factorization of the $(\det A)^{-2}$ factor in g is immediate for the same reason. The only condition that we impose on the Jacobian or other extra factors in g_n is that they also should factorize.

The only term in (5.1) which requires a more detailed discussion is f_n. The central result we need is the following:

Lemma. A variable corresponding to a line which crosses once the line corresponding to z_j has the form

$$1-A_1A_2z_j+O(z_j{}^2), \tag{5.2}$$

when A_1 is a function of variables lying to the left of the z_j line and is *determined only by the topological structure of the part of the line which lies to the left of the z_j line.*[11] and A_2 is similarly a function of right-hand variables and determined by the topological structure of the right-hand part of the line.

Thus, the two lines in Fig. 5 have the same A_1 factors but different A_2 factors. We establish the result by first considering two lines having the same left structure and differing in their right structure in the way shown in Fig. 6; that is, one of them carries on to the next point in the dual diagram, but in all other respects its right structure is the same. Application of the formulas (3.2) and (3.3) of KSV to the quadrilateral shown in the figure yields

$$X=\frac{(1-x_2\alpha_3\alpha_1)(1-x_2\alpha_3\alpha_1xX')}{(1-x_2\alpha_3\alpha_1x)(1-x_2\alpha_3\alpha_1X')}$$

$$=1+\left(\frac{x_2\alpha_3\alpha_1x}{1-x_2\alpha_3\alpha_1x}-\frac{x_2\alpha_3\alpha_1}{1-x_2\alpha_3\alpha_1}\right)(1-X')+O((1-X')^2), \tag{5.3}$$

where

$$x_1=\frac{(1-\alpha_1)(1-\alpha_1x_2X')}{(1-\alpha_1X')(1-\alpha_1x_2)}, \tag{5.4}$$

$$x_3=\frac{(1-\alpha_3)(1-\alpha_3x_2x)}{(1-\alpha_3x_2)(1-\alpha_3x)}. \tag{5.5}$$

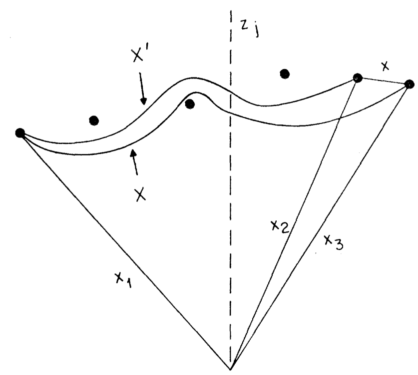

FIG. 6. A quadrilateral to be considered. Variables associated with the lines are indicated.

We suppose that we already know that X' has the form

$$X'=1-A_1A_2z_j+O(z_j{}^2), \tag{5.6}$$

because, as $z_j\to 0$, $X'\to 1$ and, in (5.6), x_1 is not, in general, equal to 1, we must also have $\alpha_1\to 1$. The value of α_3 determined from (5.5) clearly depends only on right-hand variables. Thus (5.3) shows that

$$X=1-A_1A_2'z_j+O(z_j{}^2), \tag{5.7}$$

where A_1 is the same A_1 as in (5.6) but A_2' is different.

It is quite straightforward to show by similar arguments that the line having the desired left structure and ending at the first point to the right of the z_j line has the form (5.6). The lemma then follows from a repeated application of the result (5.7).

We now use the lemma to show the desired factorization of f_n, which has the form explained by KSV:

$$\prod z_j\times f_n=-F_n/\det A_n. \tag{5.8}$$

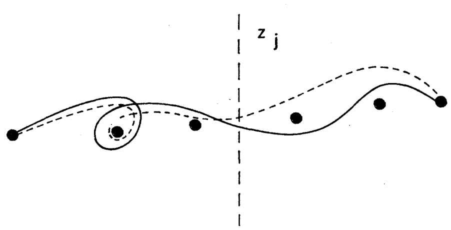

FIG. 5. Two lines having the same A_1 factor but different A_2 factors.

[11] We will call this left structure; right structure is similarly defined.

