Solutions of Partial-Wave Dispersion Relations. II*

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The study of the partial-wave dispersion relation (PWDR) is continued, being extended to the relativistic problem in the presence of inelasticity. Less restrictive sufficient conditions on the left-hand-cut discontinuity ΔT are obtained to guarantee a solution of the PWDR problem when ΔT has both positive and negative parts. The effect of Castillejo-Dalitz-Dyson poles on solutions is also discussed.

I. INTRODUCTION

In a previous paper we considered nonrelativistic partial-wave dispersion relations (PWDR)[1] and obtained sufficient conditions upon the left-hand-cut discontinuity ΔT which guarantee that N/D equations without Castillejo-Dalitz-Dyson (CDD) poles lead to a solution to the problem. The N/D equations in this context are

$$N(k^2)=\frac{1}{\pi}\int_{-\infty}^{k^2}\frac{dq^2}{q^2-k^2}D(q^2)\Delta T(q^2), \qquad (1.1)$$

$$D(k^2)=1-\frac{1}{\pi}\int_{0}^{\infty}\frac{dq^2}{q^2-k^2}qN(q^2). \qquad (1.2)$$

(In general one has to avoid zeros of D on the physical sheet of the k^2 plane so that $f=N/D$ will have the desired analytic structure.) It was shown in I that a solution exists if ΔT is subject to the condition

$$\frac{2}{\pi}\int_{-\infty}^{-\mu^2}\frac{dq^2}{\sqrt{(-q^2)}}|\Delta T(q^2)|<1. \qquad (1.3)$$

It was also shown that, for the case in which the left-hand-cut discontinuity is replaced by a finite number of poles, and in which the residues of these poles are either all negative (attractive case) or all positive (repulsive case), a solution of the PWDR exists under the less stringent conditions (2.1) and (2.2), respectively.

In Sec. II of this paper, we extend this latter result to the case in which ΔT has poles of both positive and negative residues. [Eq. (2.3)] It is shown that if the attractive part of ΔT is subject to the restriction (2.1), while the repulsive part is subject to (2.2), the solution of the N/D equations exists for the full discontinuity, such that D has no zeros on the physical sheet of the k^2 plane. In Sec. III we again replace ΔT by poles and examine the effect of CDD poles on the zeros of D. The purely attractive and purely repulsive cases are examined in detail, and the mixed case is discussed qualitatively.

In Sec. IV we consider the relativistic version of the N/D equations. It is shown that condition (4.11) upon the left-hand-cut discontinuity ΔT is sufficient to guarantee that $f=N/D$ satisfies the PWDR. The analysis here is similar to that in the nonrelativistic case as carried out in I, which established the sufficiency of condition (1.3). In Sec. V the inelastic version of the relativistic N/D equations is examined. Finally, in Sec. VI the results are discussed and various extensions of them are considered.

II. N/D EQUATIONS WITHOUT CDD POLES

We are considering the PWDR in the case for which the left-hand-cut discontinuity is replaced by poles. The question of when there exists a solution with only the required singularities (i.e., no ghost or bound-state poles) has been considered in I for the case in which the residues of the poles are all of the same sign.

(i) Attractive case:

$$\Delta T=-\pi\sum_{m=1}^{M}\lambda_m\delta(k^2+a_m^2) \quad (a_m,\lambda_m>0).$$

It was shown that a sufficient condition to guarantee a solution with the required singularities is that the following quantity be positive for $0\leq g\leq 1$:

$$A(g)=\det_M\{\delta_{ij}-g[\lambda_i/(a_i+a_j)]\}. \qquad (2.1)$$

(Note: $\det_N\{A_{ij}\}$ is used to represent the determinant of the $N\times N$ matrix A_{ij}.) When the λ_m are increased so that $A(g)=0$ at $g=1$, a bound-state zero of $D(k^2)$ is located at $k^2=0$. As the λ_m are further increased, this bound state moves to positive imaginary k, i.e., to the negative k^2 axis on the physical sheet. The residue of this bound-state pole is negative.

(ii) Repulsive case:

$$\Delta T=\pi\sum_{n=1}^{N}\mu_n\delta(k^2+b_n^2) \quad (b_n,\mu_n>0).$$

It was shown that a sufficient condition to guarantee a solution with no other singularities is that the following quantity be positive for $0\leq h\leq 1$:

$$R(h)=\det_N\{\delta_{ij}-[h\mu_i/(b_i+b_j)]\}. \qquad (2.2)$$

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[1] P. Johnson, Phys. Rev. 181, 2006 (1969). Hereafter we will refer to this paper as I.

When the μ_n are increased so that $R(h)=0$ at $h=1$, a ghost pole of the scattering amplitude is located at $k^2=\infty$. As the μ_n are further increased, this singularity moves to finite, positive imaginary values of k. The residue of this ghost pole is positive.

