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Relativistic Two-Pion-Exchange Effects in Nucleon-Nucleon Scattering up to 350 MeV

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The two-pion-exchange (TPE) contributions to all of the nucleon-nucleon scattering phase parameters corresponding to values of L between 1 and 5 are presented at the scattering energies from 10 to 350 MeV at intervals of 20 MeV. The phase parameters are derived from the complete fourth-order relativistic scattering matrix, and it is found that the TPE correction to the one-pion-exchange group of phases is large and generally improves the agreement with the phenomenological values. In addition, the effect of the pion mass difference in TPE is tested by calculating the phase parameters at both the neutral- and charged-pion masses. The effect of the nucleon mass difference is tested similarly.

I. INTRODUCTION

THE partial-wave analysis of the exact relativistic two-pion-exchange (TPE) scattering matrix derived by Gupta, Haracz, and Kaskas[1] has been carried out by Haracz and Sharma[2] at nucleon-nucleon scattering energies of 95 and 310 MeV. This analysis is now extended to cover the energy range 10–350 MeV at intervals of 20 MeV. Moreover, the TPE contribution to the phase parameters are evaluated without approximation and to a greater accuracy than was done in Ref. 2. In addition, the effect of the pion and nucleon mass differences on the TPE phase parameters is tested at several energies by calculating these parameters using the charged and neutral particle masses and comparing the results.

Phenomenological evidence for the TPE effect was first established by Breit *et al.*[3] in 1961 by a partial-wave analysis of the nonrelativistic TPE interaction derived by Gupta[4] in 1960 from his exact relativistic TPE scattering operator which is presented in the same paper. These results of Gupta were used by Breit[5] in suggesting that a natural qualitative explanation of the empirical evidence would be achieved by combining the higher-order pion-nucleon effects with the then hypothetical vector mesons. The analysis of Ref. 2 made use of the complete fourth-order relativistic calculation of Ref. 1 to obtain phase parameters in qualitative agreement with those given in Ref. 3. Similar agreement is noted in the results of Wortman,[6] which follow from the evaluation of the fourth-order diagrams by the method of Cutkosky.[7] In this work of Wortman, geometric unitarization[8] is applied to obtain graphs of a selected list of the phase parameters. In the present work, the fourth-order diagrams are evaluated exactly, and the TPE contribution to the phase parameters is tabulated for easy application, while the list of the TPE phase parameters for $0<L\leqslant 5$ is not abbreviated as in the references noted. Although qualitative agreement with the earlier results is apparent, significant differences occur especially for the low-L phase parameters.

II. TPE W MATRIX AND PHASE PARAMETERS

The phase parameters corresponding to the one-pion-exchange (OPE) and TPE effects are conveniently related to a spin and isospin matrix W, as shown in Sec. II of Ref. 2. The W matrix was introduced in Ref. 1 and is directly related to the Hermitian operator K. The operator K is related to the scattering operator S in such a way that a unitary approximation to S is obtained by keeping any number of terms in the expansion of K in powers of the pion-nucleon coupling constant $g^2/4\pi ch=14$.[9] If the incident nucleons have propagation four-vectors p and q and the scattered nucleons have p' and q', respectively, the W matrix in the c.m. system, where $\mathbf{p}=-\mathbf{q}$, $\mathbf{p}'=-\mathbf{q}'$, and $p_0=q_0$

[1] S. N. Gupta, R. D. Haracz, and J. Kaskas, Phys. Rev. **138**, B1500 (1965).
[2] R. D. Haracz and R. D. Sharma, Phys. Rev. **176**, 2013 (1968).
[3] G. Breit, K. E. Lassila, H. M. Ruppel, and M. H. Hull, Jr., Phys. Rev. Letters **6**, 138 (1961). A justification for the way the quantities derived in field theory are used in partial-wave analyses is presented in G. Breit, Ann. Phys. (N. Y.) **16**, 346 (1961).
[4] S. N. Gupta, Phys. Rev. **117**, 1146 (1960). The spin-orbit part of the TPE potential is further studied in the nonrelativistic limit by S. N. Gupta, Phys. Rev. **122**, 1923 (1961).
[5] G. Breit, Proc. Natl. Acad. Sci. U. S. **46**, 746 (1960); Phys. Rev. **120**, 287 (1960). A preliminary report of the first of these papers was presented at the 1960 annual meeting of the National Academy of Science in Washington, D. C. An independent account from a different starting point is found in J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960); Nuovo Cimento **16**, 388 (1960); Phys. Rev. **119**, 1784 (1960).
[6] W. R. Wortman, Phys. Rev. **176**, 1762 (1968). The contribution from the TPE continuum is calculated by evaluating the integrals corresponding to the fourth-order diagrams by the method of Cutkosky. The author points out that the results of Ref. 1 are used to resolve an ambiguity in sign.
[7] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).
[8] M. J. Moravcosik, Ann. Phys. (N. Y.) **30**, 1 (1964).
[9] See the first article of Ref. 4 for a discussion of the operator K and its role in the perturbation theory.

$=p_0'=q_0'$, is parametrized as[10]

$$W=BS+iC\sin\theta(\sigma_n{}^{(1)}+\sigma_n{}^{(2)})+\tfrac{1}{2}G(\sigma_l{}^{(1)}\sigma_l{}^{(2)}+\sigma_m{}^{(1)}\sigma_m{}^{(2)})T+\tfrac{1}{2}H(\sigma_l{}^{(1)}\sigma_l{}^{(2)}-\sigma_m{}^{(1)}\sigma_m{}^{(2)})T+N\sigma_n{}^{(1)}\sigma_n{}^{(2)}T. \quad (1)$$

In this expression, σ_l, σ_m, and σ_n are the components of the Pauli spin matrices along the directions of $\mathbf{p}'-\mathbf{p}$, $\mathbf{p}'+\mathbf{p}$, and $(\mathbf{p}'-\mathbf{p})\times(\mathbf{p}'+\mathbf{p})$, respectively, the superscripts denote nucleon 1 or 2, and S and T are the spin-singlet and spin-triplet projection operators. The real coefficients B, C, G, H, and N depend on the scattering energy, the c.m. scattering angle θ, coupling constants, particle masses, and isospin.

