Eikonal Approximation in Quantum Field Theory*

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The eikonal approximation for high-energy collisions, long familiar in the theory of potential scattering, is considered from the viewpoint of relativistic quantum field theory. We study, in particular, the Feynman amplitude $M(s,t)$ describing the scattering of two spin-0 particles, a and b, interacting by the exchange of spin-0 mesons. We show that if $M_n(s,t)$, the contribution to $M(s,t)$ arising from *all* nth-order Feynman diagrams in which exactly n mesons are exchanged between a and b, is written in an appropriately symmetrized way, and if the terms in any a or b particle propagator which are quadratic in the internal momenta are then dropped, the resulting expression, $M_n^{\text{eik}}(s,t)$, may be evaluated in closed form, and the sum over n, which defines $M^{\text{eik}}(s,t)$, may be carried out. The representation of $M^{\text{eik}}(s,t)$ found in this way involves the exponential of a function χ of a relative space-time variable $x=(x^0,\mathbf{x})$ and the external momenta; χ is a relativistic generalization of the eikonal χ_{pot} familiar from the theory of high-energy potential scattering. $M^{\text{eik}}(s,t)$ is both crossing-symmetric and time-reversal-invariant. In the static limit ($m_b\to\infty$), χ tends to χ_{pot} for the appropriate Yukawa potential and $f^{\text{eik}}=-M^{\text{eik}}/8\pi\sqrt{s}$ has a limiting form $f_{\text{pot}}^{\text{eik}}$, which we also derive directly from the theory of potential scattering; for small scattering angles, $f_{\text{pot}}^{\text{eik}}$ coincides with the standard result. The amplitude for particle-antiparticle scattering is studied in the same model. It is shown that the eikonal $\chi_2(x)$ associated with the contribution of all annihilation-type diagrams has a logarithmic singularity at $x=0$ whose coefficient is proportional to $\alpha(t)+1$, where $\alpha(t)$ is the Regge-trajectory function obtained from the asymptotic behavior of the ladder-type diagrams alone. Another connection with Regge behavior is made by showing that the summation of a certain infinite class of radiative corrections to the lowest-order γ-e Compton amplitude gives rise, in our eikonal approximation, to an eikonal $\chi(x)$ which has a similar logarithmic singularity with strength $1+\beta(t)$; here $\beta(t)$ is the trajectory function, introduced less directly in earlier work, which reproduces the major part of the spectrum of positronium on setting $\beta(t)=l=n-1$. A generalization of a simple algebraic identity used in the derivation of the above results, in the form of an integral representation, permits their extension to the case where one or more particles are off the mass shell. This is illustrated by a computation of an eikonal-type approximation to the Green's function for a relativistic particle moving in an external scalar field and by the summation of an infinite class of contributions to the vertex function in the model referred to above. The possibility of applying an off-shell eikonal approximation to the analysis of production processes is emphasized.

I. INTRODUCTION

In recent years, there has been great interest in high-energy approximations to scattering amplitudes which exhibit an exponential dependence on some of the kinematical variables, especially in connection with the revival of Regge theory. Simple approximations of this type have been known for a long time in the theory of nonrelativistic potential scattering: These are the so-called eikonal type of approximations.[1–3] For example, for a spinless particle of mass m scattered by an external potential $V(\mathbf{x})$ the scattering amplitude $f(\mathbf{p}',\mathbf{p})$ may be approximated, for large $|\mathbf{p}|$ and small scattering angle and under suitable restrictions on V,[4] as

$$f(\mathbf{p}',\mathbf{p})\approx\frac{|\mathbf{p}|}{2\pi i}\int d^2b\; e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{b}}(e^{i\chi(\mathbf{b})}-1)\,, \qquad (1.1)$$

where $\mathbf{b}$ is a 2-component vector orthogonal to $\hat{p}=\mathbf{p}/|\mathbf{p}|$ and the "eikonal" χ is defined by

$$\chi(\mathbf{b})=\frac{-m}{|\mathbf{p}|}\int_{-\infty}^{\infty}V(\mathbf{b}+\hat{p}\xi)d\xi. \qquad (1.2)$$

It seems worthwhile to ask to what extent analogous approximations for a two-body scattering amplitude may be obtained in quantum field theory. In the present paper we show that there is indeed a natural relativistic generalization of the eikonal approximation. The techniques used are rather simple and may be useful for deeper investigations of the asymptotic behavior of scattering amplitudes.

The usual derivations of equations such as (1.1) are based on calculations which start with expressions for the scattering amplitude and the Schrödinger wave function in position space. In Sec. II, we reconsider the problem of nonrelativistic potential scattering in momentum space, starting with an exact expression for the nth-order term f_n in the Born expansion of f. We show that if in the energy denominators appearing in this expression, terms of the form $\mathbf{K}^2$ are dropped relative to terms of the form $\mathbf{p}\cdot\mathbf{K}$, where $\mathbf{K}$ is a partial sum of internal momenta, the resulting approximation to f_n may be evaluated in closed form, with the help of an identity used in earlier, closely related work in quantum

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[1] G. Molière, Z. Naturforsch. 2, 133 (1947).

[2] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Britten and L. G. Dunham (Wiley-Interscience, Inc., New York, 1959), Vol. I, p. 315.

[3] L. I. Schiff, Phys. Rev. **103**, 443 (1956); see also D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957).

[4] See, e.g., Ref. 2, pp. 342–344.

electrodynamics.[5] The sum over n may then be carried out, yielding

$$f(\mathbf{p}',\mathbf{p}) \to f_{\text{pot}}^{\text{eik}}(\mathbf{p}',\mathbf{p}),$$

where

$$f_{\text{pot}}^{\text{eik}}(\mathbf{p}',\mathbf{p}) \equiv \frac{im}{2\pi}\int d\mathbf{x}\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} V(\mathbf{x}) \frac{e^{i\chi_{\text{pot}}}-1}{\chi_{\text{pot}}}, \quad (1.3)$$

with

$$\chi_{\text{pot}}(\mathbf{x}) \equiv \frac{-m}{|\mathbf{p}|}\int_0^\infty d\xi [V(\mathbf{x}+\hat{p}'\xi)+V(\mathbf{x}-\hat{p}\xi)]. \quad (1.4)$$

For small scattering angles, (1.3) and (1.4) reduce to (1.1) and (1.2), respectively; as distinct from (1.1), (1.3) is invariant under time reversal.

The advantage of an approach based on perturbation expansions in $\mathbf{p}$ space is that it is relatively straightforward to extend it to quantum field theory, using Feynman diagrams. In Sec. III, we consider the scattering of two spinless particles a and b interacting by the exchange of a scalar meson. We study M_n, the contribution to the Feynman amplitude $M(s,t)$ arising from *all* diagrams in which exactly n mesons are exchanged between a and b, and show that if the particle propagators are approximated by dropping terms quadratic in the internal four-momenta, the resulting amplitude $M_n^{\text{eik}}(s,t)$ may again be evaluated in closed form and the sum over n carried out. This yields as an approximation to the relativistic amplitude $f(s,t) = -M/8\pi\sqrt{s}$, the result

$$f^{\text{eik}}(s,t) = \frac{-g^2}{8\pi\sqrt{s}}\int d^4x\, e^{-iq\cdot x}\Delta_F(x)\frac{e^{i\chi}-1}{\chi}, \quad (1.5)$$

where $\Delta_F(x)$ is the meson propagator and χ is a "relativistic eikonal," defined by

$$\chi = \frac{ig^2}{4m_am_b}\int_0^\infty\int_0^\infty d\xi_a d\xi_b [\Delta_F(x+u_a'\xi_a-u_b\xi_b) + \Delta_F(x+u_a'\xi_a+u_b'\xi_b)+\Delta_F(x-u_a\xi_a-u_b'\xi_b) + \Delta_F(x-u_a\xi_a+u_b\xi_b)], \quad (1.6)$$

with $q=p_a-p_a'=-(p_b-p_b')$, $u_a=p_a/m_a$, $u_a'=p_a'/m_a$, etc. In the static limit, e.g., $m_b\to\infty$, with $f^2=g^2/4m_am_b$ kept fixed, Eqs. (1.5) and (1.6) reduce to (1.3) and (1.4), respectively, with $V(\mathbf{x})=-(f^2/4\pi)e^{-\mu|\mathbf{x}|}/|\mathbf{x}|$—the Yukawa potential appropriate to the case at hand. Hence these equations may be regarded as relativistic generalizations of (1.3) and (1.4). In addition, (1.5) also preserves time-reversal invariance, as well as crossing symmetry.

After these developments, still in Sec. III, we consider the special case of particle-antiparticle scattering, in which annihilation type of diagrams enter in an important way, permitting a connection with Regge behavior to be made. In Sec. IV, we consider further extensions of these ideas to the propagator in an external field and to the vertex function. The main technical tool here is a generalization of the identity referred to above (proved in the Appendix) which permits the relevant summations to be carried out in closed form even when some of the external momenta are off the mass shell. We also study an infinite class of radiative corrections to the lowest-order Compton amplitude in spinor electrodynamics and examine a connection previously drawn between Regge behavior and the spectrum of positronium. A concluding discussion is given in Sec. V.

[5] M. Lévy, Phys. Rev. **130**, 791 (1963).

II. EIKONAL APPROXIMATION FOR POTENTIAL SCATTERING

The purpose of this section is to consider an approximation to the scattering amplitude in nonrelativistic potential scattering which leads to an eikonal type of formula and which can be readily generalized to quantum field theory.

Let $\mathbf{p}$ and $\mathbf{p}'$ denote the initial and final momenta of a spin-0 particle of mass m scattered by an external potential $V(\mathbf{x})$. The Born expansion for the scattering amplitude $f(\mathbf{p}',\mathbf{p})$ is given by

$$f(\mathbf{p}',\mathbf{p}) = \sum_{n=0}^{\infty} f_{n+1}(\mathbf{p}',\mathbf{p}), \quad (2.1)$$

where

$$f_{n+1}(\mathbf{p}',\mathbf{p}) = (-m/2\pi)\langle\mathbf{p}'|VG_0V\cdots G_0V|\mathbf{p}\rangle, \quad (2.2)$$

with n factors G_0V, and $G_0=[E_0+(\nabla^2/2m)+i\epsilon]^{-1}$ with $E_0=\mathbf{p}^2/2m=\mathbf{p}'^2/2m$. We note that f_{n+1} may be written in any of the $n+1$ equivalent ways

$$f_{n+1} = (-m/2\pi)\langle\phi_{\mathbf{p}';-}{}^{(n-l)}|V|\phi_{\mathbf{p};+}{}^{(l)}\rangle, \quad (2.3)$$

where

$$|\phi_{\mathbf{p};-}{}^{(l)}\rangle = (G_0V)^l|\mathbf{p}\rangle,$$
$$|\phi_{\mathbf{p}';-}{}^{(n-l)}\rangle = (G_0^\dagger V)^{(n-l)}|\mathbf{p}'\rangle, \quad (2.4)$$

and $l=0, 1, \cdots, n$. On writing

$$V(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\hat{V}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad (2.5)$$

for each factor V in (2.4), we get

$$\phi_{\mathbf{p};+}{}^{(l)}(\mathbf{r}) = \left(\frac{m}{4\pi^3}\right)^l \int \prod_{i=1}^{l} d\mathbf{k}_i g_+(\mathbf{p},\mathbf{K}_i) \times \hat{V}(\mathbf{k}_i)e^{i(\mathbf{p}-\mathbf{K}_l)\cdot\mathbf{r}},$$
$$\phi_{\mathbf{p}';-}{}^{(n-l)}(\mathbf{r}) = \left(\frac{m}{4\pi^3}\right)^{n-l} \int \prod_{j=l+1}^{n} \frac{d\mathbf{k}_j}{(2\pi)^3} g_-(\mathbf{p}',\mathbf{K}_j') \times \hat{V}(\mathbf{k}_j)e^{i(\mathbf{p}'-\mathbf{K}_n')\cdot\mathbf{r}}, \quad (2.6)$$

where

$$g_\pm(\mathbf{p},\mathbf{K}) = [\mathbf{p}^2-(\mathbf{p}-\mathbf{K})^2\pm i\epsilon]^{-1}, \quad (2.7)$$

and

$$\mathbf{K}_i=\mathbf{k}_1+\mathbf{k}_2+\cdots+\mathbf{k}_i,\quad \mathbf{K}_j'=\mathbf{k}_j+\mathbf{k}_{j+1}+\cdots+\mathbf{k}_n.$$