In this section we will consider the more general left-hand-cut discontinuity with both attractive and repulsive parts, namely,

$$\Delta T=\pi\left[-\sum_{m=1}^{M}\lambda_m\delta(k^2+a_m^2)+\sum_{n=1}^{N}\mu_n\delta(k^2+b_n^2)\right]. \quad (2.3)$$

Our object here is to show that if condition (2.1) is met upon the attractive part of ΔT, and if condition (2.2) is met upon the repulsive part of ΔT, then the N/D solution of the composite left-hand-cut discontinuity will have *no* singularities in the upper-half k plane caused by spurious zeros of D.

The proof of this result is crucially dependent upon certain theorems about determinants. We will state these theorems here and prove them in the appendices. First, let us recall the definition of a positive definite matrix.

Definition. A real, symmetric $n\times n$ matrix $A=\{a_{ij}\}$ is positive definite if all its eigenvalues are positive. A well-known[2] necessary and sufficient condition for positive definiteness of a real, symmetric matrix is that the determinants of the principal submatrices A_k be positive for $k=1,\cdots,n$ (A_k consists of the first k columns and rows of A).

Next, we state a useful theorem about the matrix appearing in (2.1). A proof by induction of this theorem is given in Appendix A.

Theorem I. Define[3] the determinant D of the $N\times N$ matrix M

$$D=\det M=\det_N\{\delta_{ij}+\lambda_i/(a_i+a_j)\},$$

with $a_i>0$. Let us consider the domain of λ_i for which the determinants of the principal submatrices of M are positive. Then $\partial D/\partial\lambda_i>0$ for λ_i in this domain. [Remark: The theorem applies if all $\lambda_i>0$. Also, if conditions (2.1) are met, the theorem applies for negative coefficients λ_i.] The following theorem on determinants is needed.

Theorem II. Suppose that a real matrix B may be decomposed into $B=S+A$, where S is symmetric and positive definite while A is antisymmetric. Then $\det B\geq\det S$.

This theorem, first proved by Ostrowski and Taussky,[4] does not appear frequently in the literature; thus we present a proof of it in Appendix B.

[2] E. F. Beckenbach and R. Bellman, *Inequalities* (Springer-Verlag, Berlin, 1965), pp. 57–59.

[3] This determinant is explicitly evaluated in I.

[4] A. Ostrowski and O. Taussky, Ned. Akad. Wet. Proc. (A) **54**, 383 (1951).

Now let us consider the equations for N and D when the left-hand-cut discontinuity is given by (2.3). The equations are of the form

$$N(k^2)=\sum_{m=1}^{M}\frac{\lambda_m}{k^2+a_m^2}D(-a_m^2)-\sum_{n=1}^{N}\frac{\mu_n}{k^2+b_n^2}D(-b_n^2), \quad (2.4)$$

$$D(k^2)=1-\sum_{m=1}^{M}\frac{\lambda_m}{a_m-ik}D(-a_m^2)+\sum_{n=1}^{N}\frac{\mu_n}{b_n-ik}D(-b_n^2). \quad (2.5)$$

The quantities $D(-a_m^2)$ and $D(-b_n^2)$ can be determined from auxiliary conditions obtained by substituting $k^2=-a_m^2$ and $k^2=-b_n^2$ into (2.5).

It is useful to consider the auxiliary functions $D_a(k^2)$ and $D_r(k^2)$, defined through the equations

$$D_a(k^2)=1-g\sum_{m=1}^{M}\frac{\lambda_m}{a_m-ik}D_a(-a_m^2),$$
$$D_r(k^2)=1+h\sum_{n=1}^{N}\frac{\mu_n}{b_n-ik}D_r(-b_n^2). \quad (2.6)$$

Let us also define

$$\mathfrak{D}_a=\det_M\left\{\delta_{jm}+g\frac{\lambda_j}{a_j+a_m}\right\},\quad \mathfrak{D}_r=\det_N\left\{\delta_{jn}-\frac{h\mu_j}{b_j+b_n}\right\}. \quad (2.7)$$

It is shown in Appendix C that under assumptions (2.1) and (2.2) the functions D_a and D_r satisfy the following constraints for $0\leq k^2\leq\infty$:

$$1\geq\text{Re}D_a(k^2)\geq D_a(0)>0,$$
$$1\leq\text{Re}D_r(k^2)\leq D_r(0)<\infty. \quad (2.8)$$

Note also that $\mathfrak{D}_a$ and $\mathfrak{D}_r$ are positive under the assumptions (2.1) and (2.2).

We will now show that under conditions (2.1) and (2.2) the quantity $D(k^2)$ cannot vanish on the physical sheet of the k^2 plane ($\text{Im}k\geq0$).