The phase parameters are defined in terms of the partial-wave amplitudes as in Eqs. (6) of Ref. 2. The Yale notation[11] for the phase parameters is used, where K_L is the singlet phase shift, $\delta^L{}_L$ is the triplet uncoupled phase, $\theta^L{}_J$ is a triplet coupled phase, and ρ_J is the coupling parameter. Now the expansion of the operator K in powers of the coupling constant leads directly to a similar expansion of W:

$$W=\sum_{n=1}^{\infty} W(n), \quad (2)$$

where $W(n)$ is proportional to $(g^2/4\pi c\hbar)^{n/2}$, with $W(n)$ related to $K(n)$ as shown in Eq. (3) of Ref. 2. If the phase parameters are also expanded in powers of the coupling constant, the first two orders of the spin matrix elements of W and the phase parameters are related as

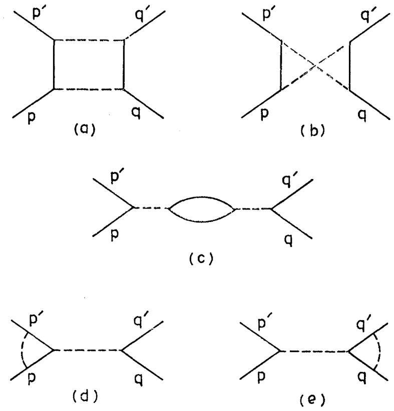

FIG. 1. Diagrams for the fourth-order pion-nucleon interaction.

$$^0W_{00}(i)=-(1/|\mathbf{p}|)(4\pi c\hbar/p_0)\times\sum_L[(4\pi)(2L+1)]^{1/2}K(i)Y_L{}^0 \quad (3)$$

for the singlet state, and

$$\begin{aligned}{}^1W_{M_S,M_{S'}}(i)=&-(1/|\mathbf{p}|)(4\pi c\hbar/p_0)\sum_L[(4\pi)(2L+1)]^{1/2}\Big[C_{L1}(L,M_{S'};M_{S'}-M_S,M_S)C_{L1}(L,M_{S'};0,M_{S'})\delta^L{}_L(i)\\&+C_{L1}(L+1,M_{S'};M_{S'}-M_S,M_S)C_{L1}(L+1,M_{S'};0,M_{S'})\theta^L{}_{L+1}(i)+C_{L1}(L-1,M_{S'};M_{S'}-M_S,M_S)\\&\times C_{L1}(L-1,M_{S'};0,M_{S'})\theta^L{}_{L-1}(i)-\frac{1}{2}\left(\frac{2L+5}{2L+1}\right)^{1/2}C_{L1}(L+1,M_{S'};M_{S'}-M_S,M_S)\\&\times C_{L+2,1}(L+1,M_{S'};0,M_{S'})\rho_{L+1}(i)-\frac{1}{2}\left(\frac{2L-3}{2L+1}\right)^{1/2}C_{L1}(L-1,M_{S'};M_{S'}-M_S,M_S)\\&\times C_{L-2,1}(L-1,M_{S'};0,M_{S'})\rho_{L-1}(i)\Big]Y_L{}^{M_{S'}-M_S} \quad (4)\end{aligned}$$

for the triplet states, where $i=2$, 4 corresponds to the order of the coupling constant. In these equations, $^0W_{00}$ is the singlet matrix element of W, $^1W_{M_S,M_{S'}}$ is a triplet matrix element,[12] M_S, $M_{S'}=1$, 0, -1, the $Y_L{}^{M_L}$ are spherical harmonics, and the $C_{LS}(J,M_J;M_L,M_S)$ are Wigner coefficients. This result follows from the well-known partial-wave decomposition of the scattering matrix.[13] The inverted relationships for the fourth-order terms in the expansions of the phase parameters in terms of the fourth-order terms of the W matrix, $W(4)$, is derived in Ref. 2 and follows from the relativistic scattering operator derived in Ref. 4. All the renormalized fourth-order contributions are included and are represented by uncrossed and crossed diagrams [Figs. 1(a) and 1(b), respectively] and by the pion self-energy and by vertex diagrams [Figs. 1(c)–1(e)]. For these diagrams, the solid lines correspond to nucleons and the dashed lines to pions. The vertex and

[10] This parametrization is similar to that used in parametrizing the M matrix by L. W. Wolfenstein, Phys. Rev. **96**, 1654 (1954).

[11] The Yale notation is similar to the nuclear bar notation, the definitions of the coupling parameters differing as $\rho_J=\sin 2\bar{\epsilon}_J$. See G. Breit, Rev. Mod. Phys. **34**, 766 (1962), especially p. 786.

[12] The relations between the spin matrix elements of W and the coefficients B, C, G, H, and N are given in Eq. (12) of Ref. 2. However, the coefficient C appears with the wrong sign. The correction of this error reveals significant qualitative changes in the triplet coupled and uncoupled phase shifts for low values of L. The singlet phase shifts and the coupling parameters are not affected, and the conclusions of Ref. 2 are not changed.

[13] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955); G. Breit, J. B. Ehrman, and M. H. Hull, Jr., *ibid.* **97**, 1051 (1955); H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, *ibid.* **105**, 302 (1957).