The factorization of $\det A_n$ is immediate, and we concentrate attention on F_n. According to KSV, it has the form of a sum of products of logarithms of sets of variables. These correspond to lines which fulfill the conditions that they are a maximal set forming a closed loop with p_2 and p_3 (or equivalently p_1 and p_4), and no other closed loops are present in the dual diagram. A term in F_n, therefore, has the structure

$$\prod \ln X_k \times \prod \ln Y_l\,, \tag{5.9}$$

where the X_k are the variables corresponding to the lines forming the closed loop and the Y_l are the rest. The X_k lines cross every one of the z lines, and their logarithms in (5.9) provide the z factors displayed on the left of (5.8). Thus, in evaluating f with $z_j=0$, we can put $z_j=0$ in all the $\ln Y$ factors in (5.9). This means that no lines crossing the z_j lines contribute to the $\ln Y$ product, which therefore can be written as a product of a left- and a right-hand factor. The $\ln X$ product can be similarly decomposed except that it contains one X whose line crosses the z_j line. We need only the terms in F which are linear in z_j, and since a variable whose line crosses the z_j line n times has a logarithm which vanishes like z_j^n, only X's corresponding to crossing that line once need be considered. The lemma then applies, and gives

$$\ln X = -A_1 A_2 z_j + O(z_j^2)\,. \tag{5.10}$$

Factors of z can be removed from A_1 and A_2 corresponding to the other z lines crossed, and also from the remaining $\ln X$ factors. The fact that A_1 and A_2 are determined solely by the topological structure of the left- and right-hand parts of the line to which they refer, taken together with the other properties discussed in this paragraph, means that each term in f_n can be written in the form

$$LR + O(z_j)\,, \tag{5.11}$$

where L (R) depends only on the left-hand (right-hand) variables and is determined by the left-hand (right-hand) topological structure of the lines corresponding to the KSV prescription for this particular term. Summing over all possible terms corresponds to summing over all possible left- and right-hand structures and gives a factored form for f_n:

$$f_n|_{z_j=0} = (\textstyle\sum L)(\sum R)\,. \tag{5.12}$$

This is the factorization condition we desired to establish. It readily extends to the case where several z_j are set equal to zero.

We denote by f_n', g_n', and d_n' the factors which correspond to n (≥ 0) nonzero z_j between two vanishing z_j, and by f_n'', g_n'', and d_n'' the similar factors corresponding to n nonzero z_j before the first or after the last vanishing z_j. Diagrammatically these correspond to Fig. 7 and all the diagrams related to Fig. 7 by duality. Then the summation of (3.2) with $m=0$ is performed

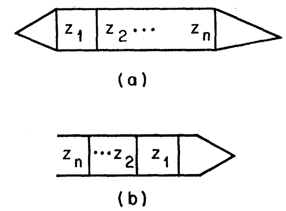

FIG. 7. Diagrams corresponding to (a) f', g', and d'; (b) f'', g'', and d''.

in exactly the same way that it is for ladder diagrams in conventional perturbation theory.[6,7] The answer is

$$\sum_{n=0}^{\infty} M_n(l,t) = \frac{\mathcal{G}(l,t)\Gamma(-l)\mathcal{G}(l,t)}{l-\alpha_0-\mathcal{F}(l,t)}\,, \tag{5.13}$$

where

$$\mathcal{F}(l,t) = \sum_{n=0}^{\infty} \bar{F}_n(l,t)\,, \tag{5.14}$$

$$\bar{F}_n(l,t) = g^{2n+2} \prod_{i=1}^{n+1} \int_0^1 dx_i \int_0^1 dy_i \times \prod_{j=1}^{n} \int_0^1 dz_j \left(\frac{z_j^{l-\alpha_0}-1}{l-\alpha_0}\right)\left(-\frac{\partial}{\partial z_j}\right) \times [f_n'^{\,l} g_n' e^{-d_n'}]\,, \tag{5.15}$$

$$\mathcal{G}(l,t) = \sum_{n=0}^{\infty} \mathcal{G}_n(l,t)\,, \tag{5.16}$$

$$\mathcal{G}_n(l,t) = g^{2n+1} \prod_{i=1}^{n+1} \int_0^1 dx_i \int_0^1 dy_i \prod_{j=1}^{n} \int_0^1 dz_j \times \left(\frac{z_j^{l-\alpha_0}-1}{l-\alpha_0}\right)\left(-\frac{\partial}{\partial z_j}\right)[f_n''^{\,l} g_n'' e^{-d_n''}]\,. \tag{5.17}$$

The vanishing of the denominator in (5.13) gives the Regge-pole trajectory.

As noted in the introduction, even the approximation which retains only $\bar{F}_0$ in (5.14) incorporates important nonperturbative features. If circling lines are omitted, $\bar{F}_0$ is given by the expression (4.30) of KSV with the exponent $\alpha_{13}(t)$ replaced by l. Higher terms in (5.16) involve nonvanishing z's and are more complicated.