The first step is to show that Eq. (2.5) has a solution. To establish this let us examine the $M+N$ equations involving $D(-a_m^2)$ and $D(-b_n^2)$:

$$\sum_{j=1}^{M}\left\{\delta_{jm}+\frac{g\lambda_j}{a_j+a_m}\right\}D(-a_j^2)-\sum_{l=1}^{N}\frac{h\mu_l}{a_m+b_l}D(-b_l^2)=1,\quad M\text{ equations}$$
$$\sum_{j=1}^{M}\frac{g\lambda_j}{b_n+a_j}D(-a_j^2)+\sum_{l=1}^{N}\left\{\delta_{ln}-\frac{h\mu_l}{b_l+b_n}\right\}D(-b_l^2)=1,\quad N\text{ equations}. \quad (2.9)$$

(We have introduced parameters g and h here, where $0\leq g,\ h\leq 1$. We will eventually set $g=h=1$.) We can prove that this set of equations has a unique solution if the determinant of coefficients of $D(-a_j^2)$ and $D(-b_l^2)$ is nonzero. Let us denote this determinant by $\mathfrak{D}$

$$\mathfrak{D}=\det_{M+N} L=\det_{M+N} M\,,$$

where

$$L=\begin{vmatrix} \delta_{jm}+g\lambda_j/(a_j+a_m) & -h\mu_l/(b_l+a_m) \\ g\lambda_j/(b_n+a_j) & \delta_{ln}-h\mu_l/(b_l+b_n) \end{vmatrix}, \quad (2.10)$$

and

$$M=\begin{vmatrix} \delta_{jm}+\dfrac{g(\lambda_j\lambda_m)^{1/2}}{(a_i+a_m)} & -(gh)^{1/2}\dfrac{(\mu_l\lambda_m)^{1/2}}{(b_l+a_m)} \\ (gh)^{1/2}\dfrac{(\lambda_j\mu_n)^{1/2}}{(b_n+a_j)} & \delta_{ln}-h\dfrac{(\mu_l\mu_n)^{1/2}}{(b_l+b_n)} \end{vmatrix}.$$

One can easily write M as the sum of a symmetric and an antisymmetric matrix:

$$M=\begin{vmatrix} \delta_{jm}+g\dfrac{(\lambda_j\lambda_m)^{1/2}}{(a_j+a_m)} & 0 \\ 0 & \delta_{ln}-h\dfrac{(\mu_l\mu_n)^{1/2}}{(b_l+b_n)} \end{vmatrix} + \begin{vmatrix} 0 & \dfrac{-(gh)^{1/2}(\mu_l\lambda_m)^{1/2}}{(b_l+a_m)} \\ (gh)^{1/2}\dfrac{(\lambda_j\mu_n)^{1/2}}{(b_n+a_j)} & 0 \end{vmatrix}.$$

The symmetric term is the direct sum of two matrices, each of which is positive definite. [The upper matrix is manifestly positive definite; positive definiteness of the lower matrix follows from condition (2.2).] The conditions of Theorem II are thus met, and one may conclude that $\det\mathfrak{D}>0$ here.

We can also show that $D(k^2=0)>0$ here. By direct application of (2.5) one can show that $D(0)$ is given by

$$\mathfrak{D}D(0) = \det_{M+N}\begin{vmatrix} \delta_{jm}-g\lambda_j/(a_j+a_m) & h\mu_l/(b_l+a_m) \\ -g\lambda_j/(b_n+a_j) & \delta_{ln}+h\mu_l/(b_l+b_n) \end{vmatrix}. \quad (2.11)$$

By an analogous argument $D(0)>0$.

We have thus established that under conditions (2.1) and (2.2), the function N/D associated with the left-hand-cut discontinuity (2.3) has neither bound states nor real ghosts. However, it is possible for zeros of D to enter the upper-half k plane at some point other than zero or infinity. We must also eliminate the possibility of these "complex-energy ghosts."

In Appendix D it is shown that for k^2 real and positive the quantity $\mathrm{Re}D(k^2)$ is positive under the assumption of conditions (2.1) and (2.2). Thus for $0\leq g,\ h\leq 1$ the quantity $\mathrm{Re}D(k^2)$ cannot vanish for positive real values of k^2. As a result no zeros of D can enter the upper-half k plane[5] for this range of values. Consequently, D cannot vanish in the upper-half plane.

It was shown in I that if g and h are sufficiently small, there are no zeros of D in the upper-half k plane. Specifically, the sufficient condition (1.3) in this context may be written

$$\sum_{m=1}^{M}\frac{\lambda_m}{2a_m}+\sum_{n=1}^{N}\frac{|\mu_n|}{2b_n}<\frac{1}{4}. \quad (2.12)$$

To see that the conditions obtained here are considerably stronger than (2.12), let us note that (2.1) and (2.2) are satisfied if the following simple conditions are met[6]:

$$\sum_{m=1}^{M}\frac{\lambda_m}{2a_m}<1\,,\qquad \sum_{n=1}^{N}\frac{|\mu_n|}{2b_n}<1. \quad (2.13)$$

We have seen that under assumptions (2.1) and (2.2), D cannot vanish in the upper-half k plane. In particular, therefore, the residues of the poles of N in (2.4) cannot vanish. Hence, the quantity $f=N/D$ is unitary, has the prescribed discontinuity ΔT, and vanishes as $k^2\to\infty$ within the cut plane.