TABLE I. OPE and TPE contributions to the phase parameters for the isosinglet ($T=0$) state. The coupled and uncoupled phase shifts are in radians. The Yale phenomenological phase parameters are included for comparison and correspond to fit $(\text{Y-IV})_{pp+np}$ in Ref. 14. The number in parentheses above each block of the phenomenological phase parameters is the parallel-shift uncertainty for that block. These uncertainties are given in Table VI of Ref. 14.

E (MeV)	OPE	OPE+ TPE	YALE	OPE	OPE+ TPE	YALE	OPE	OPE+ TPE	YALE
Phase parameter		ρ_1			K_1			$^3\theta^{D}_{1}$	
			(0.0171)			(0.0085)			(0.0074)
10	0.1949	0.1266	0.0404	−0.0603	−0.0468	−0.0450	−0.0086	−0.0059	−0.0122
30	0.5246	0.2442	0.0682	−0.1362	−0.0946	−0.0956	−0.0493	−0.0615	−0.0662
50	0.7529	0.2684	0.0792	−0.1722	−0.1126	−0.1224	−0.0940	−0.1466	−0.1189
			(0.0220)			(0.0194)			(0.0044)
70	0.9290	0.2619	0.0862	−0.1921	−0.1225	−0.1515	−0.1363	−0.2439	−0.1651
90	1.0738	0.2444	0.0904	−0.2040	−0.1310	−0.1860	−0.1757	−0.3470	−0.2016
110	1.1978	0.2237	0.0927	−0.2113	−0.1407	−0.2254	−0.2122	−0.4530	−0.2285
130	1.3068	0.2034	0.0947	−0.2159	−0.1525	−0.2689	−0.2461	−0.5606	−0.2478
150	1.4043	0.1853	0.0975	−0.2187	−0.1668	−0.3145	−0.2778	−0.6691	−0.2638
			(0.1151)			(0.1045)			(0.0970)
170	1.4928	0.1702	0.1021	−0.2203	−0.1836	−0.3599	−0.3075	−0.7779	−0.2802
190	1.5741	0.1586	0.1088	−0.2210	−0.2028	−0.4050	−0.3355	−0.8868	−0.2983
210	1.6492	0.1506	0.1179	−0.2211	−0.2244	−0.4505	−0.3620	−0.9955	−0.3174
230	1.7191	0.1462	0.1289	−0.2208	−0.2482	−0.4950	−0.3870	−1.1040	−0.3397
250	1.7847	0.1455	0.1418	−0.2201	−0.2740	−0.5408	−0.4108	−1.2121	−0.3629
270	1.8463	0.1482	0.1559	−0.2193	−0.3016	−0.5866	−0.4335	−1.3197	−0.3881
			(0.1321)			(0.0910)			(0.0584)
290	1.9045	0.1542	0.1707	−0.2182	−0.3308	−0.6318	−0.4552	−1.4268	−0.4152
310	1.9596	0.1636	0.1859	−0.2170	−0.3616	−0.6798	−0.4759	−1.5334	−0.4432
330	2.0120	0.1759	0.2009	−0.2157	−0.3939	−0.7278	−0.4957	−1.6394	−0.4720
350	2.0620	0.1912	0.2155	−0.2143	−0.4273	−0.7745	−0.5148	−1.7448	−0.5015
Phase parameter		$^3\delta^{D}_{2}$			$^3\theta^{D}_{3}$			ρ_3	
			(0.0073)			(0.0035)			(0.0266)
10	0.0140	0.0153	0.0150	−0.0005	0.0001	0.0004	0.0029	0.0029	0.0031
30	0.0734	0.0858	0.0911	−0.0055	0.0016	0.0033	0.0273	0.0271	0.0292
50	0.1326	0.1618	0.1858	−0.0131	0.0062	0.0072	0.0586	0.0577	0.0625
			(0.0078)			(0.0037)			(0.0085)
70	0.1854	0.2336	0.2552	−0.0214	0.0136	0.0116	0.0893	0.0873	0.0899
90	0.2325	0.3001	0.3053	−0.0299	0.0231	0.0168	0.1176	0.1143	0.1080
110	0.2748	0.3617	0.3443	−0.0383	0.0341	0.0230	0.1434	0.1388	0.1237
130	0.3132	0.4189	0.3776	−0.0464	0.0461	0.0301	0.1670	0.1610	0.1408
150	0.3485	0.4722	0.4056	−0.0541	0.0587	0.0379	0.1886	0.1813	0.1584
			(0.0784)			(0.0405)			(0.0826)
170	0.3810	0.5220	0.4274	−0.0616	0.0717	0.0460	0.2084	0.1999	0.1755
190	0.4112	0.5686	0.4436	−0.0688	0.0848	0.0541	0.2268	0.2171	0.1915
210	0.4395	0.6125	0.4553	−0.0757	0.0981	0.0620	0.2438	0.2331	0.2057
230	0.4660	0.6539	0.4627	−0.0824	0.1113	0.0696	0.2597	0.2480	0.2182
250	0.4909	0.6929	0.4661	−0.0888	0.1244	0.0768	0.2745	0.2621	0.2287
270	0.5145	0.7298	0.4665	−0.0949	0.1374	0.0834	0.2885	0.2753	0.2376
			(0.0860)			(0.0248)			(0.0530)
290	0.5369	0.7648	0.4644	−0.1008	0.1502	0.0895	0.3016	0.2878	0.2449
310	0.5582	0.7981	0.4602	−0.1065	0.1628	0.0951	0.3140	0.2996	0.2506
330	0.5785	0.8296	0.4546	−0.1120	0.1751	0.1002	0.3258	0.3109	0.2553
350	0.5979	0.8596	0.4478	−0.1174	0.1871	0.1048	0.3369	0.3217	0.2591
Phase parameter		K_3			$^3\theta^{G}_{3}$			$^3\delta^{G}_{4}$	
			(0.0102)			(OPE)			(OPE)
10	−0.0012	−0.0012	−0.0013	−0.0001	−0.0001	−0.0001	0.0002	0.0002	0.0003
30	−0.0103	−0.0100	−0.0110	−0.0015	−0.0015	−0.0016	0.0046	0.0047	0.0049
50	−0.0208	−0.0198	−0.0222	−0.0046	−0.0047	−0.0049	0.0126	0.0130	0.0135
			(0.0063)			(0.0047)			(0.0058)
70	−0.0301	−0.0279	−0.0301	−0.0086	−0.0091	−0.0091	0.0222	0.0231	0.0237
90	−0.0379	−0.0342	−0.0352	−0.0130	−0.0143	−0.0138	0.0321	0.0338	0.0344
110	−0.0445	−0.0390	−0.0389	−0.0176	−0.0200	−0.0188	0.0421	0.0446	0.0450
130	−0.0500	−0.0425	−0.0406	−0.0223	−0.0261	−0.0236	0.0517	0.0553	0.0553
150	−0.0547	−0.0450	−0.0404	−0.0269	−0.0325	−0.0275	0.0611	0.0657	0.0651
			(0.0647)			(0.0457)			(0.0413)
170	−0.0587	−0.0466	−0.0390	−0.0316	−0.0391	−0.0298	0.0700	0.0759	0.0728
190	−0.0621	−0.0476	−0.0368	−0.0361	−0.0457	−0.0306	0.0786	0.0857	0.0780
210	−0.0650	−0.0481	−0.0337	−0.0406	−0.0525	−0.0300	0.0869	0.0952	0.0812
230	−0.0675	−0.0481	−0.0300	−0.0449	−0.0593	−0.0285	0.0947	0.1045	0.0824
250	−0.0697	−0.0477	−0.0261	−0.0491	−0.0661	−0.0264	0.1023	0.1134	0.0819
270	−0.0716	−0.0470	−0.0223	−0.0533	−0.0729	−0.0238	0.1095	0.1220	0.0795
			(0.0326)			(0.0473)			(0.0481)
290	−0.0733	−0.0461	−0.0183	−0.0573	−0.0798	−0.0212	0.1164	0.1303	0.0755
310	−0.0748	−0.0450	−0.0144	−0.0612	−0.0866	−0.0187	0.1231	0.1384	0.0704
330	−0.0760	−0.0438	−0.0105	−0.0650	−0.0934	−0.0164	0.1295	0.1462	0.0644
350	−0.0772	−0.0424	−0.0067	−0.0687	−0.1002	−0.0142	0.1356	0.1538	0.0573