On use of (2.3), (2.6), and the relations $\hat{V}^*(\mathbf{k})=\hat{V}(-\mathbf{k})$, $g_-^*(\mathbf{p},\mathbf{K})=g_+(\mathbf{p},\mathbf{K})$ we see that (2.3) assumes the form

$$f_{n+1}(\mathbf{p}',\mathbf{p})=\frac{-m}{2\pi}\left(\frac{m}{4\pi^3}\right)^n\int d\mathbf{k}_1\cdots d\mathbf{k}_n[\hat{V}(\mathbf{k}_n)\cdots\hat{V}(\mathbf{k}_1)]\hat{V}(\mathbf{q}-\Sigma\mathbf{k}_i)\times D_l,\quad(2.8)$$

where

$$D_l=\prod_{j=l+1}^{n}g_+(\mathbf{p}',-\mathbf{K}_j')\prod_{i=1}^{l}g_+(\mathbf{p},\mathbf{K}_n).\quad(2.9)$$

Of course f_{n+1} does not depend on the value of l.

The energy denominators in (2.9) all have the form $(-2\mathbf{p}'\cdot\mathbf{K}-\mathbf{K}^2+i\epsilon)^{-1}$ or $(2\mathbf{p}\cdot\mathbf{K}-\mathbf{K}^2+i\epsilon)^{-1}$, where $\mathbf{K}$ is a partial sum of $\mathbf{k}$'s. The high-energy approximation which allows us to compute the right-hand side of (2.8) is essentially obtained by dropping the terms quadratic in the internal momenta, i.e., by making the replacement

$$g(\mathbf{p}',-\mathbf{K})=(-2\mathbf{p}'\cdot\mathbf{K}-\mathbf{K}^2+i\epsilon)^{-1}\rightarrow(-2\mathbf{p}'\cdot\mathbf{K}+i\epsilon)^{-1},\quad(2.10)$$
$$g(\mathbf{p},\mathbf{K})=(2\mathbf{p}\cdot\mathbf{K}-\mathbf{K}^2+i\epsilon)^{-1}\rightarrow(2\mathbf{p}\cdot\mathbf{K}+i\epsilon)^{-1}.$$

However, the resulting approximation to f_{n+1} then depends, for $\mathbf{q}\neq0$, on the choice of l. (See Sec. III for elaboration of this point. In the language of Feynman diagrams, this choice corresponds to the choice of vertex at which over-all momentum conservation is imposed.) To avoid this we first replace D_l in (2.8) by its average over l, i.e., we write

$$f_{n+1}(\mathbf{p}',\mathbf{p})=\frac{-m}{2\pi}\left(\frac{m}{4\pi^3}\right)^n\int d\mathbf{k}_1\cdots d\mathbf{k}_n[\hat{V}(\mathbf{k}_n)\cdots\hat{V}(\mathbf{k}_1)]\hat{V}(\mathbf{q}-\Sigma\mathbf{k}_i)\bar{D},\quad(2.11)$$

where

$$\bar{D}=\frac{1}{n+1}\sum_{l=0}^{n}D_l.$$

We then make the replacement which defines our eikonal approximation,

$$\bar{D}\rightarrow D^{\text{eik}}\equiv\frac{1}{n+1}\sum_{l=0}^{n}D_l^{\text{eik}},\quad(2.12)$$

in (2.11), where D_l^{eik} denotes the result of making the approximation (2.10) in D_l. Thus

$$D_l^{\text{eik}}=b_n^{-1}(b_n+b_{n-1})^{-1}\cdots(b_n+b_{n-1}+\cdots+b_{l+1})^{-1}\times(a_1+a_2+\cdots+a_l)^{-1}\cdots(a_1+a_2)^{-1}a_1^{-1},$$

where $a_i=2\mathbf{p}\cdot\mathbf{k}+i\epsilon$ and $b_i=-2\mathbf{p}'\cdot\mathbf{k}_i+i\epsilon$. Corresponding to (2.12) we have

$$f_{n+1}(\mathbf{p}',\mathbf{p})\rightarrow f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p}),$$

where

$$f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p})\equiv\frac{-m}{2\pi}\left(\frac{m}{4\pi^3}\right)^n\frac{1}{n+1}\sum_{l=0}^{n}\int d\mathbf{k}_1\cdots d\mathbf{k}_n[\hat{V}(\mathbf{k}_n)\cdots\hat{V}(\mathbf{k}_1)]\hat{V}(\mathbf{q}-\Sigma\mathbf{k}_i)D_l^{\text{eik}}.\quad(2.13)$$

To evaluate (2.13), we note that since the product of the V's is invariant under separate permutations of the momenta $\mathbf{k}_1,\cdots,\mathbf{k}_l$ and the momenta $\mathbf{k}_{l+1},\cdots,\mathbf{k}_n$, we can sum over all such permutations in the integrand, provided we divide by their total number: $l!(n-l)!$. Making use of the identity[6]

$$\sum_{\text{perm}}(c_1')^{-1}(c_1'+c_2')^{-1}\cdots(c_1'+c_2'+\cdots+c_n')^{-1}=(c_1c_2\cdots c_n)^{-1},\quad(2.14)$$

where $(c_1',c_2',\cdots,c_n')$ is a permutation of a sequence of numbers $(c_1,c_2,\cdots,c_n)$, we find

$$f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p})=\frac{(-m/2\pi)}{n+1}\left(\frac{m}{4\pi^3}\right)^n\sum_{l=0}^{n}\frac{1}{l!(n-l)!}\int d\mathbf{k}_1\cdots d\mathbf{k}_n\times\prod_{i=1}^{l}\frac{\hat{V}(\mathbf{k}_i)}{2\mathbf{p}\cdot\mathbf{k}_i+i\epsilon}\prod_{j=l+1}^{n}\frac{\hat{V}(\mathbf{k}_j)}{-2\mathbf{p}'\cdot\mathbf{k}_j+i\epsilon}\hat{V}(\mathbf{q}-\sum\mathbf{k}_i).\quad(2.15)$$

Recalling the definition (2.5), and substituting

$$V(\mathbf{q}-\Sigma\mathbf{k}_i)=\int d\mathbf{x}\;e^{i(\mathbf{q}-\Sigma\mathbf{k}_i)\cdot\mathbf{x}}V(\mathbf{x})$$

into (2.15), we see that the dependence of the integrand on the $\mathbf{k}_i$ factorizes so that (2.15) may be rewritten in the form

$$f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p})=\frac{(-m/2\pi)}{n+1}\sum_{l=0}^{n+1}\frac{1}{l!(n-l)!}\int d\mathbf{x}\;\hat{V}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\times[U(\mathbf{x};\mathbf{p})]^l[U(\mathbf{x};-\mathbf{p}')]^{n-l},\quad(2.16)$$

where

$$U(\mathbf{x};\,\mathbf{p})=\frac{2m}{(2\pi)^3}\int d\mathbf{k}\;\hat{V}(\mathbf{k})e^{-i\mathbf{q}\cdot\mathbf{x}}(2\mathbf{p}\cdot\mathbf{k}+i\epsilon)^{-1}.\quad(2.17)$$

The sum on l may now be carried out, giving, finally,

$$f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p})=\frac{(-m/2\pi)}{(n+1)!}\int d\mathbf{x}\;V(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\times[U(\mathbf{x};\,\mathbf{p})+U(\mathbf{x};\,-\mathbf{p}')]^n.\quad(2.18)$$

In correspondence with (2.1), we define

$$f_{\text{pot}}^{\text{eik}}(\mathbf{p}',\mathbf{p})=\sum_{n=0}^{\infty}f_{n+1}^{\text{eik}}(\mathbf{p}',\mathbf{p}).\quad(2.19)$$

On reversing the order of summation in (2.19) and inte-

[6] See Eq. (20) of Ref. 5. A proof is given in the Appendix.

gration in (2.18), we get

$$f_{\rm pot}{}^{\rm eik}(\mathbf{p}',\mathbf{p})=\frac{-m}{2\pi i}\int d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}V(\mathbf{x})\frac{e^{i\chi_{\rm pot}}-1}{\chi_{\rm pot}}, \quad (2.20)$$

where $\chi_{\rm pot}$ is defined by

$$\chi_{\rm pot}=-i[U(\mathbf{x};\mathbf{p})+U(\mathbf{x};-\mathbf{p}')]. \quad (2.21)$$

Equations (2.20) and (2.21), which coincide with (1.3) and (1.4), are the principal results of this section. To see the identity of (2.21) with (1.4), note that the representation (we suppress a convergence factor $e^{-\epsilon\xi}$)

$$(\omega+i\epsilon)^{-1}=-i\int_0^\infty d\xi\, e^{i\omega\xi} \quad (2.22)$$

may be used to write $U(\mathbf{x};\,\mathbf{p})$ and $U(\mathbf{x};\,-\mathbf{p}')$ in the form

$$U(\mathbf{x};\,\mathbf{p})=\frac{-im}{|\mathbf{p}|}\int_0^\infty d\xi\, V(\mathbf{x}-\hat{p}\xi),$$
$$U(\mathbf{x};\,-\mathbf{p}')=\frac{-im}{|\mathbf{p}|}\int_0^\infty d\xi\, V(\mathbf{x}+\hat{p}'\xi), \quad (2.23)$$

with $\hat{p}=\mathbf{p}/|\mathbf{p}|$, $\hat{p}'=\mathbf{p}'/|\mathbf{p}|$.

Our eikonal function $\chi_{\rm pot}=\chi_{\rm pot}(\mathbf{x};\,\mathbf{p}',\mathbf{p})$ is clearly invariant under the time-reversal transformation $\mathbf{p}\to-\mathbf{p}'$ and $\mathbf{p}'\to-\mathbf{p}$ and hence so is $f_{\rm pot}{}^{\rm eik}(\mathbf{p}',\mathbf{p})$. The invariance of $f^{\rm eik}$ under space inversion: $\mathbf{p}\to-\mathbf{p}$, $\mathbf{p}'\to-\mathbf{p}'$ holds if $V(\mathbf{x})=V(-\mathbf{x})$, i.e., if V has this invariance.

For sufficiently small scattering angles θ, we may let $\hat{p}'\to\hat{p}$ in (2.23), giving

$$\chi_{\rm pot}\approx\frac{-m}{|\mathbf{p}|}\int_{-\infty}^\infty d\xi\, V(\mathbf{x}+\hat{p}\xi),$$

or, choosing $\hat{p}$ as the z axis and writing $\mathbf{x}=\mathbf{b}+\hat{p}z$, we have

$$\chi_{\rm pot}\approx\chi(\mathbf{b}),$$

defined by (1.2). We may then also approximate $\exp(i\mathbf{q}\cdot\mathbf{x})$ by $\exp(i\mathbf{q}\cdot\mathbf{b})$ in (2.20) and, on writing $d\mathbf{x}=d^2bdz$, we see that Eq. (2.20) for $f_{\rm pot}{}^{\rm eik}$ reduces to (1.1).

