Pion Lifetime, $\rho\pi$ and $\sigma\pi$ Intermediate States, and Sum Rules

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The pion lifetime is calculated assuming an unsubtracted dispersion relation for the axial-vector matrix element and using $\rho\pi$ and $\sigma\pi$ intermediate states as well as the $\bar{N}N$ intermediate state. By imposing convergence conditions, a series of sum rules is obtained, including the generalized Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation and modified Goldberger-Treiman relations. By assuming that the $NN\pi$, $\rho\pi\pi$, and $\sigma\pi\pi$ vertex functions are dominated by a three-pion resonance with $I=1$ and $J^P=0^-$, it is found possible to make the Goldberger-Treiman relations agree with experiment if the resonance mass is $M_R \approx 2$ GeV. This resonance is tentatively identified with the $\bar{N}N$ resonance structure observed around 2 GeV.

1. INTRODUCTION

ONE method of deriving the Goldberger-Treiman[1] relation assumes an unsubtracted dispersion relation for the pion lifetime form factor,

$$F_\pi(\xi)=\frac{1}{\pi}\int\frac{d\xi'\,\mathrm{Im}F_\pi(\xi')}{\xi'-\xi}. \tag{1}$$

If, in the expression for $\mathrm{Im}F_\pi(\xi)$, only the $\bar{N}N$ intermediate states are kept, then the usual treatment of the matrix elements that appear leads to a logarithmic divergence in Eq. (1). Imposing the convergence condition that the coefficient of the divergent term should vanish yields the Goldberger-Treiman relation. Hyperon-antihyperon intermediate states have also been considered in this connection.[2] Although the 3π intermediate states have the correct quantum numbers to contribute, they are customarily ignored, since there is no known way to estimate their contribution.

If we substitute the experimental value $g_A=-G_A/G_V=1.18\pm0.02$ observed by Sosnovsky *et al.*[3] into the Goldberger-Treiman relation $F_\pi=g_AM/G$, we find $F_\pi=0.087M$, which differs by 13% from the observed value $F_\pi=0.10M$ obtained from measurements of the π^+ lifetime. If we use the recently observed value $g_A=1.23\pm0.01$, based on a new measurement of the neutron half-life 10.80 ± 0.16 min, reported by Christensen *et al.*[4] we find $F_\pi=0.090M$, which differs by 10% from the observed value. Thus, it is desirable to find the source of this 10% correction to the Goldberger-Treiman relation. Electromagnetic effects are only expected to provide about 1% of this correction.

In this paper, we shall assume the unsubtracted dispersion relation (1) for $F_\pi(\xi)$ and impose convergence conditions on this relation. Intermediate states other than baryon-antibaryon states are also considered. In particular, the $\rho\pi$ and $\sigma\pi$ intermediate states are studied in detail.

In general, in this paper we have assumed that the form factors of the matrix elements of $\partial^\mu A^a{}_\mu(0)$ satisfy dispersion relations which are *at most* once-subtracted. In our first calculation, in Sec. 3, we assume that the form factors of the matrix elements of $j_\pi{}^a(0)$ approach nonzero constants at infinite momentum transfer, and therefore satisfy dispersion relations which are once-subtracted. In our second calculation in Sec. 4, we assume these form factors satisfy unsubtracted dispersion relations.

The convergence conditions on the $\sigma\pi$, $\rho\pi$, and $\bar{N}N$ contributions lead to an hierarchy of sum rules. These sum rules take the form of an extended Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation[5] and generalized Goldberger-Treiman relations. This program is carried out with both once-subtracted and unsubtracted dispersion relations for the $NN\pi$, $\rho\pi\pi$, and $\sigma\pi\pi$ form factors. In the former case, only scattering contributions to the absorptive part are considered and the correction to the Goldberger-Treiman relation is found to be in the wrong direction. In the latter case, it is found that a 10% correction to the Goldberger-Treiman relation can be obtained by saturating the dispersion relations with a meson possessing the quantum numbers of the pion and a mass $M_R\approx 2$ GeV.

2. PION FORM FACTOR AND $\rho\pi$, $\sigma\pi$, AND $\bar{N}N$ INTERMEDIATE STATES

The form factor is defined by

$$\langle 0|A_\mu{}^a(0)|\pi^b(k)\rangle=(2\pi)^{3/2}\delta_{ab}k_\mu F_\pi(k^2), \tag{2}$$

where $A_\mu{}^a(0)$ is the hadronic axial-vector current. By a reduction of the pion state, we obtain

$$\delta_{ab}k_\mu\,\mathrm{Im}F_\pi(k^2)=\frac{(2\pi)^4}{2i}\sum_n[\delta^4(k-P_n)\langle 0|A_\mu{}^a(0)|n\rangle\langle n|j^b(0)|0\rangle+\delta^4(k+P_n)\langle 0|j^b(0)|n\rangle\langle n|A_\mu{}^a(0)|0\rangle]. \tag{3}$$

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[1] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); **111**, 354 (1958); for a review of techniques used to derive this relation, see J. Bernstein, *Elementary Particles and Currents* (W. H. Freeman and Co., San Francisco, 1968).

[2] M. Ida, Phys. Rev. **132**, 401 (1963).

[3] A. N. Sosnovsky *et al.*, Nucl. Phys. **10**, 395 (1959).

[4] C. J. Christensen *et al.*, Phys. Letters **26B**, 11 (1967).

[5] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

The second term vanishes for physical pion decay, since $k_0>0$ and $P_{n0}>0$, and we get

$$\delta_{ab}k^2\,\mathrm{Im}F_\pi(k^2)=\tfrac{1}{2}(2\pi)^4\sum_n \delta^4(k-P_n)\times\langle 0|\partial^\mu A_\mu{}^a(0)|n\rangle\langle n|j^b(0)|0\rangle. \quad (4)$$