TABLE I (*continued*)

E (MeV) Phase parameter	OPE	OPE+TPE ${}^3\theta_{G_5}$	YALE	OPE	OPE+TPE ρ_5	YALE	OPE	OPE+TPE K_5	YALE
			(OPE)			(OPE)			(OPE)
10	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
30	−0.0003	−0.0002	−0.0003	0.0022	0.0022	0.0023	−0.0009	−0.0009	−0.0010
50	−0.0011	−0.0009	−0.0011	0.0073	0.0074	0.0078	−0.0029	−0.0029	−0.0031
			(0.0035)			(OPE)			(OPE)
70	−0.0023	−0.0016	−0.0024	0.0142	0.0143	0.0152	−0.0055	−0.0055	−0.0059
90	−0.0037	−0.0024	−0.0040	0.0219	0.0219	0.0234	−0.0082	−0.0082	−0.0088
110	−0.0054	−0.0032	−0.0057	0.0298	0.0298	0.0318	−0.0109	−0.0108	−0.0117
130	−0.0071	−0.0038	−0.0074	0.0377	0.0377	0.0403	−0.0135	−0.0132	−0.0144
150	−0.0089	−0.0043	−0.0081	0.0454	0.0454	0.0485	−0.0159	−0.0154	−0.0170
			(0.0493)			(0.0241)			(0.0262)
170	−0.0108	−0.0046	−0.0075	0.0529	0.0528	0.0565	−0.0182	−0.0174	−0.0194
190	−0.0126	−0.0048	−0.0058	0.0601	0.0600	0.0645	−0.0203	−0.0193	−0.0217
210	−0.0145	−0.0049	−0.0032	0.0670	0.0668	0.0722	−0.0222	−0.0209	−0.0240
230	−0.0163	−0.0048	0.0001	0.0736	0.0734	0.0799	−0.0240	−0.0223	−0.0265
250	−0.0182	−0.0046	0.0039	0.0799	0.0797	0.0868	−0.0256	−0.0236	−0.0291
270	−0.0200	−0.0043	0.0081	0.0860	0.0857	0.0933	−0.0271	−0.0247	−0.0317
			(0.0173)			(0.0223)			(0.0288)
290	−0.0218	−0.0039	0.0126	0.0918	0.0915	0.0996	−0.0285	−0.0257	−0.0345
310	−0.0235	−0.0034	0.0173	0.0974	0.0970	0.1058	−0.0298	−0.0266	−0.0374
330	−0.0252	−0.0028	0.0222	0.1027	0.1023	0.1121	−0.0310	−0.0273	−0.0404
350	−0.0269	−0.0022	0.0271	0.1078	0.1074	0.1185	−0.0321	−0.0279	−0.0435

pion self-energy parts contribute the following to the coefficients of $W(4)$:

$$B(4)=-H(4)=-G(4)=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}(g^4/16\pi c\hbar p_0{}^2)\times(\mathbf{k}^2/[\mathbf{k}^2+\lambda^2])(f_1+f_2)\,, \quad (5)$$

where τ_i is an isospin matrix, $\mathbf{k}=\mathbf{p}'-\mathbf{p}$, $\lambda=m_\pi c/\hbar$, with m_π taken to be the average pion mass of 138 MeV, and f_1 and f_2 are integrals defined in Eqs. (18) and (19) of Ref. 1. The uncrossed and crossed diagrams were shown in Ref. 2 to dominate the fourth-order pion-nucleon effect. They contribute the following fourth-order coefficients:

$$B(4)=A\left\{\left[\left(2+\frac{\mathbf{s}^2}{2\kappa(\kappa+p_0)}\right)^2+\frac{\mathbf{k}^2\mathbf{s}^2}{4\kappa^2(\kappa+p_0)^2}\right]F_1+\frac{4\mathbf{p}^2+\mathbf{k}^2}{\kappa^2}F_2+\frac{\mathbf{p}^2\mathbf{s}^2}{\kappa^4}F_3\right\},$$

$$C(4)=A\left[\left(2+\frac{\mathbf{s}^2}{2\kappa(\kappa+p_0)}\right)\frac{\mathbf{p}^2}{\kappa(\kappa+p_0)}F_1+\frac{2\mathbf{p}^2}{\kappa^2}F_2+\frac{\mathbf{p}^2\mathbf{s}^2}{2\kappa^4}F_3\right],$$

$$G(4)=A\left[2\left(2+\frac{\mathbf{s}^2}{2\kappa(\kappa+p_0)}\right)^2F_1+\frac{2\mathbf{s}^2-\mathbf{k}^2}{\kappa^2}F_2+\frac{\mathbf{s}^4}{2\kappa^4}F_3\right], \quad (6)$$

$$H(4)=A(\mathbf{k}^2/\kappa^2)F_2\,,$$

$$N(4)=A\left\{\left[\left(2+\frac{\mathbf{s}^2}{2\kappa(\kappa+p_0)}\right)^2-\frac{\mathbf{k}^2\mathbf{s}^2}{4\kappa^2(\kappa+p_0)^2}\right]F_1+\frac{\mathbf{s}^2-\mathbf{k}^2}{\kappa^2}F_2+\frac{\mathbf{s}^2(\mathbf{s}^2-2\mathbf{p}^2)}{2\kappa^4}F_3\right\},$$

where $\mathbf{s}=\mathbf{p}'+\mathbf{p}$, $\kappa=Mc/\hbar$, with M the average nucleon mass taken to be 938.9 MeV, and

$$A=(g^2/4\pi c\hbar)^2 c\hbar/4p_0{}^2\,, \quad (7)$$

$$F_i=a\,\mathrm{Re}I_i+bJ_i\,,\quad i=1,2,3,$$

with

$$a=2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}-3\,,\quad b=2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}+3\,. \quad (8)$$

The multiple integrals I_i and J_i are defined in Eqs. (26) and (34) of Ref. 1, respectively. They depend on the scattering energy and angle and on the average particle masses.

The theoretical phase parameters are presented in Tables I and II at energies from 10 to 350 MeV at intervals of 20 MeV. The isosinglet ($T=0$) parameters appear in Table I and the isostriplet ($T=1$) parameters appear in Table II. The first column under each phase parameter contains the OPE contribution, and the second column contains OPE+TPE, where TPE is the total fourth-order contribution corresponding to the W-matrix coefficients of Eqs. (5) and (6). The third column contains the Yale $(\text{Y-IV})_{pp+np}$ phase parameters of Seamon *et al.*,[14] the $T=0$ parameters being taken from Table IV, and the $T=1$ parameters from Table III of that paper. In this third column, the numbers in parentheses are the uncertainties in the parameters as given in Table VI of Ref. 14. These uncertainties were obtained by parallel shifts of the phase-energy curves within the energy intervals 0–69, 69–155, 155–275, and 275–350 MeV. It should be emphasized that the OPE and the Yale phase parameters are included merely as a reference for the discussion of the fourth-order contributions. The discussion that follows is not meant to imply an expectation of agreement with the phenomenological Yale phase

[14] R. E. Seamon, K. A. Friedman, G. Breit, R. D. Haracz, J. M. Holt, and A. Prakash, Phys. Rev. **165**, 1579 (1968).

TABLE II. OPE and TPE contributions to the phase parameters for the isotriplet ($T=1$) states. The Yale phenomenological phase parameters with their parallel-shift uncertainties are included for comparison.