III. N/D EQUATIONS WITH CDD POLES

The PWDR boundary-value problem in general does not have unique solutions; the well-known CDD ambiguity illustrates the nonuniqueness of such boundary-value problems. We will discuss the case in which a CDD pole of D is placed on the positive real k^2 axis with a real residue. We are particularly concerned with how the presence of this pole affects the zeros of D.

(i) Attractive case:

$$D(k^2)=1-\sum_{m=1}^{M}\frac{\lambda_m}{a_m-ik}D(-a_m^2)-\frac{c}{p^2-k^2}. \quad (3.1)$$

In the vicinity of the CDD pole at $k^2=p^2>0$ there will be two zeros of D. If we choose $c>0$ in Eq. (1), these zeros will lie on the unphysical sheet of the k^2 plane, provided that D has no other zeros on the physical sheet. Also, the first zero of D to enter the physical sheet must do so at either $k^2=0$ or $k^2=\infty$. Let us assume that condition (2.1) is met so that a solution of (3.1) exists with

[5] D is the ratio of two polynomials in g, h, and k; thus as g and h are varied, the zeros of D move continuously on the Reimann sphere.

[6] See the Appendix of I, where these conditions are shown to be sufficient to establish the validity of (2.1) and (2.2), respectively.

$c=0$ which has no undesired zeros. As c is increased, in general, the zeros of D move so that one of them eventually passes through $k^2=0$ and onto the physical sheet. (We will see that none can enter through $k^2=\infty$ in the attractive case.) In fact, we will show here that for fixed values of λ_m and a_m,

$$(\partial/\partial c)D(k^2=0)<0. \tag{3.2}$$

In other words, increasing the residue of a CDD pole here aids the formation of bound states just as increasing the values of λ_m does.

To establish (3.2), we use (3.1) in the usual way to obtain

$$\mathfrak{D}D(k^2=0) = \det_{M+1}\begin{vmatrix} 1-c/p^2 & \lambda_i/a_i \\ 1-c/(p^2+a_j^2) & \delta_{ij}+\lambda_i/(a_i+a_j)\end{vmatrix}, \tag{3.3}$$

where

$$\mathfrak{D}=\det_M\{\delta_{ij}+\lambda_i/(a_i+a_j)\}.$$

Note that under condition (2.1), $\mathfrak{D}$ is positive, so that no ghosts can enter from infinity here. It is straightforward to show from (3.3) that

$$\frac{\partial}{\partial c}D(k^2=0)=-\frac{1}{p^2}\det_M\left\{\delta_{ij}-\frac{\lambda_i}{2a_i}\frac{p^2-a_ia_j}{p^2+a_i^2}\right\}. \tag{3.4}$$

It is also straightforward to show that the determinant in (3.4) is $\mathfrak{D}_-\,\mathrm{Re}D_-(p^2)$, where D_- obeys the equation

$$D_-(k^2)=1+\sum_{m=1}^{M}\frac{\lambda_m}{a_m-i_k}D_-(-a_m^2) \tag{3.5}$$

and

$$\mathfrak{D}_-=\det\{\delta_{ij}-\lambda_i/(a_i+a_j)\}.$$

The function D_- defined in (3.5) has the left-hand-cut discontinuity opposite in sign to the function D in Eq. (3.1). Under the assumption (2.1), one can be assured that $\mathfrak{D}_-\,\mathrm{Re}D(p^2)>0$. Thus

$$\frac{\partial}{\partial c}D(k^2=0)=-\frac{\mathfrak{D}_-\,\mathrm{Re}D_-(p^2)}{p^2\mathfrak{D}}<0. \tag{3.6}$$

(ii) Repulsive case:

$$D(k^2)=1+\sum_{n=1}^{N}\frac{\mu_n}{b_n-ik}+\frac{c}{p^2-k^2}. \tag{3.7}$$

Here it is also necessary to require that c be positive in order that the zeros of D in the neighborhood of $k^2=p^2$ occur on the unphysical sheet. It again follows that the first zero of D must enter the physical sheet either at $k^2=0$ or $k^2=\infty$. Let us assume that condition (2.2) is satisfied. Then one can easily show that no zeros of D pass through $k^2=0$. A ghost zero of D enters the physical sheet at $k^2=\infty$ when the quantity

$$\mathfrak{D}=\det_N\{\delta_{ij}-\mu_i/(b_i+b_j)\} \tag{3.8}$$

vanishes. But $\mathfrak{D}$ is independent of c, so that the CDD poles do not enhance the formation of ghosts in this case.