Let us define the form factors $K(\xi)$ and $\gamma_{\rho\pi\pi}(\xi)$ in terms of the matrix elements

$$\langle 0|\partial^\mu A_\mu{}^a(0)\left|\pi^c(q)\rho^d(p)\begin{pmatrix}\text{in}\\ \text{out}\end{pmatrix}\right\rangle=\frac{ie_{acd}}{(2\pi)^3}\epsilon_\rho\cdot(p+q)\begin{pmatrix}K(\xi)\\ K^*(\xi)\end{pmatrix}, \quad (5)$$

and

$$\langle 0|j_\pi{}^b(0)\left|\pi^c(q)\rho^d(p)\begin{pmatrix}\text{in}\\ \text{out}\end{pmatrix}\right\rangle=\frac{2i}{(2\pi)^3}\epsilon_\rho\cdot(p+q)e_{bcd}\begin{pmatrix}\gamma_{\rho\pi\pi}(\xi)\\ \gamma_{\rho\pi\pi}{}^*(\xi)\end{pmatrix}, \quad (6)$$

where $\xi=(p+q)^2$. Then, the $\rho\pi$ contribution to $\mathrm{Im}F_\pi(k^2)$ is given by

$$\delta_{ab}k^2\,\mathrm{Im}F_\pi(k^2)=\tfrac{1}{2}(2\pi)^4\sum_{\lambda=1}^{3}\sum_{c,d}\int d^4p\,d^4q\,\delta(p^2-m_\rho{}^2)\times\theta(p)\delta(q^2-m_\pi{}^2)\theta(q)\delta^4(k-p-q)\times\frac{ie_{acd}}{(2\pi)^3}\epsilon_\rho{}^\lambda\cdot(p+q)\left(\frac{-2i}{(2\pi)^3}\right)\epsilon_\rho{}^\lambda\cdot(p+q)e_{bcd}\times\left(\frac{K(\xi)\gamma_{\rho\pi\pi}{}^*(\xi)+K^*(\xi)\gamma_{\rho\pi\pi}(\xi)}{2}\right), \quad (7)$$

where we have used

$$\sum_{c,d}e_{acd}e_{bcd}=2\delta_{ab},$$

$$\sum_{\lambda=1}^{3}\epsilon_\rho{}^\lambda\cdot(p+q)\epsilon_\rho{}^\lambda\cdot(p+q)=\frac{(\xi-m_\rho{}^2-m_\pi{}^2)^2-4m_\rho{}^2m_\pi{}^2}{4m_\rho{}^2}, \quad (8)$$

and here $k=p+q$ and $k^2=\xi$. Also,

$$\int d^4p\,d^4q\,\delta(p^2-m_\rho{}^2)\delta(q^2-m_\pi{}^2)\theta(p)\theta(q)\delta^4(k-p-q)=(\tfrac{1}{2}\pi)\xi^{-1}[(\xi-m_\rho{}^2-m_\pi{}^2)^2-4m_\rho{}^2m_\pi{}^2]^{1/2}. \quad (9)$$

We therefore obtain for $\mathrm{Im}F_\pi(\xi)$ the result

$$\mathrm{Im}F_\pi(\xi)=\tfrac{1}{2}\pi^{-1}\xi^{-2}[(\xi-m_\rho{}^2-m_\pi{}^2)^2-4m_\rho{}^2m_\pi{}^2]^{3/2}\times[K(\xi)\gamma_{\rho\pi\pi}{}^*(\xi)+K^*(\xi)\gamma_{\rho\pi\pi}(\xi)]/8m_\rho{}^2. \quad (10)$$

Let us now assume that $K(\xi)$ satisfies a once-subtracted dispersion relation

$$K(\xi)=K(m_\rho{}^2)+\frac{\xi-m_\rho{}^2}{\pi}\int\frac{d\xi'\,\mathrm{Im}K(\xi')}{(\xi'-m_\rho{}^2)(\xi'-\xi)}. \quad (11)$$

The A_1 intermediate state does not contribute to $\mathrm{Im}K(\xi)$. Taking out the pion contribution gives

$$K(\xi)=K(m_\rho{}^2)-2F_\pi\gamma_{\rho\pi\pi}\frac{m_\pi{}^2}{m_\pi{}^2-m_\rho{}^2}+2F_\pi\gamma_{\rho\pi\pi}\frac{m_\pi{}^2}{m_\pi{}^2-\xi}+\frac{\xi-m_\rho{}^2}{\pi}\int\frac{d\xi'\,\tan\psi(\xi')\,\mathrm{Re}K(\xi')}{(\xi'-m_\rho{}^2)(\xi'-\xi)}, \quad (12)$$

where $F_\pi\equiv F_\pi(m_\pi{}^2)$, $\gamma_{\rho\pi\pi}\equiv\gamma_{\rho\pi\pi}(m_\pi{}^2)$, and $\psi(\xi)$ is the phase of $K(\xi)$:

$$K(\xi)=|K(\xi)|e^{i\psi(\xi)}, \quad (13)$$
$$\mathrm{Im}K(\xi)=\tan\psi(\xi)\,\mathrm{Re}K(\xi).$$

The integral equation (11) may be solved[6] to yield

$$K(\xi)=\left(K(m_\rho{}^2)A-2F_\pi\frac{\gamma_{\rho\pi\pi}m_\pi{}^2}{m_\pi{}^2-m_\rho{}^2}+2F_\pi\frac{\gamma_{\rho\pi\pi}m_\pi{}^2}{m_\pi{}^2-\xi}\right)\times\exp\left(\frac{\xi-m_\pi{}^2}{\pi}\int\frac{d\xi'\psi(\xi')}{(\xi'-m_\pi{}^2)(\xi'-\xi-i\epsilon)}\right), \quad (14)$$

where A is the real constant

$$A=\exp\left(-\frac{(m_\rho{}^2-m_\pi{}^2)}{\pi}\int\frac{d\xi'\psi(\xi')}{(\xi'-m_\pi{}^2)(\xi'-m_\rho{}^2)}\right). \quad (15)$$

The discontinuity across the cut is approximated by considering only the contribution from $\rho\pi$-$\rho\pi$ scattering. Only $\rho\pi$ intermediate states in the ${}^1P_0$ state contribute to $\mathrm{Im}K(\xi)$. Then,

$$K(\xi)=e^{2i\delta}K^*(\xi)+(\text{other contributions}) \quad (16)$$

and

$$\mathrm{Im}K(\xi)=e^{i\delta}\sin\delta K^*(\xi)+(\text{other contributions})=\mathrm{Re}[e^{i\delta}\sin\delta K^*(\xi)]. \quad (17)$$

Moreover,

$$\frac{\mathrm{Im}K(\xi)}{\mathrm{Re}K(\xi)}=\frac{\mathrm{Re}(e^{i\delta}\sin\delta)}{1-\mathrm{Im}(e^{i\delta}\sin\delta)}=\tan\psi(\xi), \quad (18)$$

where δ is the complex ${}^1P_0$ $\rho\pi$-$\rho\pi$ scattering phase shift.