VI. SINGULARITY STRUCTURE AND UNITARITY

We have used the form (2.1) for the KSV model in which the loop integrations have been performed. When one considers singularity structure it is often more convenient to retain these momentum integrations. Singularities from the x, y, and z integrations then give poles corresponding to lines in Fig. 2, and integrating over the loop momenta then gives singularities of the integral located on the Landau curves associated with Fig. 2. The implicit $i\epsilon$ prescriptions required to enable symmetric integration to be performed mean that in the physical region these singularities only occur on positive α arcs of the Landau curves.

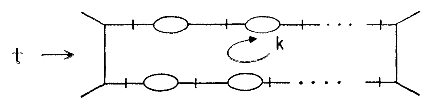

FIG. 8. Singularities being considered. The barred lines indicate pole terms.

These statements are true for any term of the form (2.1) but they need modification for the infinite sum of such terms. This is because the renormalization effects discussed in Sec. IV shift the location of poles, and the Landau singularities must be similarly displaced. This will be the case if the discussion of Sec. IV can be extended to poles which are not just direct-channel poles but lie within more complicated diagrams. In conventional renormalization theory this extension is trivial because subdiagrams behave in a way independent of their relation to the rest of the diagram. This is not the case for KSV theory, and so the extension generally is a very complicated matter. Once again we only attempt to discuss the leading trajectory.

It will be sufficient to consider the two-particle normal threshold. More complicated singularities are dealt with by an obvious extension of the same method. We first look at the set of singularities corresponding to Fig. 8. This is one of many relevant singularity configurations. The others are obtained by considering all the other ways in which self-energy loops can be assigned to the upper or lower line. The different contributions obtained in this way correspond to singularities at different points of the integration region in (2.1) and are additive. Returning to the configuration under discussion, the integrals over all the loop momenta of the self-energy parts can be performed leaving only k still to be integrated. The contributions associated with the upper and lower lines are now both very similar to that discussed in Sec. IV. The essential difference is the presence of terms corresponding to variables which are not present in the direct-channel pole case. Examples of these variables are shown by the dotted lines in Fig. 9. We shall call them extra variables.

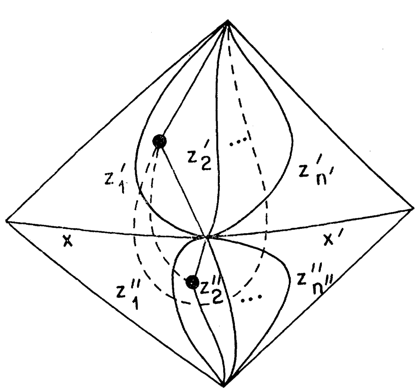

FIG. 9. The dual diagram of Fig. 8. The z' and z'' variables correspond to the dashed lines. Dotted lines represent extra variables discussed in the text.

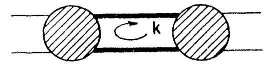

FIG. 10. The resultant singularity structure.

One now integrates by parts with respect to the z' and z'' variables to exhibit the multiple bare poles. Factorization occurs when any z_i (z_i'') is set equal to zero. At the same time, any extra variable line cutting this z' (z'') line becomes unity by duality and the variable disappears from the expression.

Alternatively, one might first perform all the integrations over loop momenta. Then the coefficient of s in the resulting exponential is of the form

$$f_1 \ln x \ln x' + (\prod z_i')f_2 + (\prod z_i'')f_3. \qquad (6.1)$$

Expanding the exponential with respect to the last two terms of (6.1) and then integrating by parts to exhibit the poles due to the z' and z'' integrations gives the contribution of the desired form.

When a sum is taken over all numbers of loops and over all assignments of self-energy loops to the top and bottom lines, the resulting singularity structure corresponds to Fig. 10. The thick lines correspond to renormalized poles located at the positions determined by the leading trajectory. The shaded blobs represent complete KSV-type scattering-amplitude expressions, except that there are modifications,

$$z'^{p-\alpha_0-1} \longrightarrow \frac{z'^{p-\alpha_0}-1}{p-\alpha_0}\frac{\partial}{\partial z'}, \qquad (6.2)$$

which prevent bare-particle poles from occuring in the squared momenta corresponding to the thick lines. In fact, exactly similar terms with $p=0$ must occur in the external lines also. This is because our external particles are supposed to be the stable spin-0 member of the leading trajectory. Before a sensible scattering amplitude is obtained, the poles in the external momenta, corresponding to Fig. 11, must be removed and external wave-function renormalization performed. Thus modifications like (6.1) must be understood throughout to be associated with these external momenta lines.