E (MeV)	OPE	OPE+ TPE	YALE	OPE	OPE+ TPE	YALE	OPE	OPE+ TPE	YALE
Phase parameter		$^3\delta^P{}_0$			$^3\delta^P{}_1$			$^3\theta^P{}_2$	
			(0.0030)			(0.0010)			(0.0009)
10	0.0775	0.0979	0.0665	−0.0488	−0.0310	−0.0441	0.0017	0.0177	0.0147
30	0.2348	0.3057	0.1660	−0.1401	−0.0703	−0.0993	0.0099	0.0786	0.0577
50	0.3602	0.4586	0.2051	−0.2088	−0.0919	−0.1436	0.0188	0.1449	0.1065
			(0.0067)			(0.0021)			(0.0016)
70	0.4648	0.5661	0.2032	−0.2644	−0.1096	−0.1840	0.0273	0.2090	0.1487
90	0.5554	0.6381	0.1808	−0.3117	−0.1278	−0.2201	0.0351	0.2691	0.1834
110	0.6357	0.6816	0.1500	−0.3531	−0.1482	−0.2527	0.0424	0.3249	0.2111
130	0.7081	0.7018	0.1181	−0.3901	−0.1714	−0.2835	0.0492	0.3765	0.2330
150	0.7743	0.7026	0.0862	−0.4236	−0.1976	−0.3107	0.0556	0.4242	0.2492
			(0.0134)			(0.0067)			(0.0040)
170	0.8354	0.6868	0.0538	−0.4544	−0.2266	−0.3372	0.0615	0.4682	0.2616
190	0.8920	0.6570	0.0195	−0.4829	−0.2583	−0.3619	0.0671	0.5089	0.2695
210	0.9450	0.6150	−0.0138	−0.5094	−0.2925	−0.3851	0.0724	0.5466	0.2742
230	0.9948	0.5623	−0.0464	−0.5342	−0.3291	−0.4066	0.0774	0.5815	0.2765
250	1.0418	0.5003	−0.0820	−0.5576	−0.3677	−0.4256	0.0822	0.6138	0.2771
270	1.0863	0.4301	−0.1202	−0.5797	−0.4083	−0.4431	0.0867	0.6437	0.2770
			(0.0246)			(0.0150)			(0.0098)
290	1.1285	0.3527	−0.1584	−0.6006	−0.4507	−0.4598	0.0910	0.6715	0.2767
310	1.1687	0.2688	−0.1987	−0.6205	−0.4946	−0.4756	0.0952	0.6972	0.2764
330	1.2071	0.1792	−0.2381	−0.6395	−0.5400	−0.4902	0.0991	0.7210	0.2754
350	1.2438	0.0844	−0.2753	−0.6576	−0.5867	−0.5040	0.1030	0.7431	0.2737
Phase parameter		ρ_2			K_2			$^3\theta^F{}_2$	
			(0.0014)			(0.0003)			(0.0017)
10	−0.0073	−0.0073	−0.0091	0.0027	0.0031	0.0035	0.0002	0.0002	0.0002
30	−0.0358	−0.0362	−0.0511	0.0121	0.0161	0.0205	0.0026	0.0027	0.0024
50	−0.0622	−0.0626	−0.0746	0.0193	0.0301	0.0354	0.0061	0.0065	0.0059
			(0.0022)			(0.0019)			(0.0027)
70	−0.0846	−0.0844	−0.0845	0.0245	0.0441	0.0486	0.0100	0.0107	0.0087
90	−0.1037	−0.1025	−0.0917	0.0284	0.0581	0.0616	0.0140	0.0150	0.0102
110	−0.1204	−0.1176	−0.0963	0.0313	0.0718	0.0739	0.0179	0.0191	0.0108
130	−0.1352	−0.1305	−0.0987	0.0336	0.0853	0.0852	0.0216	0.0231	0.0112
150	−0.1484	−0.1414	−0.0997	0.0353	0.0986	0.0964	0.0253	0.0269	0.0114
			(0.0047)			(0.0031)			(0.0061)
170	−0.1604	−0.1508	−0.0997	0.0367	0.1115	0.1069	0.0288	0.0304	0.0113
190	−0.1714	−0.1588	−0.0993	0.0378	0.1242	0.1175	0.0321	0.0336	0.0110
210	−0.1815	−0.1659	−0.0986	0.0387	0.1365	0.1277	0.0353	0.0366	0.0105
230	−0.1908	−0.1719	−0.0973	0.0395	0.1484	0.1374	0.0384	0.0393	0.0097
250	−0.1996	−0.1772	−0.0956	0.0401	0.1600	0.1473	0.0414	0.0417	0.0088
270	−0.2077	−0.1818	−0.0933	0.0405	0.1712	0.1569	0.0443	0.0439	0.0078
			(0.0141)			(0.0101)			(0.0099)
290	−0.2154	−0.1858	−0.0908	0.0409	0.1821	0.1659	0.0471	0.0457	0.0066
310	−0.2227	−0.1892	−0.0883	0.0412	0.1926	0.1749	0.0497	0.0474	0.0056
330	−0.2295	−0.1922	−0.0858	0.0414	0.2028	0.1840	0.0523	0.0487	0.0047
350	−0.2360	−0.1948	−0.0834	0.0416	0.2127	0.1927	0.0548	0.0498	0.0037
Phase parameter		$^3\delta^F{}_3$			$^3\theta^F{}_4$			ρ_4	
			(0.0011)			(0.0006)			(OPE)
10	−0.0006	−0.0006	−0.0005	0.0000	0.0000	0.0000	−0.0001	−0.0001	−0.0001
30	−0.0058	−0.0055	−0.0055	0.0004	0.0007	0.0004	−0.0025	−0.0025	−0.0023
50	−0.0128	−0.0118	−0.0126	0.0012	0.0023	0.0011	−0.0067	−0.0068	−0.0064
			(0.0024)			(0.0014)			(0.0011)
70	−0.0200	−0.0177	−0.0205	0.0022	0.0048	0.0025	−0.0116	−0.0116	−0.0112
90	−0.0269	−0.0228	−0.0269	0.0034	0.0079	0.0046	−0.0164	−0.0165	−0.0162
110	−0.0335	−0.0272	−0.0308	0.0046	0.0116	0.0069	−0.0212	−0.0213	−0.0212
130	−0.0396	−0.0309	−0.0337	0.0058	0.0157	0.0096	−0.0257	−0.0257	−0.0257
150	−0.0453	−0.0339	−0.0366	0.0070	0.0200	0.0130	−0.0299	−0.0299	−0.0293
			(0.0037)			(0.0030)			(0.0031)
170	−0.0506	−0.0365	−0.0393	0.0082	0.0247	0.0171	−0.0339	−0.0339	−0.0321
190	−0.0557	−0.0386	−0.0421	0.0094	0.0295	0.0215	−0.0377	−0.0375	−0.0341
210	−0.0604	−0.0404	−0.0448	0.0105	0.0344	0.0260	−0.0412	−0.0409	−0.0356
230	−0.0649	−0.0418	−0.0477	0.0116	0.0395	0.0305	−0.0446	−0.0441	−0.0367
250	−0.0692	−0.0430	−0.0509	0.0127	0.0446	0.0350	−0.0477	−0.0471	−0.0377
270	−0.0733	−0.0440	−0.0543	0.0138	0.0498	0.0397	−0.0507	−0.0499	−0.0389
			(0.0082)			(0.0052)			(0.0098)
290	−0.0771	−0.0447	−0.0578	0.0149	0.0550	0.0446	−0.0536	−0.0526	−0.0405
310	−0.0808	−0.0454	−0.0617	0.0159	0.0602	0.0498	−0.0563	−0.0550	−0.0422
330	−0.0844	−0.0459	−0.0656	0.0169	0.0654	0.0552	−0.0588	−0.0573	−0.0441
350	−0.0878	−0.0462	−0.0693	0.0178	0.0706	0.0606	−0.0613	−0.0595	−0.0463