If a once-subtracted dispersion relation is assumed for $\gamma_{\rho\pi\pi}(\xi)$, then since neither the pion nor the A_1 intermediate states contribute to $\mathrm{Im}\gamma_{\rho\pi\pi}(\xi)$, only the ${}^1P_0$ $\rho\pi$ intermediate state contributes, and

$$\gamma_{\rho\pi\pi}(\xi)=|\gamma_{\rho\pi\pi}(\xi)|e^{i\psi(\xi)}. \quad (19)$$

[6] J. D. Jackson, in *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, Edinburgh, 1961), p. 1.

Thus, $\gamma_{\rho\pi\pi}(\xi)$ satisfies the equation

$$\gamma_{\rho\pi\pi}(\xi)=\gamma_{\rho\pi\pi}\times\exp\left(\frac{\xi-m_\pi^2}{\pi}\int\frac{d\xi'\psi(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi-i\epsilon)}\right), \quad (20)$$

where $\psi(\xi)$ is the same function introduced earlier in connection with $K(\xi)$. The range of the integrations is $(m_\rho+m_\pi)^2\leqslant\xi\leqslant\infty$.

The expression for the $\rho\pi$ contribution to $\mathrm{Im}F_\pi(\xi)$ now becomes

$$\mathrm{Im}F_\pi(\xi)=\tfrac{1}{4}\pi^{-1}[(\xi-m_\rho^2-m_\pi^2)^2-4m_\rho^2m_\pi^2]^{3/2}|\Omega(\xi)|^2\frac{\gamma_{\rho\pi\pi}}{4m_\rho^2\xi^2}\times\left(K(m_\rho^2)A-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-m_\rho^2}+\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-\xi}\right), \quad (21)$$

where

$$\Omega(\xi)=\exp\left(\frac{\xi-m_\pi^2}{\pi}\times\int_{(m_\rho+m_\pi)^2}^{\infty}\frac{d\xi'\psi(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi-i\epsilon)}\right), \quad (22)$$

and A is given by Eq. (15).

An expression for the subtraction constant $K(m_\rho^2)$ can be obtained if $\rho\pi$ intermediate-state contributions are neglected in $\mathrm{Im}K(\xi)$. This case has been discussed by Das, Mathur, and Okubo.[7] In the notation of the present paper, we have

$$\langle\pi^a(k)|A_\mu{}^b(0)|\rho^c(p),\epsilon_\rho\rangle=[e_{abc}/(2\pi)^3]\epsilon_\rho{}^\nu(p)[K_1(\xi)g_{\mu\nu}+K_2(\xi)k_\nu(p+k)_\mu+K_3(\xi)k_\nu(p-k)_\mu], \quad (23)$$

where now $\xi=(p-k)^2$. We deduce that $K(\xi)$ discussed earlier has the form

$$K(\xi)=K_1(\xi)-(m_\rho^2-m_\pi^2)K_2(\xi)-\xi K_3(\xi). \quad (24)$$

We shall assume a once-subtracted dispersion relation for $K_1(\xi)$ and unsubtracted dispersion relations for $K_2(\xi)$ and $K_3(\xi)$. By reducing the pion state in Eq. (23), using the hypothesis of partially conserved axial-vector current (PCAC) and the current equal-time commutation relations, and taking the limit $k_\mu\to 0$, we obtain

$$K_1(m_\rho^2)=G_\rho/F_\pi, \quad (25)$$

where

$$\langle 0|V_\mu{}^b(0)|\rho_\lambda{}^d(q)\rangle=[\delta_{bd}/(2\pi)^{3/2}]G_\rho\epsilon_\mu{}^{(\rho)}(q,\lambda). \quad (26)$$

Defining

$$\langle 0|A_\mu{}^b(0)|A_{1\lambda}{}^d(q)\rangle=[\delta_{bd}/(2\pi)^{3/2}]G_{A_1}\epsilon_\mu{}^{(A_1)}(q,\lambda), \quad (27)$$

$$\langle\pi^d(q)|j_\pi{}^a(0)|\rho^c(p)\rangle=2i[e_{dac}/(2\pi)^3]\epsilon_\rho\cdot q\gamma_{\rho\pi\pi}(\eta),$$

where $\eta=(p-q)^2$ and

$$\langle\rho^d(q)|j_\pi{}^a(0)|A_1{}^c(p)\rangle=[e_{dac}/(2\pi)^3][G_S(\epsilon_\rho\cdot\epsilon_A)+G_D(\epsilon_A\cdot q)(\epsilon_\rho\cdot p)], \quad (28)$$

one finds in the pole-dominance approximation

$$\mathrm{Im}K_1(\xi)=-\pi\delta(\xi-m_A^2)G_AG_S, \quad (29a)$$

$$\mathrm{Im}K_2(\xi)=\tfrac{1}{2}\pi\delta(\xi-m_A^2)G_AG_D, \quad (29b)$$

$$\mathrm{Im}K_3=-\pi\delta(\xi-m_A^2)G_A[G_S+\tfrac{1}{2}(m_\rho^2-m_\pi^2)G_D]/m_A^2-2\pi\delta(\xi-m_\pi^2)F_\pi\gamma_{\rho\pi\pi}. \quad (29c)$$

Combining these results, we get

$$K(\xi)=\left(\frac{G_\rho}{F_\pi}-2F_\pi\gamma_{\rho\pi\pi}+\frac{G_AG_S}{m_A^2-m_\rho^2}-G_A\frac{G_S+\frac{1}{2}(m_\rho^2-m_\pi^2)G_D}{m_A^2}\right)-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{\xi-m_\pi^2}, \quad (30)$$

and

$$K(m_\rho^2)=\left(\frac{G_\rho}{F_\pi}-2F_\pi\gamma_{\rho\pi\pi}+\frac{G_AG_S}{m_A^2-m_\rho^2}-\frac{G_A}{m_A^2}[G_S+\tfrac{1}{2}(m_\rho^2-m_\pi^2)G_D]\right)-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\rho^2-m_\pi^2}. \quad (31)$$