(iii) Mixed case: Let us assume the discontinuity has both positive and negative residues. Then the equation for D is

$$D(k^2,c)=1-\sum_{m=1}^{M}\frac{\lambda_m}{a_m-ik}+\sum_{n=1}^{N}\frac{\mu_n}{b_n-ik}+\frac{c}{p^2-k^2}. \tag{3.9}$$

We will impose both conditions (2.1) and (2.2), so that $\mathrm{Re}D(k^2,0)$ is positive for k^2 positive. We will consider Eq. (3.9) only for small residue c. Now, if $\mathrm{Im}D(p^2,0)>0$ for $|c|$ sufficiently small, it is necessary to choose >0. In this case the presence of CDD poles will increase $D(0)$. On the other hand, if $\mathrm{Im}D(p^2,0)<0$ then for $|c|$ sufficiently small, it is necessary to choose $c<0$, and the presence of the CDD pole will decrease $D(0)$.

CDD poles always occur in the inhomogeneous term of the equation for D, so that their presence does not affect the possibility of ghosts appearing at $k^2=\infty$. However, CDD poles can cause complex energy zeros of D to move onto the physical sheet and become ghosts.

The considerations given here can easily be generalized to analyze the effect of more than one CDD pole in D; clearly, the consequences are similar in this more general case.

It is well known that one can formally impose the threshold condition for partial-waves amplitudes of order l by putting an lth order CDD pole in D at $k^2=0$ so that for $l=1$ the equation for D would be

$$D(k^2)=1-\sum_{m=1}^{M}\frac{\lambda_m}{a_m-ik}D(-a_m^2)+\sum_{n=1}^{N}\frac{\mu_n}{b_n-ik}D(-b_n^2)-\frac{c}{k^2}, \tag{3.10}$$

where $c>0$ is necessary to avoid a zero of D on the negative real axis. One can show that in the purely attractive case $\mu_n\equiv0$, the solution of (3.10) for D will always have zeros in the upper-half k plane. In the purely repulsive case, D will have no unwanted zeros if condition (2.2) is met. Finally, if one considers sufficiently small c in the mixed case, Eq. (10) will have a solution if $(\mathrm{Re}k)\times[\mathrm{Im}D(k^2,\,c=0)]\geq0$ in the neighborhood of $k^2=0$.

The latter result is in concurrence with the general theorem proved by Martin[7] showing that the number of oscillations in the left-hand-cut discontinuity for the lth partial wave must be greater than or equal to l.

[7] A. Martin, Nuovo Cimento **38**, 1326 (1965).

IV. RELATIVISTIC CASE WITH NO INELASTICITY

Here we will obtain sufficient conditions upon the left-hand-cut discontinuity to guarantee that the relativistic PWDR has a solution. We will make the ansatz that the partial-wave amplitude for equal-mass scattering be written in the form N/D, where D has no zeros on the physical sheet and no CDD poles, and N satisfies an unsubtracted dispersion relation. We will initially consider the "elastic" problem; i.e., we require that $f(k^2)$ obey the following relation for $0<k^2<\infty$:

$$\operatorname{Im} f(k^2)=k/(m^2+k^2)^{1/2}|f(k^2)|^2. \tag{4.1}$$

The N/D decomposition leads to the equations

$$N(k^2)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2}\frac{dq^2}{q^2-k^2}D(q^2)\Delta T(q^2), \tag{4.2}$$

$$D(k^2)=1-\frac{1}{\pi}\int_0^{\infty}\frac{dq^2}{q^2-k^2}\frac{q}{(q^2+m^2)^{1/2}}N(q^2). \tag{4.3}$$

One may decouple these equations to obtain the following integral equation for D:

$$D(k^2)=1+\frac{2}{\pi^2}\int_{-\infty}^{-\mu^2}dp^2D(p^2)\Delta T(p^2)H(p^2,k^2), \tag{4.4}$$

where

$$H(p^2,k^2)=\int_0^{\infty}\frac{dq}{(q^2+m^2)^{1/2}}\frac{q^2}{(q^2-p^2)(q^2-k^2)}. \tag{4.5}$$

The essential point here is that Eq. (4.4) is a nonsingular integral equation, in contrast to the system (4.2) and (4.3).

We wish to find conditions upon ΔT which are sufficient to guarantee that $D(k^2)$, the solution of (4.4), has no zeros on the physical sheet of the k^2 plane. From the structure of (4.3), it is implicit that D is analytic in the cut k^2 plane[8] with a right-hand cut along $k^2\geq 0$, and that $D\to 1$ as $k^2\to\infty$. Thus, if one were able to show that

$$|D(k^2)-1|<1 \tag{4.6}$$

for $k^2\mp i\epsilon>0$, he could deduce via the Phragmén-Lindelöf theorem[9] that relation (4.6) was valid everywhere in the cut k^2 plane. In particular, therefore, D would not vanish in the cut plane.