TABLE II (*continued*)

E (MeV)	OPE	OPE+TPE	YALE	OPE	OPE+TPE	YALE	OPE	OPE+TPE	YALE
Phase parameter		K_4			$^3\theta^H{}_4$			$^3\delta^H{}_5$	
			(OPE)			(OPE)			(OPE)
10	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
30	0.0010	0.0010	0.0009	0.0001	0.0001	0.0001	−0.0004	−0.0004	−0.0004
50	0.0026	0.0027	0.0025	0.0004	0.0005	0.0004	−0.0015	−0.0015	−0.0013
			(0.0009)			(OPE)			(OPE)
70	0.0042	0.0047	0.0042	0.0009	0.0010	0.0008	−0.0029	−0.0029	−0.0027
90	0.0058	0.0066	0.0058	0.0015	0.0016	0.0014	−0.0046	−0.0045	−0.0044
110	0.0073	0.0086	0.0074	0.0022	0.0023	0.0021	−0.0063	−0.0061	−0.0061
130	0.0086	0.0106	0.0090	0.0029	0.0031	0.0028	−0.0081	−0.0077	−0.0079
150	0.0097	0.0126	0.0113	0.0036	0.0040	0.0035	−0.0099	−0.0093	−0.0097
			(0.0018)			(0.0040)			(0.0027)
170	0.0108	0.0146	0.0139	0.0044	0.0048	0.0043	−0.0116	−0.0108	−0.0115
190	0.0117	0.0166	0.0163	0.0051	0.0057	0.0051	−0.0134	−0.0122	−0.0134
210	0.0125	0.0186	0.0183	0.0059	0.0066	0.0061	−0.0150	−0.0135	−0.0160
230	0.0133	0.0206	0.0196	0.0067	0.0075	0.0072	−0.0166	−0.0148	−0.0192
250	0.0139	0.0226	0.0204	0.0074	0.0085	0.0085	−0.0182	−0.0159	−0.0224
270	0.0145	0.0246	0.0207	0.0081	0.0094	0.0100	−0.0197	−0.0170	−0.0249
			(0.0047)			(0.0062)			(0.0074)
290	0.0151	0.0266	0.0208	0.0089	0.0103	0.0116	−0.0212	−0.0180	−0.0267
310	0.0156	0.0285	0.0205	0.0096	0.0111	0.0134	−0.0227	−0.0189	−0.0281
330	0.0160	0.0305	0.0200	0.0103	0.0120	0.0153	−0.0240	−0.0197	−0.0293
350	0.0164	0.0324	0.0194	0.0110	0.0129	0.0173	−0.0254	−0.0205	−0.0302
Phase parameter		$^3\theta^H{}_6$							
			(OPE)						
10	0.0000	0.0000	0.0000						
30	0.0000	0.0000	0.0000						
50	0.0001	0.0001	0.0001						
			(OPE)						
70	0.0003	0.0003	0.0002						
90	0.0005	0.0006	0.0004						
110	0.0007	0.0010	0.0007						
130	0.0010	0.0015	0.0010						
150	0.0013	0.0021	0.0013						
			(0.0027)						
170	0.0017	0.0027	0.0016						
190	0.0020	0.0034	0.0019						
210	0.0024	0.0042	0.0025						
230	0.0027	0.0050	0.0034						
250	0.0031	0.0059	0.0045						
270	0.0034	0.0068	0.0058						
			(0.0041)						
290	0.0038	0.0078	0.0074						
310	0.0041	0.0088	0.0092						
330	0.0045	0.0098	0.0112						
350	0.0048	0.0109	0.0135						

parameters since OPE and TPE effects cannot be expected to take care of the whole physical situation. On the other hand, it is hoped that the availability of the comparison may prove of value in drawing inferences regarding the relative importance of other effects.