The $\bar{N}N$ intermediate-state case was originally considered by Goldberger and Treiman.[1] Here, for convenience only, a summary of their results in the notation of this paper will be given. The form factors $C(\xi)$ and $K_{NN\pi}(\xi)$ are defined by

$$\left\langle 0\left|\partial^\mu A_\mu{}^a(0)\right|\bar{N}_1(p)\eta_1,N_2(q)\eta_2\binom{\text{in}}{\text{out}}\right\rangle=\frac{i}{(2\pi)^3}(\tfrac{1}{2}\eta_1{}^\dagger\tau^a\eta_2)\bar{V}_1(p)\gamma_5U_2(q)2M\binom{C(\xi)}{C^*(\xi)} \quad (32)$$

and

$$\left\langle\binom{\text{in}}{\text{out}}\bar{N}_1(p)\eta_1,N_2(q)\eta_2\left|j_\pi{}^b(0)\right|0\right\rangle=\frac{i}{(2\pi)^3}\eta_2{}^\dagger\tau^b\eta_1\bar{U}_2(q)\gamma_5V_1(p)G\binom{K_{NN\pi}{}^*(\xi)}{K_{NN\pi}(\xi)}, \quad (33)$$

where $\xi=(p+q)^2$ and the η's are the nucleon isospinors and τ^a $(a=1,2,3)$ are the familiar Pauli matrices. G is the pion-nucleon coupling constant $G^2/4\pi=14.6$ and $K_{NN\pi}(m_\pi^2)=1$. The form factors $a(\xi)$ and $b(\xi)$ are defined by

$$\langle N_1(p)\eta_1|A_\mu{}^a(0)|N_2(q),\eta_2\rangle=(2\pi)^{-3}\,\tfrac{1}{2}\eta_1{}^\dagger\tau^a\eta_2\bar{U}_1(p)[\gamma_\mu\gamma_5a(\xi)+(p-q)_\mu\gamma_5b(\xi)]U_2(q), \quad (34)$$

where $\xi=(p-q)^2$. Thus, $C(\xi)$ in (32) becomes

$$C(\xi)=a(\xi)+(\xi/2m)b(\xi). \quad (35)$$

[7] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 1067 (1967).

Using a once-subtracted dispersion relation for $C(\xi)$ with $C(0)=A(0)=g_A$, and including the π and $\bar{N}N$ intermediate-state contribution to $\mathrm{Im}C(\xi)$, gives

$$C(\xi)=\left(g_A B+\frac{F_\pi G}{M}\frac{\xi}{m_\pi^2-\xi}\right)\times\exp\left(\frac{\xi-m_\pi^2}{\pi}\int_{4M^2}^{\infty}\frac{d\xi'\phi(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi-i\epsilon)}\right), \quad (36)$$

where $\phi(\xi)$ is the phase of $C(\xi)$ and

$$\tan\phi(\xi)=\frac{\mathrm{Re}(e^{i\eta}\sin\eta)}{1-\mathrm{Im}(e^{i\eta}\sin\eta)}=\frac{\mathrm{Im}C(\xi)}{\mathrm{Re}C(\xi)}. \quad (37)$$

Here η is the complex ${}^1S_0$ $\bar{N}N$-$\bar{N}N$ scattering phase shift, and B is the real constant

$$B=\exp\left(\frac{m_\pi^2}{\pi}\int_{4M^2}^{\infty}\frac{d\xi'\phi(\xi')}{\xi'(\xi'-m_\pi^2)}\right). \quad (38)$$

In the once-subtracted dispersion relation for $K_{NN\pi}(\xi)$,

$$K_{NN\pi}(\xi)=1+\frac{\xi-m_\pi^2}{\pi}\int\frac{d\xi'\,\mathrm{Im}K_{NN\pi}(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi)}, \quad (39)$$

we use the same approximations as we used for $C(\xi)$. Only the $\bar{N}N$ intermediate state contributes to $\mathrm{Im}K_{NN\pi}(\xi)$, and

$$K_{NN\pi}(\xi)=\exp\left(\frac{\xi-m_\pi^2}{\pi}\int_{4M^2}^{\infty}\frac{d\xi'\phi(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi-i\epsilon)}\right), \quad (40)$$

where $\phi(\xi)$ is the phase of $K_{NN\pi}(\xi)$ and

$$\tan\phi(\xi)=\mathrm{Re}(e^{i\eta}\sin\eta)/[1-\mathrm{Im}(e^{i\eta}\sin\eta)] \quad (41)$$
$$=\mathrm{Im}K_{NN\pi}(\xi)/\mathrm{Re}K_{NN\pi}(\xi).$$

With these approximations the $\bar{N}N$ contribution to $\mathrm{Im}F_\pi(\xi)$ is

$$\mathrm{Im}F_\pi(\xi)=\frac{1}{4\pi}\left(\frac{\xi-4M^2}{\xi}\right)^{1/2}\times MG\left(g_A B+\frac{F_\pi G}{M}\frac{\xi}{m_\pi^2-\xi}\right)|K_{NN\pi}(\xi)|^2. \quad (42)$$

Let us now consider the $\sigma\pi$ intermediate state. The form factors $F_1(\xi)$, $F_2(\xi)$, and $F(\xi)$ are defined by

$$\langle\pi^b(q)|A_\mu{}^a(0)|\sigma(p)\rangle=\delta_{ab}[i/(2\pi)^3][F_1(\xi)(p+q)_\mu+F_2(\xi)(p-q)_\mu] \quad (43)$$

and

$$\langle\pi^b(q)|\partial^\mu A_\mu{}^a(0)|\sigma(p)\rangle=\delta_{ab}(2\pi)^{-3}F(\xi), \quad (44)$$

where $\xi=(p-q)^2$. They are related by

$$F(\xi)=(m_\sigma^2-m_\pi^2)F_1(\xi)+\xi F_2(\xi). \quad (45)$$

Now assume a once-subtracted dispersion relation for $F_1(\xi)$ and an unsubtracted dispersion relation for $F_2(\xi)$. Then $F(\xi)$ will satisfy a once-subtracted dispersion relation

$$F(\xi)=F(m_\sigma^2)+\frac{\xi-m_\sigma^2}{\pi}\int\frac{d\xi'\,\mathrm{Im}F(\xi')}{(\xi'-m_\sigma^2)(\xi'-\xi)}. \quad (46)$$

The subtraction constant $F(m_\sigma^2)$ can be evaluated by the standard current-algebra technique. We obtain

$$F(m_\sigma^2)=G_\sigma/F_\pi, \quad (47)$$

where

$$\langle 0|\Sigma(0)|\sigma(p)\rangle=G_\sigma/(2\pi)^{3/2} \quad (48)$$

and

$$[A_0{}^b(x),\,\partial^\mu A_\mu{}^a(0)]\delta(x_0)=i\delta^4(x)\delta_{ab}\Sigma(0). \quad (49)$$