Now we obtain conditions upon ΔT which are sufficient to establish (4.6). Let us make the definitions

$$B=\underset{\substack{k^2-i\epsilon \text{ real}\\ k^2-i\epsilon\geq 0}}{\operatorname{Max}}\left(\frac{2}{\pi^2}\int_{-\infty}^{-\mu^2}dp^2|\Delta T(p^2)|\,|H(p^2,k^2)|\right), \tag{4.7}$$

$$C=\underset{-\infty<p^2<-\mu^2}{\operatorname{Max}}\left(\frac{2}{\pi^2}\int_{-\infty}^{-\mu^2}dq^2|\Delta T(q^2)|\,|H(q^2,p^2)|\right). \tag{4.8}$$

If B is finite and C is less than one, the Neumann series of the integral equation (4.4) for D converges, and one obtains the bound

$$\underset{k^2-i\epsilon>0}{\operatorname{Max}}|D(k^2)-1|\leq B/(1-C).$$

Since D is real-analytic as a function of k^2, a similar bound is valid for $k^2+i\epsilon$ real and positive. Thus if $B+C<1$, then relation (4.6) is true; and by inference, D has no zeros on the physical sheet of the k^2 plane.

It is shown in Appendix E that if ΔT is subject to the following condition, then $B+C$ will be less than unity:

$$F=\frac{1+\sqrt{2}}{\pi}\int_{-\infty}^{-\mu^2}dp^2\frac{|\Delta T(p^2)|}{m(-p^2)^{1/2}}<1. \tag{4.9}$$

We will compare this with an explicitly soluble model in which

$$\Delta T=-\pi\lambda\delta(k^2+a^2).$$

An examination of Eqs. (4.2) and (4.3) shows that a bound state occurs at $k^2=0$ when

$$\frac{2\lambda}{\pi}\int_0^{\infty}\frac{dq}{(q^2=m^2)^{1/2}}\frac{a^2}{(q^2+a^2)^2}=1. \tag{4.10}$$

A ghost occurs at $k^2=\infty$ when

$$-\frac{2\lambda}{\pi}\int_0^{\infty}\frac{dq}{(q^2+m^2)^{1/2}}\frac{q^2}{(q^2+a^2)^2}=1. \tag{4.11}$$

To compare condition (4.9) with the exact answers, let us set $m=a$ arbitrarily. Then (4.9) guarantees that a solution exists if $|\lambda|/a^2=1/(1+\sqrt{2})$; (4.10) implies that a bound state occurs when $\lambda/a^2=\frac{3}{4}\pi$; and (4.11) indicates that a ghost is produced when $-\lambda/a^2=\frac{3}{2}\pi$.[10]

One would expect that condition (4.9) could be greatly improved by more careful analysis of this question. We have established that for sufficiently weak coupling strengths, a solution of the relativistic, elastic PWDR does exist for a large class of left-hand-cut discontinuities ΔT. We remark that Eq. (4.9) contains the implicit requirement that $(\sqrt{-p^2})\Delta T(p^2)\to 0$ as $p^2\to-\infty$. One can obtain a more general result which does not contain this implicit requirement. Here we merely state it:

For every number ϵ such that $0\leq\epsilon<1$ there exists a number $N(\epsilon)$, such that if $\Delta T(p^2)$ is subject to the

[8] Of course, one must place a Hölder condition upon T along the left-hand cut and require that $\Delta T(p^2)\to 0$ as $p^2\to-\infty$ to guarantee that the functions N and D defined by (4.2) and (4.3) are actually analytic in the desired regions.

[9] See H. McDaniel and R. L. Warnock, Phys. Rev. **180**, 1433 (1969).

[10] Note that there is no symmetry between formation of bound states with discontinuity ΔT and ghosts with discontinuity $-\Delta T$. The nonrelativistic analogs of the integrals in (4.10) and (4.11), which do not contain the factor $(q^2+m^2)^{1/2}$, happen to be equal. The symmetry exists in the nonrelativistic problem and not here.

condition

$$\frac{N(\epsilon)}{(m^2)^\epsilon}\int_{-\infty}^{-\mu^2} dp^2 \frac{|\Delta T(p^2)|}{(-p^2)^{1-\epsilon}} < 1, \qquad (4.12)$$

then the quantities B and C will obey the relation $B+C<1$. As a consequence, condition (4.12) is sufficient to guarantee a solution of the PWDR. Note that Eq. (4.12) still contains the requirement that $\Delta T(p^2) \to 0$ as $p^2 \to -\infty$. It is necessary to have $\Delta T \to 0$ in this limit in order that the unsubtracted dispersion relation for N [Eq. (4.2)] be valid.