III. RELATIVISTIC EIKONAL APPROXIMATION

In this section, we derive a relativistic form of the eikonal approximation described in Sec. II. Properties of the resulting formula are discussed and particle-antiparticle scattering is considered in the remaining parts of this section.

A. Derivation of Relativistic Eikonal Amplitude

1. Preliminaries

Consider two spinless particles, a and b, with masses m_a and m_b interacting via the exchange of neutral scalar mesons of mass μ with scalar coupling; if the fields associated with these particles are denoted by ϕ_a, ϕ_b, and ϕ, the interaction Lagrangian density takes the form

$$\mathcal{L}_I=-g\phi_a{}^\dagger\phi_b\phi+\text{H.c.}$$

Let $M(s,t)$ denote the invariant Feynman amplitude for the scattering of a and b where, as usual,

$$s=(p_a+p_b)^2,\quad t=(p_a-p_a')^2;$$

and p_a (p_a') and p_b (p_b') are the initial (final) four-momenta of a and b, respectively.

Let $M_{n+1}(s,t)$ denote the contribution to $M(s,t)$ arising from all Feynman diagrams in which precisely $n+1$ mesons are exchanged between a and b, and let $k_1,\cdots,k_{n+1}$ denote the momenta of the exchanged mesons, in the order of their emission along the world line of particle a. Then

$$-iM_{n+1}=(-ig)^{2n+2}\int\prod_{j=1}^{n+1}\frac{d^4k_j}{(2\pi)^4}\tilde{\Delta}_F(k_j)\times I\times(2\pi)^4\delta(q-\sum_{i=1}^{n+1}k_i), \quad (3.1)$$

where $\tilde{\Delta}_F(k)=i/(k^2-\mu^2+i\epsilon)$ is the meson propagator, $q=p_1-p_1'=-(p_2-p_2')$ is the four-vector momentum transfer, and I is a sum of products of propagators associated with the propagation of particles a and b. To write I as an explicit function of the external and internal momenta, we imagine, for the moment, that the δ function in (3.1) is used to eliminate k_r and designate the resulting form of I by I_r. Then

$$I_r=I_r{}^{(a)}\sum_D I_r{}^{(b)}(D), \quad (3.2)$$

where

$$I_r{}^{(a)}=\Delta_F{}^a(p_a-k_1)\cdots\Delta_F{}^a(p_a-k_1-\cdots-k_{r-1})\times\Delta_F{}^a(p_a'+k_{n+1})\cdots\Delta_F{}^a(p_a'+k_{n+1}+\cdots+k_{r+1}), \quad (3.3)$$

with $\Delta_F{}^a(p)=i/(p^2-m_a{}^2+i\epsilon)$, and $I_r{}^{(b)}(D)$ is a similar product of b particle propagators, associated with a diagram D contributing to M_{n+1}. Clearly, there are precisely $(n+1)!$ diagrams D to be considered, corresponding to $(n+1)!$ distinct orders in which the momenta $k_1\cdots k_{n+1}$ may be absorbed along the world line of b.

It proves convenient to organize the sum over the diagrams in the following way. Let $E(s)$ denote the subset of diagrams in which k_r is absorbed at the sth vertex along the world line of b. For each $D\in E(s)$, there will be, say, s_1 momenta from the set $[k_1,\cdots,k_{r-1}]$ which are absorbed *before* k_r. We denote these by $(\bar{k}_1,\cdots,\bar{k}_{s_1})$, listed, say, in the order of emission along the world line of a, and abbreviate this set by (s_1). The remaining $s_2=s-s_1-1$ momenta absorbed *before* k_r necessarily come from the set $[k_{r+1},\cdots,k_{n+1}]$; we denote these momenta by $(s_2)=(\bar{k}_{s_1+1},\bar{k}_{s_1+2},\cdots,\bar{k}_{s_1+s_2})$.

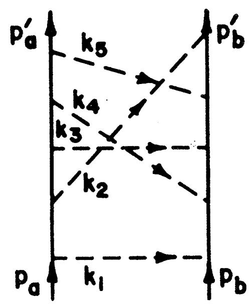

FIG. 1. A typical exchange-type Feynman diagram included in the definition of the eikonal amplitude $M^{\mathrm{eik}}(s,t)$ for the process $a+b\rightarrow a+b$.

The corresponding $I_r^b(D)$ then has the form

$$I^b(D)=\Delta_F{}^b(p_b+k_1')\Delta_F{}^b(p_b+k_1'+k_2')\cdots \Delta_F{}^b(p_b+k_1'+k_2'\cdots k_{s-1}')\Delta_F{}^b(p_2'-k_{n+1}'')\cdots \Delta_F{}^b(p_b'-k_{s+1}''-k_{s+2}''-\cdots-k_{n+1}''), \quad (3.4)$$

where $(k_1',k_2',\cdots,k_{s-1}')$ is a permutation of $(\bar{k}_1,\bar{k}_2,\cdots,\bar{k}_{s_1+s_2})$ (call it π') and $(k_{s+1}'',k_{s+2}'',\cdots,k_{n+1}'')$ is a permutation of the remaining k's, call it π''. Thus, we may write

$$I_r=I_r^{(a)}\sum_{(s_1)(s_2)}\sum_{\pi'}\sum_{\pi''}I_r^b(D). \quad (3.5)$$

From the viewpoint of the topology of the Feynman graphs contributing to M_{n+1}, the sum over π' and π'' includes all graphs in which the rth meson emitted by a is the sth meson absorbed by b and in which s_2 meson lines, emitted after r, cross the line corresponding to the rth meson to be absorbed before r, whereas the $r-1-s_1$ meson lines, emitted before r, cross the r line to be absorbed after r. As an example of the notation, for the diagram in Fig. 1, if we choose $r=3$, then we have $s=3$, $s_1=1$, $s_2=1$, $\bar{k}_1=k_1$, $\bar{k}_2=k_4$, etc. The sum over (s_1) and (s_2) may be regarded as a sum over pairs (s_1) and (s_2) with $s_1+s_2=s-1$, followed by a sum on s: $s=0, 1, \cdots, n+1$.

2. *Propagator Approximation*

We wish to make a high-p approximation in the integrand of (3.1) which will simplify it sufficiently to permit the evaluation of the integral in closed form. Basically, we wish to use, in the propagator denominators, the approximation

$$(p\pm K)^2-m^2=\pm 2p\cdot K+K^2\simeq\pm 2p\cdot K, \quad (3.6)$$

where p is an external momentum and K is a partial sum of internal momenta, thereby neglecting K^2 relative to $p\cdot K$; this is the covariant analog of (2.10), used in potential scattering. However, the integrand of (3.1) has certain symmetry properties, which we wish to preserve, but which are destroyed if such approximations are made in too naive a manner. The difficulty stems from the fact that the $n+1$ internal momenta k_i are not independent, by virtue of the δ function in (2.1), and that if, say, k_r is eliminated and then (3.6) is used, the resulting integrand depends on the chosen value of r. To make this quite clear, note that if k_r is eliminated, we must write, for the momentum of a after the emission of r, $p_a{}^{(r)}=p_a'+k_{r+1}+\cdots+k_{n+1}$, whereas, if k_{r+1} is eliminated, we must write $p_a{}^{(r)}=p_1-k_1\cdots-k_r$. Of course,

$$(p_a-k_1\cdots-k_r)^2=(p_a'+k_{r+1}+\cdots+k_{n+1})^2; \quad (3.7)$$

but, if we use (3.6), the left-hand side of (3.7) becomes, on the mass shell, $m^2-2p_a\cdot(k_1+\cdots+k_r)$, whereas the right-hand side becomes $m^2+2p_a'\cdot(k_{r+1}+\cdots+k_{n+1})$, and these quantities are no longer equal except for $q=0$.

As in the treatment of potential scattering, to avoid this feature we first write I in the form

$$I=\frac{1}{n+1}\sum_{r=1}^{n+1}I_r, \quad (3.8)$$

where $I_r=I_r(p_a'p_b';p_ap_b;k_1,k_2,\cdots,k_{r-1},k_{r+1},\cdots,k_{n+1})$ is given by (3.5) as an explicit function of the external momenta and the indicated internal momenta. Equation (3.8) provides a definition of I as a function of the external momenta and all $n+1$ internal momenta $k_1, \cdots, k_{n+1}$ which reduces to that given on the hyperplane $k_1+k_2+\cdots+k_{n+1}=q$, where, of course, each term of (3.8) makes the same contribution. We note further that $\sum I_r^b(D)$ is invariant under any permutation π_1 of $(k_1,\cdots,k_{r-1})$ and π_2 of $(k_{r+1},\cdots,k_{n+1})$. Since the product of meson propagators in (3.1) is also invariant under any such combined permutation, we may replace I_r^a in (3.5) by

$$I_{r;\mathrm{sym}}{}^a=\frac{1}{(r-1)!}\frac{1}{(n-r+1)!}\sum_{\pi_1\pi_2}I_r^a, \quad (3.9)$$

so that we consider, instead of (3.8),

$$I_{\mathrm{sym}}=\frac{1}{n+1}\sum_{r=1}^{n+1}I_{r;\mathrm{sym}}{}^a\sum_D I_r^b(D). \quad (3.10)$$

Note that Eq. (3.1), with I replaced by I_{sym}, is still exact.

We now apply the propagator approximation (3.6) to (3.10); then $I_{\mathrm{sym}}\rightarrow I^{\mathrm{eik}}$, where

$$I^{\mathrm{eik}}=\frac{1}{n+1}\sum_{r=1}^{n+1}I_{r;\mathrm{sym}}{}^{\mathrm{eik}} \quad (3.11)$$

and

$$I_{r;\,\mathrm{sym}}{}^{\mathrm{eik}}=\frac{i^{2n}}{(r-1)!(n-r+1)!}\sum_{\pi_1,\pi_2}[(a_1)^{-1}(a_1+a_2)^{-1}\cdots(a_1+a_2+\cdots+a_{r-1})^{-1}]$$
$$\times[(a_{n+1}')^{-1}(a_{n+1}'+a_n')^{-1}\cdots(a_{n+1}'+\cdots+a_{r+1}')^{-1}]\sum_{(s_1)(s_2)}\sum_{\pi',\pi''}[(b_1)^{-1}(b_1+b_2)^{-1}\cdots(b_1+b_2+\cdots+b_{s-1})^{-1}]$$
$$\times[(b_{n+1}')^{-1}(b_{n+1}'+b_n')^{-1}\cdots(b_{n+1}'+\cdots+b_{s+1}')^{-1}],\quad(3.12)$$

with the abbreviations $a_j=-2p_a\cdot k_j+i\epsilon$, $a_j'=2p_a'\cdot k_j+i\epsilon$, $b_j=2p_b\cdot k_j'+i\epsilon$, $b_{j'}=-2p_b'\cdot k_j''+i\epsilon$. Then we use the identity (3.12) to carry out the sum over the permutations π_1, π_2, π', and π'', in (3.12), giving

$$I_{r;\,\mathrm{sym}}{}^{\mathrm{eik}}=i^{2n}[(r-1)!(n-r+1)!]^{-1}[a_1a_2\cdots a_{r-1}]^{-1}[a_{r+1}'a_{r+2}'\cdots a_{n+1}']^{-1}$$
$$\times\sum_{(s_1)(s_2)}[\bar{b}_1\bar{b}_2\cdots\bar{b}_{s-1}]^{-1}[\tilde{b}_{s+1}\tilde{b}_{s+2}\cdots\tilde{b}_n]^{-1},\quad(3.13)$$

where $\bar{b}_j=2p_b\cdot\bar{k}_j+i\epsilon$ and $\tilde{b}_j=-2p_b'\cdot\tilde{k}_j+i\epsilon$; here, $(\tilde{k}_{s+1},\tilde{k}_{s+2},\cdots,\tilde{k}_n)$ is the complement of the set $(\bar{k}_1,\cdots,\bar{k}_{s-1})$ in the set $(k_1,k_2,\cdots,k_{r-1},k_{r+1},\cdots,k_{n+1})$.