In the σ model of Gell-Mann and Lévy,[8] one has

$$\Sigma(0)=m_\pi^2F_\pi\phi_\sigma(0), \quad (50)$$

where $\phi_\sigma(x)$ is the σ-meson field operator. In this case

$$G_\sigma=m_\pi^2F_\pi. \quad (51)$$

The expression for $\mathrm{Im}F(\xi)$ is given by

$$\delta_{ab}\,\mathrm{Im}F(\xi)=\tfrac{1}{2}(2\pi)^{11/2}\sum_n\delta^4(q+P_n-p)\times\langle 0|\partial^\mu A_\mu{}^a(0)|n\rangle\langle n|j_\pi{}^b(0)|\sigma(p)\rangle. \quad (52)$$

The only pole term comes from the pion intermediate state. We obtain

$$F(\xi)=\frac{G_\sigma}{F_\pi}+2F_\pi\frac{m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}-2F_\pi\frac{m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{\xi-m_\pi^2}+\frac{\xi-m_\pi^2}{\pi}\int_{(m_\sigma+m_\pi)^2}^{\infty}\frac{d\xi'\,\mathrm{Im}F(\xi')}{(\xi'-m_\sigma^2)(\xi'-\xi)}, \quad (53)$$

where we have used

$$\langle\pi^a(l)|j_\pi{}^b(0)|\sigma(p)\rangle=2\delta_{ab}(2\pi)^{-3}m_\sigma\gamma_{\sigma\pi\pi}(\eta). \quad (54)$$

Also, $\eta=(p-l)^2$ and $\gamma_{\sigma\pi\pi}=\gamma_{\sigma\pi\pi}(m_\pi^2)$.

We now have

$$F(\xi)=\left(\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}-\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{\xi-m_\pi^2}\right)\Lambda(\xi), \quad (55)$$

where

$$\Lambda(\xi)=\exp\left(\frac{\xi-m_\pi^2}{\pi}\int_{(m_\sigma+m_\pi)^2}^{\infty}\frac{d\xi'\chi(\xi')}{(\xi'-m_\pi^2)(\xi'-\xi-i\epsilon)}\right) \quad (56)$$

and

$$C=\exp\left(-\frac{(m_\sigma^2-m_\pi^2)}{\pi}\int_{(m_\sigma+m_\pi)^2}^{\infty}\frac{d\xi'\,\chi(\xi')}{(\xi'-m_\pi^2)(\xi'-m_\sigma^2)}\right). \quad (57)$$

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[8] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

Only $\sigma\pi$ scattering contributions to the Omnès factors are considered, and thus,

$$\tan\chi(\xi)=\mathrm{Re}(e^{i\eta_0}\sin\eta_0)/[1-\mathrm{Im}(e^{i\eta_0}\sin\eta_0)], \quad (58)$$

where η_0 is the $^1S_0$ complex $\sigma\pi$ scattering phase shift. If a once-subtracted dispersion relation with only $\sigma\pi$ scattering contributions is assumed for $\gamma_{\sigma\pi\pi}(\xi)$, then

$$\gamma_{\sigma\pi\pi}(\xi)=\gamma_{\sigma\pi\pi}\Lambda(\xi). \quad (59)$$

The $\sigma\pi$ contribution to $\mathrm{Im}F_\pi(\xi)$ is thus given by

$$\mathrm{Im}F_\pi(\xi)=\tfrac{1}{8}\pi^{-1}\xi^{-2}\times[(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2]^{1/2}|\Lambda(\xi)|^2m_\sigma\gamma_{\sigma\pi\pi}\times\left(\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}-\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{\xi-m_\pi^2}\right). \quad (60)$$

3. SUM RULES

We can now combine the various contributions to $\mathrm{Im}F_\pi(\xi)$ arising from the $\rho\pi$, $\sigma\pi$, and $\bar{N}N$ intermediate states. The total result is

$$\begin{aligned}\mathrm{Im}F_\pi(\xi)=&\tfrac{1}{8}\pi^{-1}\xi^{-2}[(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2]^{1/2}m_\sigma\gamma_{\sigma\pi\pi}\left(\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}-\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{\xi-m_\pi^2}\right)\\&\times|\Lambda(\xi)|^2\theta(\xi-(m_\sigma+m_\pi)^2)+\frac{1}{4\pi}\left(\frac{\xi-4M^2}{\xi}\right)^{1/2}MG\left(g_AB+\frac{F_\pi G}{M}\frac{\xi}{m_\pi^2-\xi}\right)|K_{NN\pi}(\xi)|^2\theta(\xi-4M^2)\\&+\frac{1}{4\pi}[(\xi-m_\rho^2-m_\pi^2)^2-4m_\rho^2m_\pi^2]^{3/2}\frac{\gamma_{\rho\pi\pi}}{4m_\rho^2\xi^2}\\&\times\left(K(m_\rho^2)A-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-m_\rho^2}+\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-\xi}\right)|\Omega(\xi)|^2\theta(\xi-(m_\rho+m_\pi)^2). \quad (61)\end{aligned}$$

Let us assume that $F_\pi(\xi)\sim-(K/\xi)$ as $\xi\to\infty$, where K is a finite constant. Then Eq. (1) leads to the superconvergence condition

$$\int \mathrm{Im}F_\pi(\xi)d\xi=K. \quad (62)$$

Substitution of (61) into (62) yields a relation containing three terms which are, respectively, quadratically, linearly, and logarithmically divergent. By demanding that the coefficients of these divergent terms vanish, we obtain three sum rules. We shall assume that the coefficients of ξ^{-1} and ξ^{-2} in the expansion of $\Lambda(\xi)$ and $K_{NN\pi}(\xi)$, in decreasing powers of ξ, are either zero or very small and may be neglected. The expansions

$$\left(\frac{(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2}{\xi^2}\right)^{1/2}=1+R_\sigma(\xi),\quad [(\xi-4m^2)/\xi]^{1/2}=1-2m^2/\xi+R_N(\xi),$$

$$\left(\frac{(\xi-m_\rho^2-m_\pi^2)^2-4m_\rho^2m_\pi^2}{\xi^2}\right)^{3/2}=1-\frac{3(m_\rho^2+m_\pi^2)}{\xi}+R_\rho(\xi),$$