The results are sensibly described by discussing the phase shifts according to the value of L. The isosinglet phase shift K_1 and the isotriplet phase shifts $^3\delta^P{}_0$, $^3\delta^P{}_1$, and $^3\theta^P{}_2$ all have a TPE contribution that is a large correction to OPE. However, the theoretical OPE+TPE phase shifts are not close to the Yale values except at the lowest energies. For the D-state phase shifts K_2, $^3\theta^D{}_1$, $^3\delta^D{}_2$, and $^3\theta^D{}_3$, TPE is again a large correction to OPE, and for the isotriplet singlet phase shift K_2 and the isosinglet triplet $^3\theta^P{}_3$, the application of TPE is a striking improvement over OPE. The opposite is the case for the phase shift $^3\theta^D{}_1$. However, this phase shift is coupled to the S state, hence it should be strongly affected by higher-order effects. In the case of the F-state phase shifts K_3, $^3\delta^F{}_3$, and especially $^3\theta^F{}_4$, TPE improves the theoretical result, but the theoretical and Yale phases again diverge above 150 MeV. The G-state phase shifts K_4, $^3\theta^G{}_3$, $^3\delta^G{}_4$, and $^3\theta^G{}_5$ are reasonably close to the Yale values to about 200 MeV, but the values of OPE+TPE diverge beyond this energy. The same situation occurs for the H-state phase shifts K_5, $^3\theta^H{}_4$, $^3\delta^H{}_5$, and $^3\theta^H{}_6$, except that the divergence at the high energies is generally within the parallel-shift uncertainties.

The theoretical coupling parameter ρ_1 is strongly affected by TPE, and the correction is in the right direction. Moreover, the TPE correction to OPE brings the remaining coupling parameters ρ_2, ρ_3, ρ_4,

TABLE III. The TPE $T=1$ phase parameters evaluated using the average nucleon mass of 938.9 MeV and the neutral and charged pion masses of 135.0 and 139.6 MeV, respectively, at the scattering energies of 50, 170, and 350 MeV.

E (MeV)	50		170		350	
m_π (MeV)	135.0	139.6	135.0	139.6	135.0	139.6
${}^3\delta^P{}_0$	0.1047	0.0953	−0.1389	−0.1534	−1.1522	−1.1629
${}^3\delta^P{}_1$	0.1233	0.1135	0.2428	0.2200	0.0906	0.0607
${}^3\theta^P{}_2$	0.1327	0.1227	0.4233	0.3981	0.6626	0.6284
ρ_2	−0.0003	−0.0004	0.0101	0.0094	0.0423	0.0407
K_2	0.0117	0.0104	0.0791	0.0727	0.1788	0.1671
${}^3\theta^F{}_2$	0.0005	0.0004	0.0023	0.0013	−0.0033	−0.0058
${}^3\delta^F{}_3$	0.0012	0.0010	0.0153	0.0136	0.0443	0.0401
${}^3\theta^F{}_4$	0.0012	0.0011	0.0178	0.0158	0.0559	0.0512
ρ_4	−0.0001	−0.0001	0.0001	0.0000	0.0019	0.0017
K_4	0.0002	0.0001	0.0042	0.0037	0.0172	0.0154
${}^3\theta^H{}_4$			0.0005	0.0004	0.0022	0.0017
${}^3\delta^H{}_5$			0.0010	0.0008	0.0054	0.0046
${}^3\theta^H{}_6$			0.0012	0.0010	0.0066	0.0058

and ρ_5 into better agreement within the Yale values, although TPE is not a large correction in these cases.

III. EFFECT OF PION AND NUCLEON MASS DIFFERENCES

The calculations of Sec. II were made using an average pion mass of 138 MeV and an average nucleon mass of 938.9 MeV. However, the difference between the charged and neutral pion masses has been discussed by Breit *et al.*,[15] and shown to produce appreciable changes in the OPE contribution to nucleon-nucleon scattering phase parameters. It thus seems advisable to test the sensitivity of the TPE phase parameters to the pion mass difference. A rough evaluation of the effect of the pion mass difference should suffice, and this is done by using the neutral and charged pion masses in Eqs. (5) and (6) with the average nucleon mass. The results are compared with the hope that this treatment provides at least an order-of-magnitude estimate of the effect. A similar test by Signell[16] was applied to the semitheoretical model of Saylor *et al.*[17] at 213 MeV, and large changes in some of the low-L phase shifts for proton-proton scattering were noted.

The results of the two ways of calculating the $T=1$ TPE phase parameters are presented in Table III at scattering energies of 50, 170, and 350 MeV. It is seen that the TPE phase parameters are generally affected to the second significant figure. The difference is about 10% of the value of the phase, with some exceptions above and below this amount.

The TPE phases were also calculated using the average pion mass of 138 MeV and first the neutron mass and then the proton mass at the energy of 170 MeV. The results indicate that the effect of the nucleon mass difference is generally an order of magnitude smaller than the effect of the pion mass difference.

IV. DISCUSSION

The exact fourth-order terms in the expansion of the relativistic W matrix for nucleon-nucleon scattering have been unambiguously related to the fourth-order terms in the expansions of the partial-wave amplitudes and phase parameters. These contributions to all the phase parameters corresponding to $0<L\leqslant 5$ are evaluated at 18 different energies in the energy range 10–350 MeV. It is found that in most cases the TPE contribution is a large correction to the OPE contribution for $L\leqslant 4$, and it is a significant correction for $L=5$. For values of L greater than 2, the TPE correction to OPE generally brings about better agreement between the theoretical and the phenomenological values. For values of L less than 3, the TPE contribution is especially large and provides the correct modification of the OPE phases at the low energies. As already pointed out, these results are in qualitative agreement with those of Ref. 3, which are based on the nonrelativistic approximation contained in the first article of Ref. 4. A comparison of the present results with those of Ref. 6 indicates good qualitative agreement, but differences up to about 30% are noted for the P- and D-state phase parameters at the higher energies.

The effect of the pion mass difference on the TPE phase parameters was found to be large, possibly 10% of the values of the phases. On the other hand, the effect of the nucleon mass difference is about one order of magnitude smaller than the pion effect.

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