3. Definition and Evaluation of $M^{\mathrm{eik}}(s,t)$

We define $M_{n+1}{}^{\mathrm{eik}}$, the eikonal approximation to M_{n+1}, as the result obtained by substituting (3.11) into (3.1). To carry out the integrations we write

$$\tilde{\Delta}_F(k_r)=\int\Delta_F(x)e^{-ik_r\cdot x}d^4x,\quad(3.14)$$

in computing the contribution of the rth term in the sum (3.11) to (3.1), with $k_r=q-\sum_{j\neq r}k_j$, on eliminating the δ function. The integrand then factorizes and each of the remaining k integrations can be carried out. Let us define

$$U(x;\,p,p')=g^2\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{\Delta}_F(k)e^{ik\cdot x}}{(-2p\cdot k+i\epsilon)(2p'\cdot k+i\epsilon)},\quad(3.15)$$

and introduce the abbreviations

$$\begin{aligned}U_1&=U(x;\,p_a,p_b),& U_2&=U(x;\,p_a,\,-p_b'),\\ U_3&=U(x;\,-p_a',p_b),& U_4&=U(x;\,-p_a',\,-p_b').\end{aligned}\quad(3.16)$$

For a given choice of (s_1) and (s_2), there will be, in (3.12), s_1 factors of the type $(-2p_a\cdot k)^{-1}$ $(+2p_b\cdot k)^{-1}$, giving s_1 factors U_1 on integrations over the corresponding k's (i.e., a factor $U_1{}^{s_1}$). Similarly, there will be s_2 factors of the type $(+2p_a'\cdot k)^{-1}$ $(2p_b\cdot k)^{-1}$, giving a factor $U_3{}^{s_2}$; $(r-1)-s_1$ factors of the type $(-2p_a\cdot k)^{-1}$ $(-2p_b'\cdot k)^{-1}$, giving a factor $U_2{}^{r-1-s}$; and $(n-r+1)-s_2$ factors of the type $(2p_a'\cdot k)^{-1}$ $(-2p_b'\cdot k)^{-1}$, giving a factor $U_4{}^{n-r+1-s_2}$. It follows that the contribution of one of the terms in the sum (3.12) will be proportional to the integral over x, with a factor $e^{-iq\cdot x}\Delta_F(x)$, of the function

$$\frac{1}{(r-1)!}\frac{1}{(n-r+1)!}U_1{}^{s_1}U_2{}^{r-1-s_1}U_3{}^{s_2}U_4{}^{n-r+1-s_2}.$$

Thus each choice of (s_1) and (s_2) gives the *same* contribution. Since the number of ways of choosing s_1 momenta from $r-1$ is $(r-1)!/s_1!(r-1-s_1)!$, and s_2 momenta from $n-r+1$ is $(n-r+1)!/s_2!(n-r+1-s_2)!$, the sum over sets (s_1) and (s_2) with fixed values of s_1 and s_2 gives, using binomial notation,

$$\frac{1}{(r-1)!}\frac{1}{(n-r+1)!}\binom{r-1}{s_1}U_1{}^{s_1}U_2{}^{s-1-r_1}\times\binom{n-r+1}{s_2}U_3{}^{s_2}U_4{}^{n-r+1-s_2}.$$

The sum over all s_1 and s_2 then gives

$$\frac{1}{(r-1)!}\frac{1}{(n-r+1)!}(U_1+U_2)^{r-1}(U_3+U_4)^{n-r+1}.$$

The sum over r ($r=1,\,2,\,\cdots,\,n+1$) of the last expression is

$$(1/n!)(U_1+U_2+U_3+U_4)^n.$$

Collecting the constant factors, we get, finally,

$$-iM_{n+1}{}^{\mathrm{eik}}=\frac{g^2}{(n+1)!}\int d^4x\;e^{-iq\cdot x}\Delta_F(x)(i\chi)^n,\quad(3.17)$$

where

$$\chi\equiv-i(U_1+U_2+U_3+U_4).\quad(3.18)$$

It follows that $M^{\mathrm{eik}}(s,t)$, defined by

$$M^{\mathrm{eik}}(s,t)\equiv\sum_{n=0}^{\infty}M_{n+1}{}^{\mathrm{eik}},\quad(3.19)$$

is given, on reversing the order of summation and integration, by

$$M^{\mathrm{eik}}(s,t)=ig^2\int d^4x\;e^{-iq\cdot x}\Delta_F(x)\sum_{n=0}(i\chi)^n/(n+1)!$$

or

$$M^{\mathrm{eik}}(s,t)=g^2\int d^4x\;e^{-iq\cdot x}\Delta_F(x)\frac{e^{i\chi}-1}{\chi}.\quad(3.20)$$

This is the result stated in the Introduction, since the definition (3.18) of χ is equivalent to (1.6), as we show below.

B. Properties of Relativistic Eikonal Function χ and Eikonal Amplitude M^{eik}

1. *Alternative Forms of χ and Symmetry Properties*

It is interesting to examine the various forms of the function χ, which represents a relativistic generalization of the eikonal function as defined in the study of nonrelativistic potential scattering. From (3.15), (3.16), and (3.12), we have

$$\chi=-ig^2\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\tilde{\Delta}_F(k)\times d(k;\,p_a',p_a)d(-k;\,p_b',p_b)\,, \quad (3.21)$$

where

$$d(k;\,p',p)=[d_+(2p'\cdot k)+d_-(2p\cdot k)]\,, \quad (3.22)$$

with $d_\pm(\omega)\equiv(\pm\omega+i\epsilon)^{-1}$.

It is convenient to introduce the four-velocities $u_a=p_a/m_a$, $u_a'=p_a'/m_a$, $u_b=p_b/m_b$, $u_b'=p_b'/m_b$ and to use (2.22) to write

$$d(k;\,p_a',p_a)=\frac{-i}{2m_a}\int_0^\infty d\zeta_a(e^{i(u_a'\cdot k)\zeta_a}+e^{-iu_a\cdot k\zeta_a})\,,$$

with a similar representation for $d(-k;\,p_b',p_b)$. We then get, on substituting into (3.21),

$$\chi=if^2\int_0^\infty\int_0^\infty d\zeta_a d\zeta_b[\Delta_F(x+u_a'\zeta_a-u_b'\zeta_b)+\Delta_F(x+u_a'\zeta_a+u_b\zeta_b)+\Delta_F(x-u_a\zeta_a-u_b'\zeta_b)+\Delta_F(x-u_a\zeta_a+u_b\zeta_b)]\,, \quad (3.23)$$

with $f^2=g^2/4m_am_b$; this coincides with (1.6). For small-angle deflections, $u_a'\approx u_a$, $u_b'\approx u_b$ and (3.23) becomes simply

$$\chi\approx if^2\int_{-\infty}^\infty\int_{-\infty}^\infty d\zeta_a d\zeta_b\Delta_F(x+u_a\zeta_a+u_b\zeta_b)\,, \quad (3.24)$$

which is a covariant analog of (1.2), if V is a Yukawa potential.

The symmetry properties of $\chi=\chi(p_a',p_b';\,p_a,p_b;\,x)$ are easily established, either from (3.21), using the relations $d(k;\,p',p)=d(k;\,-p,\,-p')=d(-k;\,p,p')$, or from (3.23), using $\Delta_F(\eta)=\Delta_F(-\eta)$, together with appropriate change of integration variables. Thus one finds that χ is crossing-symmetric,

$$\chi(p_a',p_b';\,p_a,p_b;\,x)=\chi(-p_a,p_b';\,-p_a',\,p_b;\,x)=\chi(p_a',\,-p_b;\,p_a,\,-p_b';\,x)\,, \quad (3.24')$$

that χ is invariant under space inversion of x and the p's,

$$\chi(p_a',p_b';\,p_a,p_b;\,x)=\chi(\bar{p}_a',\bar{p}_b';\,\bar{p}_a,\bar{p}_b;\,\bar{x})\,, \quad (3.24'')$$

with $\bar{p}=(p^0,\,-p)$ and $\bar{x}=(x^0,\,-x)$, and that χ is invariant under the interchange of initial and final momenta,

$$\chi(p_a',p_b';\,p_a,p_b'\,x)=\chi(p_a,p_b;\,p_a',p_b';\,x)\,. \quad (3.24''')$$

Using (3.21) and (3.24′)–(3.24‴), we see that $M^{\text{eik}}=M^{\text{eik}}(s,t)=M^{\text{eik}}(p_a',p_b';\,p_a,p_b)$ is crossing-symmetric:

$$M(p_a',p_b';\,p_a,p_b)=M(-p_a,\,p_b';\,-p_a',\,p_b)=M(p_a'-p_b;\,p_a,\,-p_b')\,; \quad (3.25\text{a})$$

invariant under space inversion:

$$M(p_a',p_b';\,p_a,p_b)=M(\bar{p}_a',\bar{p}_b';\,\bar{p}_a,\bar{p}_b)\,; \quad (3.25\text{b})$$

and invariant under time reversal:

$$M(p_a',p_b';\,p_a,p_b)=M(\bar{p}_a,\bar{p}_b;\,\bar{p}_a',\bar{p}_b')\,. \quad (3.25\text{c})$$

2. *Static Limit*

It is instructive to see how χ reduces to the form expected from potential scattering in the static limit, i.e., when, say, $m_b\to\infty$. To this end, it is necessary to observe first, that for slowly moving particles a and b, the use of second-order perturbation theory with the interaction (3.1) gives an interparticle potential V of the Yukawa type,

$$V(\mathbf{r})=-\frac{f^2}{4\pi}\frac{e^{-\mu|\mathbf{r}|}}{|\mathbf{r}|}\,, \quad (3.26)$$

with f a reduced coupling constant,

$$f^2=g^2/4m_am_b\,. \quad (3.27)$$

Thus, to make contact with potential scattering, we keep f^2 fixed as $m_b\to\infty$. Since both u_b' and u_b approach (1,0,0,0) as $m_b\to\infty$, the first two and last two terms in (3.23) can be combined by letting the ζ_b integration run from $-\infty$ to ∞:

$$\chi\to if^2\int_0^\infty d\zeta_a\int_{-\infty}^\infty d\zeta_b[\Delta_F(x^0+u_a'^0\zeta_a-\zeta_b;\,\mathbf{x}+\mathbf{u}_a'\zeta_a)+\Delta_F(x^0-u_a^0\zeta_a-\zeta_b;\,\mathbf{x}-\mathbf{u}_a\zeta_a)]\,. \quad (3.28)$$

On use of the relation

$$\int_{-\infty}^\infty dx^0\,\Delta_F(x^0;\,\mathbf{x})=-i\frac{e^{-\mu|\mathbf{x}|}}{4\pi|\mathbf{x}|}\,, \quad (3.29)$$

the right-hand side of (3.28) is seen to be equal to

$$-\int_0^\infty d\zeta_a[V(\mathbf{x}+\mathbf{u}_a'\zeta_a)+V(\mathbf{x}-\mathbf{u}_a\zeta_a)]\,, \quad (3.30)$$

where V is defined by (3.26). Noting that $m_b\to\infty$ implies $q^0=-(p_b^0-p_b'^0)\to 0$, which in turn implies that $|u_a'|=|u_a|$, we see, on putting $\zeta_a=|\mathbf{u}_a|^{-1}\xi$, and dropping the subscript a, that (3.30) is just equal to the expected χ_{pot}. The same conclusion can be reached by starting with (3.21) and using the fact that in the static limit the last factor in the integrand is proportional to $d_+(k^0)+d_-(k^0)=-2\pi i\delta(k^0)$.