are substituted into Eq. (61) to give

$$\begin{aligned}4\pi\,\mathrm{Im}F_\pi(\xi)=&(1/2\xi)[1+R_\sigma(\xi)]|\Lambda(\xi)|^2\theta(\xi-(m_\sigma+m_\pi)^2)m_\sigma\gamma_{\sigma\pi\pi}\left(\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}-\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{\xi-m_\pi^2}\right)\\&+\left[1-\frac{2M^2}{\xi}+R_N(\xi)\right]MG\left[g_AB+\frac{F_\pi G}{M}\left(-1-\frac{m_\pi^2}{\xi}+\frac{m_\pi^4}{\xi(m_\pi^2-\xi)}\right)\right]|K_{NN\pi}(\xi)|^2\theta(\xi-4M^2)\\&+\left[1-\frac{3(m_\rho^2+m_\pi^2)}{\xi}+R_\rho(\xi)\right]|\Omega(\xi)|^2\theta(\xi-(m_\rho+m_\pi)^2)\frac{\xi\gamma_{\rho\pi\pi}}{4m_\rho^2}\left[K(m_\rho^2)A-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-m_\rho^2}\right.\\&\left.+2F_\pi\gamma_{\rho\pi\pi}m_\pi^2\left(-\frac{1}{\xi}-\frac{m_\pi^2}{\xi^2}+\frac{m_\pi^4}{\xi^2(m_\pi^2-\xi)}\right)\right]. \quad (63)\end{aligned}$$

Setting the coefficients of ξ, ξ^0, and ξ^{-1} equal to zero gives the sum rules

$$K(m_\rho^2)A-2F_\pi\gamma_{\rho\pi\pi}m_\pi^2/(m_\pi^2-m_\rho^2)=0\,, \tag{64}$$

$$MG\left(g_AB-\frac{F_\pi G}{M}\right)|K_{NN\pi}(\infty)|^2-\frac{2F_\pi\gamma_{\rho\pi\pi}{}^2m_\pi^2}{4m_\rho^2}|\Omega(\infty)|^2=0\,, \tag{65}$$

$$\tfrac{1}{2}|\Lambda(\infty)|^2m_\sigma\gamma_{\sigma\pi\pi}\left(\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}\right)-\left[2M^3G\left(g_AB-\frac{F_\pi G}{M}\right)+F_\pi G^2m_\pi^2\right]|K_{NN\pi}(\infty)|^2$$
$$-|\Omega(\infty)|^2(2F_\pi\gamma_{\rho\pi\pi}{}^2m_\pi^2/4m_\rho^2)[m_\pi^2-3(m_\rho^2+m_\pi^2)]=0\,. \tag{66}$$

The first sum rule Eq. (64) is the generalized KSRF relation as can be seen by using the pole-dominance approximation Eq. (30) for $K(m_\rho^2)$ and adopting the resonance approximation $A=1$. Thus, we have[7,9,10]

$$\frac{G_\rho}{F_\pi}-2F_\pi\gamma_{\rho\pi\pi}+\frac{G_AG_S}{m_A^2-m_\rho^2}=\frac{G_A}{m_A^2}[G_S+\tfrac{1}{2}(m_\rho^2-m_\pi^2)G_D]\,, \tag{67}$$

where G_S and G_D are the S- and D-wave coupling constants for the $A_1\rightarrow\rho\pi$ decay process. If we neglect the terms involving the A_1 meson and use the ρ dominance of the electromagnetic form factor

$$G_\rho\gamma_{\rho\pi\pi}=m_\rho^2\,, \tag{68}$$

we obtain the KSRF relation

$$\gamma_{\rho\pi\pi}{}^2F_\pi^2=\tfrac{1}{2}m_\rho^2\,. \tag{69}$$

The second sum rule Eq. (65) may be solved for F_π to yield

$$F_\pi=B\frac{g_AM}{G}\left[1+\frac{1}{2}\left(\frac{\gamma_{\rho\pi\pi}m_\pi}{Gm_\rho}\right)^2\left|\frac{\Omega(\infty)}{K_{NN\pi}(\infty)}\right|^2\right]^{-1}. \tag{70}$$

By using the definition of B in Eq. (38) and the fact that $0<\phi(\xi)<\pi$, it can easily be shown that $1<B<1.01$.

The correction to the Goldberger-Treiman relation given by (70) is in the wrong direction. We can estimate this correction by using the values $m_\rho=0.769$ GeV, $\gamma_{\rho\pi\pi}=5.14$ (corresponding to a ρ width $\Gamma_\rho=112$ MeV). We get

$$\tfrac{1}{2}(\gamma_{\rho\pi\pi}m_\pi/Gm_\rho)^2=0.0024\,. \tag{71}$$

If $|\Omega(\infty)/K_{NN\pi}(\infty)|\sim1$, then the correction is very small. A ratio $|\Omega(\infty)/K_{NN\pi}(\infty)|$ of 10 would increase the error from 13% to 30%.

An inspection of the terms appearing in the third sum rule, Eq. (66), shows that the $\bar{N}N$ terms dominate the sum rule; therefore, another modified Goldberger-Treiman relation is obtained.

The sum rule[9] obtained upon dispersing the matrix element

$$\langle\pi|\partial^\mu A_\mu|\sigma\rangle$$

would be

$$\frac{G_\sigma}{F_\pi}C+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{m_\sigma^2-m_\pi^2}=0\,, \tag{72}$$

with $G_\sigma=m_\pi^2F_\pi$. This result is not in disagreement with our sum rules (64)–(66), but it cannot be deduced directly from the pion lifetime calculation.

4. RESULTS BASED ON POLE DOMINANCE OF $K_{NN\pi}(\xi)$, $\gamma_{\rho\pi\pi}(\xi)$, AND $\gamma_{\sigma\pi\pi}(\xi)$

We shall now evaluate the matrix element of the axial vector current assuming that the form factors $K_{NN\pi}(\xi)$, $\gamma_{\rho\pi\pi}(\xi)$, and $\gamma_{\sigma\pi\pi}(\xi)$ are dominated by a three-pion resonance possessing the quantum numbers of the pion. In this section, pole-dominance approximations for all form factors will be used. Let us consider[11]

$$K_{NN\pi}(\xi)=G(M_R^2-M_\pi^2)/(M_R^2-\xi)\,, \tag{73}$$

$$\gamma_{\rho\pi\pi}(\xi)=\gamma_{\rho\pi\pi}(M_R^2-M_\pi^2)/(M_R^2-\xi)\,, \tag{74}$$

and

$$\gamma_{\sigma\pi\pi}(\xi)=\gamma_{\sigma\pi\pi}(M_R^2-M_\pi^2)/(M_R^2-\xi)\,. \tag{75}$$