V. RELATIVISTIC CASE WITH INELASTICITY

To obtain unique solutions of the PWDR in the absence of CDD poles, it is necessary to have some information regarding contributions to the unitarity relation from inelastic intermediate states. This information, regarded as input to the boundary-value problem along with ΔT, is customarily given in one of two ways:

Case A. One is given the function $\eta(k)$, subject to the constraint $0<\eta(k)\leq 1$. The partial-wave unitarity relation may be written in the form

$$f(k^2+i\epsilon) = \frac{(m^2+k^2)^{1/2}}{k}\,\frac{\eta(k)e^{2i\delta(k)}-1}{2i}, \quad \text{for} \quad k\geq 0, \qquad (5.1)$$

where $\delta(k)$ is a real function to be determined by solving the problem.

Case B. One is given the function $\lambda(k)$, subject to $\lambda(k)\geq 1$. In terms of λ, the statement of unitarity is

$$\operatorname{Im} f(k^2+i\epsilon) = k\lambda(k)/(k^2+m^2)^{1/2}|f(k^2+i\epsilon)|^2, \quad \text{for } k^2\geq 0. \qquad (5.2)$$

If one is given $\eta(k)$ as input, he may use a technique discovered by Froissart[11] to reduce the problem to the elastic case considered in Sec. IV, where the new left-hand-cut discontinuity $\Delta T(k^2)$ is dependent upon $\Delta T(k^2)$ and $\eta(k)$.

We will consider here only case B, where one is given the function $\lambda(k)$. The N/D decomposition here is similar to that in Sec. IV, except that in Eq. (4.4) the function H is replaced by H_λ, where

$$H_\lambda(p^2,k^2) = \int_0^\infty \frac{dq\,\lambda(q)}{(q^2+m^2)^{1/2}}\,\frac{q^2}{(q^2-k^2)(q^2-p^2)}. \qquad (5.3)$$

We wish to restrict $\lambda(q)$ so that H_λ is analytic in k^2 in the plane with a cut along $k^2\geq 0$. For simplicity we impose the following condition of Hölder type on λ:

For q and q' greater than or equal to zero, let there exist a constant g such that

$$|\lambda(q)-\lambda(q')| \leq g|q-q'|^\alpha |\tfrac{1}{2}(q+q')|^\beta, \qquad (5.4)$$

where α and β are positive numbers such that $\alpha+\beta=1$. Let us also assume that $\lambda(0)=1$.

[11] M. Froissart, Nuovo Cimento **22**, 191 (1961).

It is implicit in these requirements that $\lambda(q)\leq 1+gq$ for all values of q. Let us define the quantities B_λ and C_λ in analogy to Sec. IV:

$$B_\lambda = \operatorname*{Max}_{\substack{k^2-i\epsilon \text{ real} \\ k^2-i\geq 0}} \left(\frac{2}{\pi^2}\int_{-\infty}^{-\mu^2} dp^2 |\Delta T(p^2)|\,|H_\lambda(p^2,k^2)|\right), \qquad (5.5)$$

$$C_\lambda = \operatorname*{Max}_{-\infty<p^2<-\mu^2} \left(\frac{2}{\pi^2}\int_{-\infty}^{-\mu^2} dq^2 |\Delta T(q^2)|\,|H_\lambda(q^2,p^2)|\right). \qquad (5.6)$$

It is clear in analogy with Sec. IV that a solution of the boundary-value problem will exist if $B_\lambda+C_\lambda<1$. It is shown in Appendix F that $B_\lambda+C_\lambda<1$ if the following conditions on ΔT, g, and α are met:

$$\left(\frac{3}{\pi m}+\frac{3g}{\pi}+\frac{2g}{\pi\alpha}\right)\int_{-\infty}^{-\mu^2} \frac{dp^2}{(-p^2)^{1/2}}|\Delta T(p^2)| + \frac{g}{\pi m}\int_{-\infty}^{-\mu^2} dp^2|\Delta T(p^2)|<1. \qquad (5.7)$$

We have thus established that a solution does exist in the relativistic, inelastic case if conditions (5.4) and (5.7) are met.

VI. CONCLUSIONS

In Sec. II we limited considerations to the case in which the left-hand cut of the partial-wave amplitude was approximated by poles [Eq. (2.3)]. It was shown that if conditions (2.1) and (2.2) are met by the positive and negative parts of ΔT, respectively, the nonrelativistic, elastic N/D equations will lead to solutions for which D does not vanish on the physical sheet of the k^2 plane.

One expects that these results should generalize to the case in which ΔT is continuous, at least for a large class of functions ΔT.[12] Roughly speaking, it is reasonable to require that the integral equation for D [the continuous analog of Eq. (2.5)] be essentially of Fredholm type and that the Fredholm determinant of this equation be nonzero. Also, ΔT must satisfy a Hölder condition if the Cauchy integrals in (1.1) and (1.2) are to lead to functions analytic in the respective cut planes.