For completeness, we note that the scattering amplitude in the relativistic eikonal approximation

$$f^{\rm eik}=(-1/8\pi\sqrt{s})M^{\rm eik}(s,t) \tag{3.31}$$

also assumes the expected form in the static limit. Using (3.20) and (3.30), we have, for $m_b\to\infty$,

$$M^{\rm eik}\sim g^2\int d^4x\, e^{-iq\cdot x}\Delta_F(x)\frac{e^{i\chi_{\rm pot}}-1}{\chi_{\rm pot}}.$$

Since $\chi_{\rm pot}$ is independent of time, and $q^0\to 0$ for $m_b\to\infty$, we find, using (3.29) again, that

$$M^{\rm eik}\sim -i(4m_am_b)\int d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}V(\mathbf{x})\frac{e^{i\chi_{\rm pot}}-1}{\chi_{\rm pot}}. \tag{3.32}$$

Since $m_b/\sqrt{s}\to 1$ as $m_b\to\infty$, we have from (3.31) and (3.32),

$$f^{\rm eik}\to f_{\rm pot}{}^{\rm eik}(\mathbf{p}_a{}',\mathbf{p}_a), \tag{3.33}$$

where $f_{\rm pot}{}^{\rm eik}$ is defined by (2.20).

C. Particle-Antiparticle Amplitude

It is instructive to consider the particle-antiparticle scattering amplitude $M_{a\bar{a}}$ within the framework of our model and eikonal approximation. In addition to the amplitude obtained from diagrams of the type shown in Fig. 1, we now also include the corresponding annihilation-type diagrams, as typified by Fig. 2. Let us denote the initial and final momenta by p_a, $p_{\bar{a}}$ and $p_a{}'$, $p_{\bar{a}}{}'$, respectively. The contribution of the annihilation-type diagrams can be obtained from the exchange-type diagrams by the crisscrossing transformation $p_{\bar{a}}\leftrightarrow -p_a{}'$. Thus, we define an eikonal approximation to $M_{a\bar{a}}$ by

$$M_{a\bar{a}}{}^{\rm eik}=M_1{}^{\rm eik}+M_2{}^{\rm eik}, \tag{3.34}$$

where $M_1{}^{\rm eik}$ is just the amplitude defined by Eq. (3.20), with $b=\bar{a}$ and $M_2{}^{\rm eik}$ is obtained from M_1 by crisscrossing:

$$M_1{}^{\rm eik}=-g^2\int d^4x\,\Delta_F(x)e^{-iq\cdot x}(e^{i\chi_1}-1)/\chi_1, \tag{3.35}$$

with $q=p_a-p_a{}'$ and $\chi_1=\chi(p_a{}',p_{\bar{a}}{}';\, p_a,p_{\bar{a}})$, and

$$M_2{}^{\rm eik}=-g^2\int d^4x\,\Delta_F(x)^{-iP\cdot x}(e^{i\chi_2}-1)/\chi_2, \tag{3.36}$$

with $P=p_a+p_{\bar{a}}$ and $\chi_2=\chi(-p_{\bar{a}},p_{\bar{a}}{}';\, p_a,-p_a{}';\, x)$. In terms of the function $U(x;\,p,p')$ defined by Eq. (3.15), we have, using the definition (3.18) of χ,

$$\chi_1=-i[U(x;\,p_a,p_{\bar{a}})+U(x;\,p_a,-p_{\bar{a}}{}')+U(x;\,-p_a{}',p_{\bar{a}}) +U(x;\,-p_a{}',-p_{\bar{a}}{}')] \tag{3.37}$$

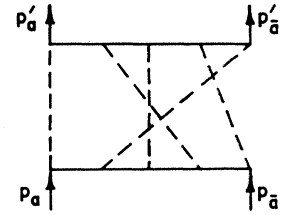

FIG. 2. A typical annihilation-type Feynman diagram included in the definition of $M_2{}^{\rm eik}(s,t)$, part of the eikonal amplitude for the process $a+\bar{a}\to a+\bar{a}$. Its contribution is obtainable from Fig. 1 with $b=\bar{a}$ by the criss-crossing transmation $p_a{}'\leftrightarrow -p_a$.

and

$$\chi_2=-i[U(x;\,p_a,-p_a{}')+U(x;\,p_a,-p_{\bar{a}}{}'), +U(x;\,p_{\bar{a}},-p_a{}')+U(x;\,p_{\bar{a}},-p_{\bar{a}}{}')]. \tag{3.38}$$

It is particularly interesting to investigate the high-energy behavior of $M_2{}^{\rm eik}$, since it is known that $M_2{}^{\rm ladder}$, the sum of all ladder-type annihilation diagrams, exhibits an asymptotic behavior of the Regge type.[7] We note that for $s=P^2\to\infty$, major contributions to the integral over x in (3.36) will come from small values of x, because of the oscillations in $e^{iP\cdot x}$ for large P and because χ_2 has a logarithmic singularity at $x=0$. In particular, it can be shown that[8] if in $U(x;\,p,p')$ we set $x\cdot p=x\cdot p'=0$, then χ_2 assumes the form

$$\bar{\chi}_2=i[\bar{\alpha}(t)+\bar{\alpha}(u)]\ln|x^2|, \tag{3.39}$$

where $t=(p_a-p_a{}')^2$, $u=(p_a-p_a{}')^2$, and

$$\bar{\alpha}(\Delta^2)=\frac{g^2}{4\pi^2}\int_0^1\frac{d\zeta}{4p^2-\Delta^2(1-\zeta^2)}. \tag{3.40}$$

The existence of this singularity may be connected to a Regge-like behavior (another example of this mechanism is given in Sec. IV C in our discussion of Compton scattering). The precise asymptotic behavior of $M_2{}^{\rm eik}$ will be discussed elsewhere. Here, we remark only that $\alpha(t)=-1+\alpha(t)$ coincides with the trajectory function obtained by summing the asymptotic parts of all ladder diagrams contributing to the annihilation amplitude.[7]

IV. EXTENSIONS TO OFF-SHELL PROCESSES AND APPLICATIONS

In this section, we show how our method can be extended to off-shell processes by calculating, with the same approximation, the propagator of a scalar particle in an external field and a class of radiative corrections to the vertex function. In another application, we discuss a class of radiative corrections to the Compton scattering amplitude.

A. Propagator of a Scalar Particle in an External Field

Let us consider the Green's function $K^A(x,x')$ of a scalar particle of mass m in a time-dependent external field $A(x)$, whose Fourier transform we denote by

[7] See, e.g., R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge University Press, Cambridge, 1967), p. 140, and references quoted there.

[8] See the Appendix of Ref. 5.

$\hat{A}(p)$. In momentum space, this can be written as

$$K^A(p',p)=(2\pi)^4\tilde{\Delta}_F(p)\delta(p'-p)+\tilde{\Delta}_F(p')M^A(p,p)\tilde{\Delta}_F(p)\,, \quad (4.1)$$

where $M^A(p',p)$ is a generalized scattering amplitude which can be expanded in a power series of the Fourier components of the external field (we call $q=p-p'$):

$$M^A(p',p)=\sum_{n=0}^{\infty}(-i)^{n+1}\frac{1}{(2\pi)^{4n}}\int\hat{A}(k_n)\times\tilde{\Delta}_F(p'+p_n)\hat{A}(k_{n-1})\cdots\tilde{\Delta}_F(p'+k_n+\cdots+k_{l+1})\times\hat{A}(q-\Sigma k_i)\tilde{\Delta}_F(p-k_1-\cdots-k_l)\hat{A}(k_l)\cdots\tilde{\Delta}_F(p-k_1)\hat{A}(k_1)d^4k_1\cdots d^4k_n\,. \quad (4.2)$$

In a way similar to that of the previous sections, we approximate the $\tilde{\Delta}_F$ functions by neglecting terms of order k_i^2. But this time, we take into account the fact that p and p' are off shell. Calling $p^2-m^2=2mz$ and $p'^2-m^2=2mz'$ and averaging over the choices of l, as before, we define

$$M_{\text{eik}}{}^A(p',p)=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{l=1}^{n+1}M_{n+1,l}{}^A(p',p)\,, \quad (4.3)$$

where

$$M_{n+1,l}{}^A(p',p)=\frac{1}{(2\pi)^{4n}}\int\frac{\hat{A}(k_n)}{2mz'+2p'\cdot k_n+i\epsilon}\cdots\frac{\hat{A}(k_{l+1})}{2mz'+2p'\cdot(k_n+\cdots+k_{l+1})+i\epsilon}\times\hat{A}(q-\Sigma k_i)\frac{\hat{A}(k_l)}{2mz-2p\cdot(k_1+\cdots+k_l)+i\epsilon}\cdots\frac{\hat{A}(k_1)}{2mz-2p\cdot k_1+i\epsilon}d^4k_1\cdots d^4k_n\,. \quad (4.4)$$

We now use the general identity proved in the Appendix [Eq. (A2)] to permute separately the momenta $k_1\cdots k_l$ on the one hand, and the momenta $k_{l+1}\cdots k_n$ on the other. $M_{n+1,l}{}^A(p',p)$ is then given, with $u=p/m$, $u'=p'/m$, by

$$M_{n+1,l}{}^A(p',p)=\frac{1}{(2\pi)^{4n}}\frac{1}{(2m)^n}\frac{-zz'}{l!(n-l)!}\int_0^\infty\int_0^\infty e^{i\beta z+i\beta'z'}d\beta\,d\beta'\times\int\prod_{i=l+1}^{n}\frac{\hat{A}(k_i)[1-e^{i\beta'u'\cdot k_i}]}{(2p'\cdot k_i/2m+i\epsilon)}\hat{A}(q-\Sigma k_i)\times\prod_{i=1}^{l}\frac{\hat{A}(k_i)[1-e^{-i\beta u\cdot k_i}]}{(-2p\cdot k_i/2m+i\epsilon)}\prod_{j=1}^{n}d^4k_j\,. \quad (4.5)$$

Calling

$$U(x;\beta)=\frac{1}{(2\pi)^4}\int e^{-ik\cdot x}\frac{\hat{A}(k)}{-2p\cdot k+i\epsilon}[1-e^{-i\beta u\cdot k}]d^4k \quad (4.6)$$

and

$$U'(x;\beta')=\frac{1}{(2\pi)^4}\int e^{-ik\cdot x}\frac{\hat{A}(k)}{2p'\cdot k+i\epsilon}[1-e^{i\beta'u'\cdot k}]d^4k\,, \quad (4.7)$$

we obtain, in the same way as before,

$$M_{n+1,l}{}^A(p',p)=\frac{-zz'}{l!(n-l)!}\int_0^\infty\int_0^\infty e^{i\beta z+i\beta'z'}d\beta\,d\beta'\times\int[U'(x;\beta')]^{n-l}A(x)[U(x;\beta)]^l e^{iq\cdot x}d^4x\,, \quad (4.8)$$

so that, putting

$$\chi(x;\beta,\beta')=-i(U+U')\,, \quad (4.9)$$

we obtain for $M_{\text{eik}}{}^A(p',p)$

$$M_{\text{eik}}{}^A(p',p)=izz'\int_0^\infty\int_{e}^\infty e^{i\beta z+i\beta'z'}d\beta\,d\beta'\times\int A(x)e^{iqx}\left(\frac{e^{i\chi}-1}{\chi}\right)d^4x\,. \quad (4.10)$$