Here M_R denotes the mass of the dominant three-pion resonance. We then obtain for the $\bar{N}N$, $\rho\pi$, and $\sigma\pi$

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[11] J. G. Cordes and J. W. Moffat, Phys. Rev. **164**, 1787 (1967).

contributions in $\mathrm{Im}F_\pi(\xi)$,

$$4\pi\,\mathrm{Im}F_\pi(\xi)=[(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2]^{1/2}\theta(\xi-(m_\sigma+m_\pi)^2)(m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2/2\xi^2)$$
$$\times\left(1+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}}{m_\sigma^2-m_\pi^2}+\frac{M_R^2}{m_\pi^2}\frac{F_\pi M_R\gamma_{R\sigma\pi}}{m_\sigma^2-M_R^2}-\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}}{\xi-m_\pi^2}-\frac{M_R^2}{m_\pi^2}\frac{F_\pi M_R\gamma_{R\sigma\pi}}{\xi-M_R^2}\right)$$
$$\times\frac{M_R^2-m_\pi^2}{M_R^2-\xi}+\left(\frac{\xi-4M^2}{\xi}\right)^{1/2}MG\left(g_A+\frac{F_\pi G}{M}\frac{\xi}{m_\pi^2-\xi}+\frac{F_RG_{NNR}}{M}\frac{\xi}{M_R^2-\xi}\right)$$
$$\times\frac{M_R^2-m_\pi^2}{M_R^2-\xi}\theta(\xi-4M^2)+[(\xi-m_\rho^2-m_\pi^2)^2-4m_\rho^2m_\pi^2]^{3/2}\theta(\xi-(m_\rho+m_\pi)^2)(\gamma_{\rho\pi\pi}/4m^2\xi^2)$$
$$\times\left(K(m_\rho^2)-\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-m_\rho^2}+\frac{F_R\gamma_{R\rho\pi}M_R^2}{M_R^2-m_\rho^2}+\frac{2F_\pi\gamma_{\rho\pi\pi}m_\pi^2}{m_\pi^2-\xi}-\frac{F_R\gamma_{R\rho\pi}M_R^2}{M_R^2-\xi}\right)\frac{M_R^2-m_\pi^2}{M_R^2-\xi},\quad(76)$$

where F_R, G_{NNR}, $\gamma_{R\rho\pi}$, and $\gamma_{R\sigma\pi}$ are defined analogously to F_π, G, $\gamma_{\rho\pi\pi}$, and $\gamma_{\sigma\pi\pi}$ defined in Eqs. (2), (33), (27), and (54), respectively. This result is substituted into the unsubtracted dispersion relation Eq. (1). The terms containing the double pole $(M_R^2-\xi)^{-2}$ are evaluated assuming that the resonance has a width Γ_R so that the double pole becomes $[(M_R^2-\xi)^2+\Gamma_R^2M_R^2]^{-2}$. In the terms containing only the single pole $(M_R^2-\xi)^{-1}$, the principal value is taken, a procedure which corresponds to the narrow-width approximation. The $\rho\pi$ contribution contains a logarithmically divergent part. Applying the convergence condition and using the pole-dominance approximation for $K(m_\rho^2)$, we obtain the relation

$$G_\rho/F_\pi-2F_\pi\gamma_{\rho\pi\pi}+F_R\gamma_{R\rho\pi}=0\,.$$

In order that the KSRF relation be well satisfied, the term $F_R\gamma_{R\rho\pi}$ must be small. If we assume that $F_R(\xi)$ is dominated by the pion pole, consistency conditions yield the relations

$$F_R\gamma_{R\rho\pi}=\alpha 2F_\pi\gamma_{\rho\pi\pi}\,,\quad F_RG_{NNR}=-\alpha F_\pi G\,,$$

and

$$M_R\gamma_{R\sigma\pi}F_R=-\alpha 2m_\sigma\gamma_{\sigma\pi\pi}F_\pi\,,$$

where

$$\alpha=g_{R\pi}(m_\pi^2)/g_{R\pi}(M_R^2)\,.$$

Also,

$$\langle 0|\,j_\pi{}^a(0)\,|R^b(q)\rangle\equiv[\delta_{ab}/(2\pi)^{3/2}]g_{R\pi}(q^2)\,.$$

Thus, in order to retain the KSRF relation, α must be small. Since α is unknown, we assume that it is so small that the terms containing it may be dropped. In the remaining terms, we shall neglect the pion mass whenever numerically justified in order to simplify the calculations. The remaining contributions for $\xi=m_\pi^2$ are then of the form

$$4\pi^2F_\pi=MG\left(g_A-\frac{F_\pi G}{M}\right)I_1-\frac{2F_\pi\gamma_{\rho\pi\pi}{}^2m_\pi^2}{4m_\rho^2}I_2$$
$$+\frac{m_\sigma\gamma_{\sigma\pi\pi}m_\pi^2}{2(m_\sigma+m_\pi)^2}\left(1+\frac{2F_\pi m_\sigma\gamma_{\sigma\pi\pi}}{m_\sigma^2-m_\pi^2}\right)I_3$$
$$-\frac{m_\sigma^2\gamma_{\sigma\pi\pi}{}^2m_\pi^2F_\pi}{(m_\sigma+m_\pi)^4}I_4,\quad(77)$$

where

$$I_1=\int_{4M^2}^\infty d\xi\left(\frac{\xi-4M^2}{\xi}\right)^{1/2}\frac{M_R^2-m_\pi^2}{\xi(M_R^2-\xi)},\quad(78)$$

$$I_2=\int_{(m_\rho+m_\pi)^2}^\infty d\xi\times\frac{[(\xi-m_\rho^2-m_\pi^2)^2-4m_\rho^2m_\pi^2]^{3/2}}{\xi^4}\frac{(M_R^2-m_\pi^2)}{(M_R^2-\xi)},\quad(79)$$

$$I_3=(m_\sigma+m_\pi)^2\int_{(m_\sigma+m_\pi)^2}^\infty d\xi\times\frac{[(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2]^{1/2}(M_R^2-m_\pi^2)}{(\xi-m_\pi^2)\xi^2(M_R^2-\xi)},\quad(80)$$

and

$$I_4=(m_\sigma+m_\pi)^4\int_{(m_\sigma+m_\pi)^2}^\infty d\xi\times\frac{[(\xi-m_\sigma^2-m_\pi^2)^2-4m_\sigma^2m_\pi^2]^{1/2}(M_R^2-m_\pi^2)}{(\xi-m_\pi^2)^2\xi^2(M_R^2-\xi)}.\quad(81)$$

Wherever necessary, the principal value of the integral is implied.