The shaded blobs themselves contain the t-channel normal threshold. Exactly as in conventional perturbation theory, this leads to a total discontinuity round the normal threshold which is exactly in the form required by unitarity. In a similar way, Cutkosky discontinuity formulas consistent with unitarity can be established for any Landau singularity.

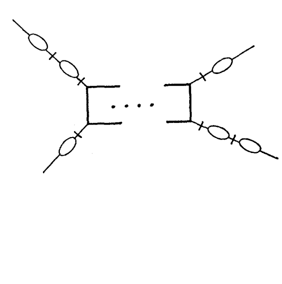

FIG. 11. Singularities generating poles associated with external wave function renormalization.

Finally, one might examine the singularity structure of terms in the sum (5.14) defining the function $\mathfrak{F}$ which gives the correct Regge trajectory. It is easy to see that individual terms have singularities at the bare normal thresholds.[12] When the sum is performed, they must be translated to the renormalized normal thresholds. However, an explicit verification of this seems complicated and we do not attempt it. That it must be true follows for the leading trajectory from the unitarity properties already established.

VII. CONCLUSION

Our investigation has essentially been concerned with renormalization effects in a KSV-type theory. The bare leading trajectory of the Veneziano model is renormalized into a new nonlinear trajectory which becomes complex at the first normal threshold. We have verified that the complete amplitude obtained by infinite summation has the correct Landau-Cutkosky singularity structure corresponding to the particles lying on this renormalized leading trajectory. In particular, this is true for the singularities corresponding to the lowest stable member of this trajectory. Then singularities are real and are those required by unitarity in the physical region. Note that these results follow from simple duality requirements of the KSV type. It is only necessary to invoke the existence of encircling lines in dual diagrams in order to obtain daughter-trajectory factorization.

As far as its leading trajectory is concerned, there is only one major requirement of a sensible theory which remains unestablished. This is that it is possible to define the detailed form of (2.1) so that the Regge pole found in the limit $s\to-\infty$ remains the dominant asymptotic contribution as $s\to+\infty$. It seems very likely that this is possible, but it would clearly be of great interest to prove that this is so. We are unable at present to do this.

While the theory treats the leading trajectory poles satisfactorily and, in particular, has the real singularity structure required by unitarity, it seems that if similar properties were required for all the daughter trajectories, one would again find the encircling lines which lead to the difficulties noticed by KSV and BHS. There are three possible ways out of the problem.

[12] Only the lowest such threshold was noted by KSV in the discussion of their approximation, but it is easy to see that all are present. A straightforward way to do this is to reintroduce loop momenta into the expression (5.15).

One is that the infinities encountered by KSV and BHS are a property of the type of expansion used and are not present in the correctly summed theory. It appears that the infinities are connected with the rapidly increasing degeneracy of daughters, which is found in the Veneziano model. This degeneracy is broken in a KSV theory as the daughters move off to different points on unphysical sheets. In order to investigate the effect of this, it would be desirable to develop an analog to renormalized perturbation theory for the KSV model, which at present is formulated in terms of bare particles.

An alternative possibility depends upon what really happens to the daughter trajectories if full factorization is not imposed. It seems natural to suppose that their effects are removed from the real axis onto unphysical sheets. Without full factorization they cannot become simply a displaced pole. A reasonable conjecture is that each becomes a sequence of displaced poles. If these sequences had points of the boundary of the physical region as limit points, care would be needed that unitarity was not upset in the neighborhood of these points. The relationship between unitarity and the real Landau-Cutkosky singularity structure depends upon being able to make analytic continuations in the neighborhood of the physical region. Near such points, this would not be possible. Examples of such behavior consistent with unitarity have been discussed by Martin in a rather different context.[13]

Finally, there is the possibility that satellite Veneziano terms might modify the theory in a way that removed some of the daughter difficulties.

Obviously none of these possibilities is more than a pious hope in our present state of knowledge. However, the beautiful way in which the KSV model produces a consistent structure associated with the renormalized leading trajectory gives ground for thinking that this approach has value and that the little understood daughter phenomena may not prove fatal to its ultimate utility.

ACKNOWLEDGMENTS

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[13] A. Martin, CERN Report No. Th-727, 1967 (unpublished)