The phase χ can be transformed as before, using the integral representation of the δ_+ function. For example, we have

$$U=-\frac{i}{2m}\frac{1}{(2\pi)^4}\int_0^\infty ds\int\hat{A}(k)e^{-iu\cdot ks}(1-e^{-i\beta u\cdot k})e^{-ik\cdot x}d^4k=-\frac{i}{2m}\int_0^\infty ds\,[A(x+us)-A(x+us+u\beta)]\,,$$

or

$$U=-\frac{i}{2m}\int_0^\beta A(x+us)ds\,. \quad (4.11)$$

We have a similar expression for U', so that altogether

$$\chi=-\frac{i}{2m}\left[\int_0^\beta A(x+us)ds+\int_0^{\beta'}A(x-u's)ds\right]. \quad (4.12)$$

The on-shell limit of Eq. (4.10), that is, the limit when z and z' tend to zero, is obtained by replacing $\chi(x;\beta,\beta')$ with $\chi(x;+\infty,+\infty)=\chi_0$ and by doing the β and β' integrations, which supply a factor $(-zz')^{-1}$. This gives, of course, the relativistic external field analog of Eq. (2.20):

$$M_{\text{eik}}{}^A(p',p)=-i\int A(x)e^{iq\cdot x}\left(\frac{e^{i\chi_0}-1}{\chi_0}\right)d^4x\,. \quad (4.13)$$

Our expression for the propagator in the eikonal approximation is, using (4.10),

$$K_{\mathrm{eik}}{}^{A}(p',p)=(2\pi)^4\Delta_F(p)\delta(p-p')$$
$$-\frac{i}{4m^2}\int_0^\infty\int_0^\infty e^{i\beta z+i\beta' z'}d\beta\, d\beta'\int A(x)e^{iq\cdot x}\left(\frac{e^{i\chi}-1}{\chi}\right)d^4x. \tag{4.14}$$

In the limit where z and z' become infinite (extreme off-shell behavior), the phase χ tends to $\chi(x;\,0,0)=0$, so that the second term of the right-hand side of (4.14) tends to zero like $(zz')^{-1}$. The propagator then tends to its free limit, as one would expect.

B. Radiative Corrections to the Vertex Function

We now use the techniques of Sec. III to sum, in closed form, a certain class of diagrams contributing to the amplitude $\Gamma(p',p)$ for emission of a virtual meson by particle a or b, in the framework of our propagator approximation.

Thus, let D denote a vertex diagram of order $2n+1$ with the property that if one follows the world line of the particle, there are n successive emissions of mesons, followed by the emission of the "vertex" meson, followed by the absorption of the n mesons. (See Fig. 3.) Let $\Gamma_{2n+1}(p_a',p_a)$ denote the sum of the contributions $\Gamma_{2n+1}(D)$ from the diagrams with the prescribed topology. Then

$$\Gamma_{2n+1}(p_a',p_a)=g(-ig)^{2n}\times\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\prod_{i=1}^n\tilde{\Delta}_F(k_i)\times J, \tag{4.15}$$

where

$$J=\frac{1}{n!}\sum_{\pi'}\Delta_F{}^a(p-k_1')\Delta_F{}^a(p-k_1'-k_2')\cdots\Delta_F{}^a(p-k_1'-k_2'-\cdots-k_n')$$
$$\times\sum_{\pi''}\Delta_F{}^a(p'-k_1'')\Delta_F{}^a(p'-k_1''-k_2'')\cdots\Delta_F{}^a(p'-k_1''-k_2''-\cdots-k_n''), \tag{4.16}$$

and $(k_1',\cdots,k_n')$ and $(k_1'',\cdots,k_n'')$ are permutations π' and π'' of $(k_1,\cdots,k_n)$.

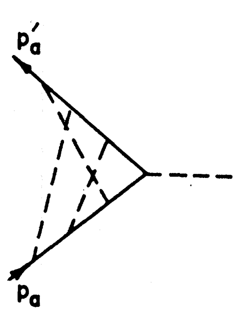

FIG. 3. Typical diagram contributing to $\Gamma_s(p_a',p_a)$—a part of the vertex function. The mesons emitted before the "vertex" meson are required to be absorbed after it, along the world line of the particle.

Let us consider first the on-mass-shell case: $p^2=p'^2=m_a^2$. The propagator approximation (3.6) followed by use of the identity (2.14) then gives $J\rightarrow J^{\mathrm{eik}}$, where

$$J^{\mathrm{eik}}\equiv(i^{2n}/n!)[(-2p_a\cdot k_1+i\epsilon)^{-1}\times(-2p_a\cdot k_2+i\epsilon)^{-1}\cdots(-2p_a\cdot k_n+i\epsilon)^{-1}]$$
$$\times[(-2p_a'\cdot k_1+i\epsilon)^{-1}(-2p_a\cdot k_2+i\epsilon)^{-1}\cdots(-2p_a'\cdot k_n+i\epsilon)^{-1}]. \tag{4.17}$$

Correspondingly,

$$\Gamma_{2n+1}(p_a',p_a)\rightarrow\Gamma_{2n+1}{}^{\mathrm{eik}}(p_a',p_a),$$

where

$$\Gamma_{2n+1}{}^{\mathrm{eik}}=(g/n!)[iK(p_a',p_a)]^n. \tag{4.18}$$

Here $K(p_a',p_a)$ is just an abbreviation for $U(0;\,p_a,-p_a')$ [Eq. (3.15)], i.e.,

$$K(p_a',p_a)=g^2\int\frac{d^4k}{(2\pi)^4}\tilde{\Delta}_F(k)(-2p_a'\cdot k+i\epsilon)^{-1}\times(-2p_a\cdot k+i\epsilon)^{-1}. \tag{4.19}$$

Thus $\Gamma_s{}^{\mathrm{eik}}$, defined by

$$\Gamma_s{}^{\mathrm{eik}}(p_a',p_a)\equiv\sum_{n=0}^\infty\Gamma_{2n+1}{}^{\mathrm{eik}}(p_a',p_a), \tag{4.20}$$

is given by

$$\Gamma_s{}^{\mathrm{eik}}(p_a',p_a)=ge^{iK(p_a',p_a)}. \tag{4.21}$$

To obtain an extension of this result to the case where a is not on the mass shell, we use again the generalization of (2.14), since there is then a constant additive term $p_a^2-m_a^2$ or $p_a'^2-m_a^2$ in the denominators in J. Thus, on making the propagator approximation in Eq. (4.16), we get $J\rightarrow J^{\mathrm{eik}}$, where now

$$J^{\mathrm{eik}}=gL(p_a,\eta_a;\,k_1,k_2,\cdots,k_n)\times L(p_a',\eta_a';\,k_1,k_2,\cdots,k_n), \tag{4.22}$$

with

$$\eta_a=-i(p_a^2-m_a^2),\quad \eta_a'=-i(p_a'^2-m_a^2), \tag{4.23}$$

and

$$L(p,\eta;\,k_1,k_2,\cdots,k_n)=\sum_\pi\frac{1}{a_1'+\eta}\,\frac{1}{a_1'+a_2'+\eta}\cdots\frac{1}{a_1'+a_2'+\cdots+a_n'+\eta}, \tag{4.24}$$

with $a_i=i(2p\cdot k_i)+\epsilon$ and $(a_1',\cdots,a_n')$ a permutation π of $(a_1,\cdots,a_n)$. Using, for variety, Eq. (3) of the Appendix to rewrite L, we get, on substitution into (4.15),

$$\Gamma_{2n+1}{}^{\mathrm{eik}}(p_a',p_a)=\frac{g}{n!}\int_0^\infty d\beta'\int_0^\infty d\beta\, e^{-\beta'\eta'}e^{-\beta\eta}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\beta'}\times[iK(p_a',p_a;\,\beta'\beta)]^n, \tag{4.25}$$

where

$$K(p',p;\beta',\beta)=-ig^2\int\frac{d^4k}{(2\pi)^4}\tilde{\Delta}_F(k)\times\frac{1-e^{i\beta'(-2p'\cdot k-i\epsilon)}}{-2p'\cdot k+i\epsilon}\,\frac{1-e^{i\beta(-2p\cdot k+i\epsilon)}}{-2p\cdot k+i\epsilon}. \quad (4.26)$$

Using (4.15) in Eq. (4.20), we now find that

$$\Gamma_s^{\text{eik}}(p_a',p_a)=g\int_0^\infty d\beta'\int_0^\infty d\beta\times e^{-\beta'\eta'}e^{-\beta\eta}\frac{\partial}{\partial\beta'}\frac{\partial}{\partial\beta}e^{iK(p_a',p_a;\beta'\beta)}. \quad (4.27)$$

Since $K(p',p;\infty,\infty)=K(p',p)$, it is clear that (4.27) reduces to (4.21) on the mass shell, $\eta'=\eta=0$.

It must be noted that we have been rather cavalier in the above discussion—the integrals (4.19) and (4.26) are actually logarithmically divergent. This could be remedied, for example, by modifying the denominators as described in Ref. 14. The combinatorics remain the same.

C. Class of Radiative Corrections to Compton Scattering Amplitude

Another application of our techniques consists in summing a series of radiative corrections to the Compton scattering amplitude, represented in Figs. 4(a) and 4(b). If we call p and p' the initial and final electron momenta, q and q' the photon momenta, then $P=p+q=p'+q'$, $Q=p-q'$, $P^2=s$, $Q^2=u$, $(p-p')^2=t$. The amplitude corresponding to the exchange of n virtual photons and represented in Fig. 4(a) can be written

$$M_n^{(a)}=\left(\frac{-i\alpha}{4\pi^3}\right)^n\sum_{\text{perm}\{k_i',\mu_i'\}}\int\gamma_{\mu_1}\frac{1}{\not p'-\not k_1'-m}\times\gamma_{\mu_2}\frac{1}{\not p'-\not k_1-\not k_2-m}\cdots\gamma_{\mu_n}\frac{1}{\not p'-\Sigma\not k_i-m}\times M_0^{(a)}(P-\Sigma k_i)\gamma_{\mu_n'}\frac{1}{\not p-\Sigma\not k_i'-m}\gamma_{u_{n-1}'}\cdots\frac{1}{\not p-\not k_1'-m}\gamma_{\mu_1'}\prod_i\frac{d^4k_i}{k_i^2-\lambda^2}. \quad (4.28)$$

In Eq. (4.28), the polarizations $\mu_1'\cdots\mu_n'$ and the momenta $k_1'\cdots k_n'$ are an arbitrary permutation of $\mu_1\cdots\mu_n$ and $k_1\cdots k_n$, respectively. We have introduced a small mass λ for the photon. $M_0^{(a)}(P)$ is the lowest-order Compton scattering amplitude corresponding to Fig. 4(a) without radiative corrections. The amplitude corresponding to Fig. 4(b) can be obtained from Eq. (4.28) by replacing $M_1^{(a)}(P-\Sigma k_i)$ with $M_0^{(b)}(Q-\Sigma k_i)$. Since $M_n^{(a)}$ is to be calculated between Dirac spinors,

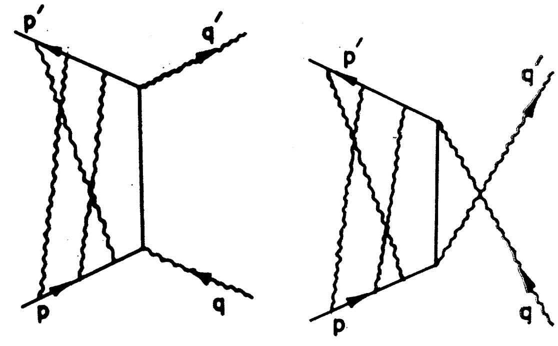

FIG. 4. Typical diagrams included in the study of radiative corrections to Compton scattering. In type (a), the virtual photons emitted before the absorption of the initial photon are required to be absorbed after the emission of the final photon. Type (b) is obtained from (a) by crossing.