The roots in (79)–(81) may be evaluated by making the approximations $(m_\rho-m_\pi)/(m_\rho+m_\pi)\approx1$ and

$(m_\sigma - m_\pi)/(m_\sigma + m_\pi) \approx 1$. Performing the integrations in (78)–(81), we get

$$I_1 = 2[(M_R^2 - m_\pi^2)/M_R^2][a \arctan(1/a) - 1] \quad (M_R < 2M), \quad (82)$$

where

$$a = [(2M/M_R)^2 - 1]^{1/2}$$

and

$$I_1 = 2[(M_R^2 - m_\pi^2)/M_R^2][\tfrac{1}{2} a \ln|(1+a)/(1-a)| - 1] \quad (M_R > 2M), \quad (83)$$

where

$$a = [1 - (2M/M_R)^2]^{1/2}.$$

Also, we have

$$I_2 = [(M_R^2 - m_\pi^2)/M_R^2] \times [(1-z)^3 \ln|(1-z)/z| - \tfrac{1}{6}(6z^2 - 15z + 11)], \quad (84)$$

where

$$z = [(m_\rho + m_\pi)/M_R]^2$$

and

$$I_3 = [(M_R^2 - m_\pi^2)/M_R^2] \times [(\tfrac{1}{2} - y) + (y - y^2) \ln|(y-1)/y|], \quad (85)$$

$$I_4 = [(M_R^2 - m_\pi^2)/M_R^2] \times [\tfrac{1}{6}(1 + 3y - 6y^2) + (y^2 - y^3) \ln|(y-1)/y|], \quad (86)$$

where

$$y = [(m_\sigma + m_\pi)/M_R]^2.$$

As $M_R \to \infty$, we have $a \to 1$, $z \to 0$, and $y \to 0$. Thus $I_1 \to 2 \ln M_R$, $I_2 \to 2 \ln M_R$, $I_3 \to \frac{1}{2}$, and $I_4 \to \frac{1}{6}$. Therefore, in the limit $M_R \to \infty$ the sum rule Eq. (77) reduces to our earlier result Eq. (65).

We have performed a numerical calculation in order to determine the value of M_R for which the modified Goldberger-Treiman relation Eq. (77) can be satisfied experimentally to give $F_\pi = 0.10M$. In Eq. (77), we shall choose a σ-meson mass $m_\sigma = 0.75$ GeV and a width $\Gamma_\sigma = 0.42$ GeV.[12] For the ρ meson we choose $m_\rho = 0.769$ GeV and $\Gamma_\rho = 0.112$ GeV. This latter width is the latest value obtained by the Orsay group from the $\rho^0 \to e^+e^-$ process.[13] These input values correspond to $\gamma_{\sigma\pi\pi} = -2.25$ and $\gamma_{\rho\pi\pi} = 5.14$, where the choice of the sign of the $\sigma\pi\pi$ coupling is in accordance with Eq. (72), i.e., the result obtained from current algebra and PCAC. For $g_A = 1.18$ the result of the calculation shows that Eq. (77) gives $F_\pi = 0.10M$ for $M_R = 1.87$ GeV or $M_R = 2.0$ GeV. The value of M_R required to make Eq. (77) satisfied is insensitive to the $\sigma\pi$ contribution for either sign of $\gamma_{\rho\pi\pi}$.

Resonance structure in the $\bar{N}N$ S-wave channel with $I = 1$ has been observed around 2 GeV. In particular a resonance designated by $T(2195)$[14] has been seen which could be tentatively identified with our resonance.

5. CONCLUSIONS

In both calculations, it turns out that in order to obtain finite results, a convergence condition must be imposed which has the effect of requiring that $K(\xi)$ be unsubtracted. From this point, our derivation of the KSRF relation is essentially the same as that in Ref. 7. The two derivations of the KSRF relation differ in that ours is obtained as a necessary convergence condition, while in Ref. 7 it follows from PCAC.

We note that in the first calculation in Sec. 3, the imposition of a second convergence condition leads to the result that $C(\xi)$ must be once-subtracted for consistency because of the admixture of the $\rho\pi$ contribution. In the second calculation, in Sec. 4, we do not find it necessary to impose further restrictions on $C(\xi)$.

If one insists on using once-subtracted dispersion relations for the $NN\pi$, $\rho\pi\pi$, and $\sigma\pi\pi$ vertex functions, the results of Sec. 3 indicate that it is not possible to obtain from the pion lifetime calculation a set of results consistent with experiment. The generalized KSRF relation can be derived, but the modified Goldberger-Treiman relation obtained has corrections which are in the wrong direction. Although these corrections may be very small, they could be quite large.

However, the results of Sec. 4 indicate that it is possible to obtain a set of results from the pion lifetime calculation which are in complete accord with experiment, if unsubtracted dispersion relations are assumed for the $NN\pi$, $\rho\pi\pi$, and $\sigma\pi\pi$ vertex functions, and these dispersion relations are saturated by a three-pion resonance with a mass around 2 GeV. This approach has the advantage of deriving both the generalized KSRF relation and a modified Goldberger-Treiman relation in a consistent manner from the pion lifetime calculation; in particular, the modified Goldberger-Treiman relation can then be brought into agreement with the experimental value of F_π.

There is no direct relation between our work and that of Pagels.[15] In our calculation of the pion lifetime we assume an unsubtracted dispersion relation for $F_\pi(\xi)$ and consider the $\bar{N}N$, $\rho\pi$, and $\sigma\pi$ intermediate-state contributions to $\mathrm{Im}F_\pi(\xi)$. On the other hand, Pagels uses PCAC to relate the form factor $C(\xi)$ to the form factor $K_{NN\pi}(\xi)$. Then the problem of calculating F_π is reduced to the determination of $K_{NN\pi}(0)$. Pagels assumes a once-subtracted dispersion relation for $K_{NN\pi}(\xi)$ and considers various contributions to $\mathrm{Im}K_{NN\pi}(\xi)$, a number of which involve the ρ and σ mesons. Also, in this approach $C(\xi)$ must satisfy an unsubtracted dispersion relation, which is not the case in our calculation, as we have pointed out in the second paragraph of this section.

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