we can get rid of the γ_μ matrices as follows: We first write the first factor of the integrand, in Eq. (4.28),

$$\gamma_{\mu_1}\frac{(\not p'-\not k_1-m)}{k_1^2-2p'\cdot k_1+i\epsilon}\rightarrow\gamma_{\mu_1}\frac{(\not p'-m)}{-2p'\cdot k_1+i\epsilon}, \quad (4.29)$$

where the approximation consists in neglecting k_1 in the numerator and k_1^2 in the denominator. If we apply the first factor of Eq. (4.28) on $\bar{u}(p')$ on the left and make use of the Dirac equation, the complete factor becomes simply $2p_{\mu_1}'(-2p'\cdot k_1+i\epsilon)^{-1}$. We then use the same method for the second factor, and so on. Similarly, we replace the first factor on the right, in the last line of Eq. (4.28), by $2p_{\mu_1'}(-2p\cdot k_1'+i\epsilon)^{-1}$, etc. Correspondingly, we define

$$M_{n;\text{eik}}^{(a)}=\left(-\frac{i\alpha}{4\pi^3}\right)\sum_{\text{perm}}\int\frac{2p_{\mu_2}'}{-2p'k_1}\frac{2p_{\mu_2}'}{-2p'\cdot(k_1+k_2)}\cdots\frac{2p_{\mu_n}'}{-2p'\cdot(k_1+\cdots+k_i)}M_0^{(a)}(P-\Sigma k_i)\frac{2p_{\mu_1}'}{-2p\cdot k_1'}\cdots\frac{2p_{\mu_{n'}}}{-2p\cdot(k_1'+\cdots+k_n')}\prod_i\frac{d^4k_i}{k_i^2-\lambda^2}. \quad (4.30)$$

The same technique which has been used previously enables us to write then that

$$M_{n;\text{eik}}^{(a)}=\frac{1}{n!}\int M_0^{(a)}(x)e^{iP\cdot x}[\bar{U}(x)]^n d^4x, \quad (4.31)$$

where

$$\bar{U}(x)=\frac{-i\alpha}{4\pi^3}\int e^{-ik\cdot x}\times\frac{4p\cdot p'}{(-2p\cdot k+i\epsilon)(-2p'\cdot k+i\epsilon)}\frac{d^4k}{k^2-\lambda^2} \quad (4.32)$$

and

$$M_0^{(a)}(x) = \int M_0^{(a)}(P) e^{-iP\cdot x} \frac{d^4P}{(2\pi)^4}. \qquad (4.33)$$

In order to preserve gauge invariance, it is necessary[9] to replace $\bar{U}(x)$ by $U(x)$ defined as

$$U(x) = \frac{i\alpha}{8\pi^3} \int e^{-ik\cdot x} \frac{d^4k}{k^2-\lambda^2} \times \left(\frac{2p_\mu}{-2p\cdot k+i\epsilon} - \frac{2p_\mu'}{-2p'\cdot k+i\epsilon} \right)^2. \qquad (4.34)$$

This amounts to including an appropriate set of radiative corrections where the virtual photons are emitted and absorbed by the "same" electron. The sum over n of the modified $M_{n;\,\mathrm{eik}}{}^{(a)}$ is then

$$M_{\mathrm{eik}}{}^{(a)} = \int M_0^{(a)}(x) e^{U(x)} e^{iP\cdot x} d^4x. \qquad (4.35)$$

Here again, in the spirit of the discussion of Sec. II C, we are interested in the behavior of $U(x)$ when $x \to 0$. It is readily found that when $x\cdot p = x\cdot p' = 0$ and $x^2 \to 0$ we have

$$U(x) \simeq -\tfrac{1}{2}\gamma(t) \ln(\lambda^2 |x^2|), \qquad (4.36)$$

where

$$\gamma(t) = \frac{\alpha t}{\pi} \int_0^1 \frac{(1+z^2)dz}{4m^2 - t(1-z^2)} \qquad (4.37)$$

is precisely the function which is connected with the positronium spectrum.[10] Putting $\beta(t) \equiv -1+\gamma(t) = l = n-1$ reproduces the major part of the positronium spectrum. This is not surprising, since the diagrams of Figs. 4(a) and 4(b) enter importantly, in the crossed channel, in any description of the annihilation into two photons of a bound electron-positron system.

If only (4.36) is used in (4.35), the corresponding contribution to M_0^{eik} is proportional to $s^{\beta(t)}$. However, even if a Regge-like behavior were found to occur in the region $0 \leq t \leq 4m^2$, this would not, it should be noted, mean that the amplitude which we have calculated is the dominant one in the physical region.[11]

V. DISCUSSION

In Sec. V A, we summarize and comment on the results of the preceding sections. In Sec. V B, we conclude with remarks on possible generalizations and lines for further work.

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A. Summary and Commentary

1. Potential Scattering

We have seen, in Sec. II, that the simple approximation (2.10), when applied to a suitably symmetrized form of f_n, the nth-order term in the Born expansion of f, permits its evaluation in closed form; the sum over n then leads to the eikonal-type formula (2.20), with the eikonal function χ_{pot} given by (2.21). The integral for χ_{pot} may be written in the form

$$\chi_{\mathrm{pot}}(\mathbf{x}) = \frac{-m}{|\mathbf{p}|} \int_{-\infty}^{\infty} d\xi \; V[\mathbf{x}+\mathbf{s}(\xi)], \qquad (5.1)$$

where $\mathbf{s}(\xi) = \hat{p}\theta(-\xi)\xi + \hat{p}'\theta(\xi)\xi$. Thus, $\chi_{\mathrm{pot}}(\mathbf{x})$ is obtained by integrating along a path consisting of two half-lines, one from infinity to $\mathbf{x}$, in the direction of the initial momentum $\mathbf{p}$, the other from $\mathbf{x}$ to infinity, in the direction of the final momentum $\mathbf{p}'$. Equation (5.1) may be compared with a time-reversal-invariant modification of (1.2) suggested by Glauber,[12] obtained by replacing $\hat{p}$ by $\hat{K} = (\mathbf{p}+\mathbf{p}')/|\mathbf{p}+\mathbf{p}'|$. This is equivalent to replacing $\mathbf{s}(\xi)$ by $\hat{K}\xi$ in (5.1), i.e., to integrating along the straight line through $\mathbf{x}$, in the average direction $\hat{K}$. It would seem that the form (5.1) is closer to what one would expect on the basis of a wave-packet picture of the scattering process. Of course, for very small angles, the difference disappears.

The eikonal (5.1) also occurs in the work of Schiff,[3] in his approximation to f for the case of large-angle potential scattering, viz.,

$$f(\mathbf{p}',\mathbf{p}) \approx \frac{-m}{2\pi} \int d\mathbf{x} \; e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) e^{i\chi_{\mathrm{pot}}}. \qquad (5.2)$$

The same result would have been obtained in our approach if, instead of *averaging* over the $n+1$ alternative forms of f_{n+1} [Eq. (2.3)], we had simply *summed* over them, i.e., if we had not divided by $n+1$ in Eq. (2.11). The basis of Schiff's arguments leading to (5.2) is a physical picture in which the large-angle scattering of order $n+1$ takes place primarily by a single scattering through a large angle, accompanied by n scatterings through small angles.

It would seem to be a worthwhile task to carry out some numerical calculations for, e.g., Yukawa potentials. This would enable us to make a comparison of the relative and absolute accuracy of all these eikonal-type approximations, at least in simple cases, and to delineate the regions of validity of (1.3).

2. Relativistic Eikonal Approximation

In Sec. III, we found a natural extension to quantum field theory of the results for potential scattering. Use of the propagator approximation (3.6), analogous to (2.10), leads to Eq. (3.10) for $M^{\mathrm{eik}}(s,t)$, as an approxima-

[12] See Ref. 2, p. 345.

tion to the sum of all contributions from spin-0 meson exchanges between spin-0 particles a and b. The quantity $\chi(x)$ occurring in (3.20) is a relativistic generalization of $\chi_{\text{pot}}(\mathbf{x})$, reducing to it in the static limit[13] (m_a or $m_b\to\infty$). A form of $\chi(\mathbf{x})$, alternative to those already given [(3.21), (3.13)] with perhaps greater heuristic value, may be obtained as follows. We regard the quantity $gd(k;p_b',p_b)$, appearing in (3.21), as the Fourier transform with respect to k of a transition current $\langle p_b'|j(x)|p_b\rangle$, which we abbreviate as $J_b(x)$. This current serves as the source of a field $A_b(x)=\int\Delta_F(x-x')J_b(x')d^4x'$ with which the transition current $J_a(x)$ interacts to produce the eikonal χ, via

$$\chi(x)=-i\int d^4y\, J_a(y)A_b(x+y).$$

Thus, we see that $\chi(0)$ is the space-time integral of an effective interaction density $J_a(y)A_b(y)$ and hence may be regarded as an "action," or as a relevant piece of the action, defined as the space-time integral of the Lagrangian density. Yet another point of view, more symmetric between a and b, is to regard $\chi(x)$ as a field, satisfying

$$(\Box+m^2)\chi(x)=-J_{ab}(x),$$

where the source current $J_{ab}(x)$ is defined by

$$J_{ab}(x)=\int d^4y\, J_a(y-\tfrac{1}{2}x)J_b(y+\tfrac{1}{2}x).$$

We should note that the crossing symmetry exhibited by χ and M^{eik} is simply a consequence of the fact that the crossing operation, say, for b, is equivalent to reversing the order of absorption of mesons along the world line of b, and that all permutations of these orders have been kept. It is amusing to note that our approximation would *not* have led to a simple result if we had confined ourselves to ladder-type diagrams, since then the identity (2.14), which leads to a factorized integrand, does not come into play.

In Sec. III C, the particle-antiparticle amplitude $M_{a\bar{a}}$ was studied and an eikonal approximation $M_{a\bar{a}}^{\text{eik}}=M_1+M_2$ was defined [Eqs. (3.34)–(3.36)] with M_1 and M_2 arising, respectively, from the exchange-type and annihilation-type diagrams, shown in Figs. 2 and 3.

Some contact with Regge theory was made by showing that, with $x\cdot p_i=0$, the eikonal χ_2 has a logarithmic singularity at $x^2=0$, with strength equal to $\bar{\alpha}(t)+\bar{\alpha}(u)$; here $\bar{\alpha}(t)$, defined by (3.4), is such that $\alpha(t)=-1+\bar{\alpha}(t)$ coincides with the trajectory found from the ladder-diagram contributions.

3. *Extensions*

Further insight into the relation of our eikonal approximation with Regge theory was obtained in Sec. IV, where it was shown that a certain class of radiative corrections to the lowest-order Compton amplitude when treated in the eikonal approximation leads to an eikonal with a logarithmic singularity of strength $1+\beta(t)$. Here $\beta(t)$ is just the function, introduced previously by a less direct approach,[10] which reproduces the spectrum of positronium on setting $\beta(t)=l=n-1$.

In addition, a generalization of the algebraic identity (2.14) to the integral representation derived in the Appendix, which takes into account an additive constant in the denominator, was shown to permit an extension of the eikonal approximation to a variety of processes in which one or more particles are off the mass shell. As an example, such an approximation was derived for the Green's function K^A for a particle moving in an external time-dependent scalar field $A(x)$ [Eq. (4.14)] and, *en passant*, for the generalized scattering amplitude $M(p',p)$ describing the scattering in such a field [Eq. (4.13)]. This latter formula generalizes the corresponding formula (2.20) for nonrelativistic scattering in a time-independent potential $V(\mathbf{x})$. It should be noted that the generalization to relativistic kinematics is trivial if $A(x)$ is time-independent because of the simple connection between the Schrödinger and Klein-Gordon equations for that case.[13] As another example, an eikonal approximation Γ_s^{eik} was derived for $\Gamma_s(p',p)$, the contributions of an infinite class of diagrams, typified by Fig. 3, to the vertex function $\Gamma(p',p)$, in both the on-shell and off-shell cases [Eqs. (4.21) and (4.22)].

B. Concluding Remarks

Some generalization of the results obtained in this paper can be had very easily. To include mesonic exchanges with different masses one simply replaces $g^2\Delta_F(x)$ by $\Sigma g_i^2\Delta_F(x;\mu_i)$ in the definition of χ and M^{eik}. Radiative corrections to the meson propagators can obviously be included by replacing $\Delta_F(x)$ by the full propagator $\Delta_F'(x)$. Vertex corrections can be included without any more labor, only to the extent that $\Gamma_a(p_i,p_{i-1})$ the proper vertex for emission of a meson of momentum k at the ith vertex can be approximated by a function of p_a and k only. Generalizations to in-

[13] It may strike the reader as curious that although χ_{pot} was defined within the framework of potential scattering, using nonrelativistic kinematics throughout, nevertheless the equality and proportionality, respectively, of χ with χ_{pot} and of M^{eik} with $f_{\text{pot}}^{\text{eik}}$ in the static limit arose without any nonrelativistic approximation, such as $E_{\mathbf{p}}=(\mathbf{p}^2+m^2)^{1/2}\approx m$, being made for particle a. The explanation for this lies in the simple connection between the solutions of the external-field Klein-Gordon equation, $[\Box+m^2+2mV(x)]\psi(x)=0$, and the corresponding Schrödinger equation, $[-i\partial_t-(\nabla^2/2m)+V(\mathbf{x})]\psi(x)=0$, in the case where $V(x)\to V(\mathbf{x})$, a time-independent potential. For then, to each stationary-state solution $\psi_p(x)=\psi_{\mathbf{p}}(\mathbf{x})\exp(-i\mathbf{p}^2t/2m)$ of the latter equation there corresponds an exact solution $\psi_p(x)=\psi_{\mathbf{p}}(\mathbf{x})\times\exp(-iE_{\mathbf{p}}t)$ of the former equation. An equivalent way to look at this is to note that the Green's operators $(\Box+m^2)^{-1}$ and $(-2im\partial_t-\nabla^2)^{-1}$ characteristic of the two cases become identical if the only time dependence is that which enters from the corresponding free-particle wave functions.

clude spin do not seem very hard to come by, as the example of the Compton effect indicates.[14]

One of the most interesting aspects of the results obtained is the technique for including off-shell effects within the framework of eikonal-type approximations. It would seem, for example, that in the analysis of high-energy production processes of the type $a+b\to c+d'+e'$, where important contributions come from the two-step process $a+b\to c+d$, $d\to d'+e'$, with d virtual, an approach based on the off-shell eikonal approximation for $M_{a+b\to c+d}$ might form a useful first step. Although we have not written it down explicitly for the scattering amplitude, it is obvious from the examples of the propagator and vertex function in Sec. IV that analogous expressions can be given for this case also: Each off-shell external particle leads to an integration over an additional parameter.

It is clear that much further work remains to be done to assess the validity of relativistic eikonal-type approximations within a given model and to test their relevance to actual physical problems.

We note finally that, after completing our work, we came across a paper of Erickson and Fried,[15] who use functional derivative techniques to study similar mathematical problems in a quite different context. Such techniques could be used to provide alternative derivations of eikonal-type approximations, which might provide additional insight into their meaning.[16]

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[14] It may be worthwhile to note that in approximation (3.6) the *squared* terms in internal momenta could be kept, and all summations could still be carried out. Thus, if we approximate $(p-\sum k_i)^2$ by $(p^2-2p\cdot\sum_i k_i+\sum_i k_i^2)$, dropping only the interference terms $\sum_{i\neq j} k_i\cdot k_j$, the subsequent considerations remain essentially unchanged: To include such terms one need only replace $-2p\cdot k$ and $2p'\cdot k$ by $-2p\cdot k+k^2$ and $2p'\cdot k+k^2$ in the definition (3.15) of $U(x;\, p,p')$. Further study is needed to determine the significance of such an approximation.

[15] G. W. Erickson and H. M. Fried, J. Math. Phys. **6**, 414 (1965).

[16] In fact, while this paper was being typed, we received a report by H. D. I. Abarbanel and C. Itzykson [Phys. Rev. Letters **23**, 53 (1969)] in which such techniques are used to derive a relativistic eikonal approximation. These authors mention some closely related papers of which we were unaware: R. Torgerson, Phys. Rev. **143**, 1194 (1966), and R. Sugar and R. Blankenbecler, *ibid.* **183**, 1387 (1969). We thank Dr. Abarbanel for informing us of his work with Itzykson, prior to publication.

APPENDIX: SOME USEFUL IDENTITIES

Consider the function

$$F_n(x)=\sum_\pi \frac{1}{a_1'+x}\,\frac{1}{a_1'+a_2'+x}\times\frac{1}{a_1'+a_2'+\cdots+a_n'+x}, \qquad \text{(A1)}$$

where the sum is over all permutations π of the sequence $(a_1,\cdots,a_n)$ with $a_i'=a_{\pi(i)}$. Each of the denominators in (A1) is assumed to have a positive real part. Using the representation

$$\frac{1}{z}=\int_0^\infty e^{-\alpha z}d\alpha,$$

valid whenever $\mathrm{Re}z>0$, we may write

$$\frac{1}{x}F_n(x)=\sum_\pi \int_0^\infty d\alpha_0\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_n \times e^{-\alpha_0 x}e^{-\alpha_1(a_1'+x)}\cdots e^{-\alpha_n(a_1'+a_2'+\cdots+a_n'+x)}.$$

Introducing new variables β, β_1, $\cdots$, β_n via

$$\beta=\alpha_0+\alpha_1+\cdots+\alpha_n,\qquad \beta_1=\alpha_1+\cdots+\alpha_n,\cdots,\beta_n=\alpha_n,$$

we have

$$\frac{1}{x}F_n(x)=\sum_\pi \underset{\beta\ge\beta_1\ge\cdots\ge\beta_n\ge0}{\int\int\cdots\int} d\beta d\beta_1\cdots d\beta_n \times e^{-\beta x}e^{-\beta_1 a_1'}\cdots e^{-\beta_n a_n'}.$$

On putting $\beta_k'=\beta_{\bar{k}}$ with $\bar{k}=\pi^{-1}(k)$ $(k=1,\,2,\,\cdots,\,n)$ in the term corresponding to the permutation π and noting that

$$e^{-\beta_1 a_1'}e^{-\beta_2 a_2'}\cdots e^{-\beta_n a_n'}=e^{-\beta_1' a_1}e^{-\beta_2' a_2}\cdots e^{-\beta_n' a_n},$$

we get

$$\frac{1}{x}F_n(x)=\sum_\pi \underset{\beta\ge\beta_{\pi(1)}'\ge\cdots\ge\beta_{\pi(n)}'\ge0}{\int\cdots\int} d\beta d\beta_1'\cdots d\beta_n' e^{-\beta x}e^{-\beta_1' a_1}\cdots e^{-\beta_n' a_n}.$$

On dropping the primes, and noting that for fixed β the union of the domains

$$\beta\ge\beta_{\pi(1)}\ge\beta_{\pi(2)}\ge\cdots\ge\beta_{\pi(n)}\ge0$$

is just the hypercube $\beta\ge\beta_k\ge0$, we see that

$$\frac{1}{x}F_n(x)=\int_0^\infty d\beta\, e^{-\beta x}\int_0^\beta d\beta_1\cdots\int_0^\beta d\beta_n e^{-\beta_1 a_1}\cdots e^{-\beta_n a_n}.$$

On integrating over the β_k, we get

$$F_n(x)=x\int_0^\infty d\beta\, e^{-\beta x}\prod_{i=1}^n\left(\frac{1-e^{-\beta a_i}}{a_i}\right), \qquad \text{(A2)}$$

which is the result used in Sec. IV. Equation (2) may be recast, on integration by parts, into the form

$$F_n(x)=\int_0^\infty d\beta\, e^{-\beta x}\frac{\partial}{\partial\beta}\prod_{i=1}^{n}\frac{1-e^{-\beta a_i}}{a_i}. \tag{A3}$$

From Eq. (3) we have immediately, as a simple check,

$$F_n(0)=\prod_{i=1}^{n}\frac{1-e^{-\beta a_i}}{a_i}\bigg|_{\beta=0}^{\beta=\infty}=\frac{1}{a_1a_2\cdots a_n}, \tag{A4}$$

a result used in Secs. II and III. Using mathematical induction, it is easy to provide a purely algebraic proof of (4). We leave this task to the interested reader.

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Renormalization of Regge Trajectories and Singularity Structure in Kikkawa-Sakita-Virasoro–Type Theories*

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An investigation is made of theories which satisfy the duality principle using the Veneziano amplitude as a Born term. In constructing the theory, it is found necessary to average over different ways of assigning the loop momenta to the points of the duality diagram. The Regge-pole terms in the asymptotic behavior are identified, and transcendental equations which express the full renormalization of the leading trajectory are recorded. (It is necessary to assume that the integrals can be so defined that this asymptotic behavior, found in the limit Re$s\rightarrow-\infty$, continues to be the dominant behavior as Re$s\rightarrow+\infty$.) The amplitude is shown to have the Landau-Cutkosky singularity structure corresponding to poles lying on the renormalized leading trajectory. In particular, if low-lying particles on this trajectory are the only stable particles in the theory, the real singularity structure required by unitarity is correctly obtained. It is then possible that the failure in a finite theory of exact factorization for all daughters would not spoil the theory.

I. INTRODUCTION

RECENTLY Kikkawa, Sakita, and Virasoro (KSV)[1] have proposed a way of constructing a new form of perturbation theory, consistent with duality, in which the Veneziano amplitude[2] plays the role of a Born term. Such a series appears likely to be formally unitary and to correct the most glaring deficiency of the Veneziano model itself. However, KSV in a note added in proof, and also Bardakci, Halpern, and Shapiro (BHS)[3] have pointed out that in order to obtain full factorization of even the single-loop KSV expression in a way which is consistent with Veneziano-type functions associated with tree diagrams,[4] the integrand in the KSV integral must contain an infinite product which leads to an exponential divergence.

This disastrous conclusion is enforced by the requirement that factorization, and consequent unitarity-like discontinuity formulas around normal threshold singularities, is required for *all* poles contained in the Veneziano amplitude whatever their level in the daughter sequence. While this would be an agreeable property if it were obtainable, it is not clear that its failure robs the KSV approach of all its utility. Two lines of thought suggest that this is not necessarily the case. One is that the daughter properties of a Veneziano amplitude can be modified by the addition of nonleading terms. Bardakci and Mandelstam[5] have conjectured that these nonleading additions cannot be used in a way which leads to a simpler, and so probably less divergent, daughter sequence, but a proof has not, at present, been given that this is so. Secondly, the effect of unitarizing the theory will be to destroy the narrow-resonance approximation of the Veneziano amplitude. Resonance poles should move onto unphysical sheets, leaving only the stable-particle poles renormalized to locations which are still real. For simplicity, we shall always consider the model in which the only stable particle is the spin-0 member of the leading trajectory. If that leading trajectory factorizes properly, then the real normal thresholds corresponding to stable particles will have Cutkosky discontinuity formulas which correspond to physical unitarity. This will not be true for singularities involving daughter-trajectory particles, if the latter do not factorize

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