We make the working hypothesis that if ΔT is nonpathological in the above sense, one may uniformly approximate ΔT by poles in the N/D equations. Thus, in the continuous case one expects there will be acceptable solutions if the continuous analogs of (2.1) and (2.2) are met. Since these latter conditions are rather unwieldy, we consider the more restrictive simpler conditions (2.13). The continuous analogs of these conditions should be sufficient to eliminate zeros of D in non-

[12] A discussion of the relativistic version of this question is given in a paper by A. K. Common, CERN Th. 977 (unpublished).

pathological cases. In other words, one would expect that if $\Delta T=\Delta T_r+\Delta T_a$, where $\Delta T_a<0$ and $\Delta T_r>0$, such that

$$\frac{1}{2\pi}\int_{-\infty}^{-\mu^2}\frac{dq^2}{\sqrt{(-q)^2}}\Delta T_r(q^2)<1 \tag{6.1}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{-\mu^2}\frac{dq^2}{\sqrt{(-q^2)}}\Delta T_a(q^2)>-1\,, \tag{6.2}$$

there would be no zeros of D on the physical sheet of the k^2 plane.

We expect a similar generalization of the results about CDD poles obtained in Sec. III to the case in which ΔT is continuous.

In Sec. IV it was shown that if ΔT is nonpathological and subject to condition (5.9), the N/D equations (4.2) and (4.3) have a solution for which no zeros of D appear on the physical sheet. We expect that the results of Sec. II should have some generalization to the relativistic case. The results obtained in Sec. II are obtained as consequences of symmetry, monotonicity, and positive definiteness, and hence are rather independent of the form of the matrices appearing there. Details will not be given here; we merely state the result, which is analogous to (2.13).

Consider the relativistic N/D equations in which ΔT is given by Eq. (2.3):

$$\Delta T=\pi\Big[-\sum_{m=1}^{M}\lambda_m\delta(k^2+a_m^2)+\sum_{n=1}^{N}\mu_n\delta(k^2+b_n^2)\Big]. \tag{6.3}$$

Then sufficient conditions for a solution to the PWDR problem are

$$\sum_{m=1}^{M}\frac{2\lambda_m}{\pi}\int_0^\infty\frac{dq}{(q^2+m^2)^{1/2}}\frac{a_m^2}{(q^2+a_m^2)}<1 \tag{6.4}$$

and

$$\sum_{n=1}^{N}\frac{2\mu_n}{\pi}\int_0^\infty\frac{dq}{(q^2=m^2)^{1/2}}\frac{q^2}{(q^2+b_n^2)}<1. \tag{6.5}$$

One expects analogous relations to be true for the case in which ΔT is continuous. A generalization to the inelastic case should also be possible.

Finally, we note that the N/D equations are often used to calculate the location and residues of bound states for cases in which ΔT is strong enough to form bound states. (It is usually assumed that CDD poles are absent when this is done.) In effect, one is assuming that the partial-wave amplitude $f(g,k^2)$ corresponding to the discontinuity $g\Delta T$ is meromorphic in some sufficiently large domain of g and k^2. The question of whether such a domain exists will not be considered here. Such a domain does exist in nonrelativistic scattering off a superposition of Yukawa potentials.

In scattering off a superposition of Yukawa potentials, one does not get ghosts; ghosts are also unwanted in N/D calculations. There exists some class of left-hand-cut discontinuities ΔT which produce bound states but not ghosts. Not much is known about it, however. The appearance of ghosts at infinity can be avoided by making sure that the Fredholm determinant of the integral equation for D does not vanish. Complex-energy ghosts on the physical sheet are more difficult to avoid.

APPENDIX A

Here we consider the matrix

$$M=\{\delta_{ij}+\lambda_j/(a_i+a_j)\}\,.$$

Let us assume that the determinants of this matrix and the principal submatrices are positive. We will show that

$$\frac{\partial}{\partial\lambda_i}(\det_N M)\geq 0\,.$$

We will prove this by induction on N, the order of the matrix M. For $N=1$ the result is trivially true. Let us assume the theorem is true for N (let $b=a_{N+1}$); then

$$\frac{\partial}{\partial\lambda_{N+1}}(\det_{N+1} M)=\det_{N+1}\begin{vmatrix}\delta_{ij}+\lambda_i/(a_i+a_j) & 1/(a_j+b)\\ \lambda_i/(a_i+b) & 1/2b\end{vmatrix}$$

$$=\det_N\left\{\delta_{ij}+\frac{\lambda_i}{a_i+a_j}\left(\frac{a_i-b}{a_i+b}\right)^2\right\}.$$

Since the theorem is assumed to be true for Nth-order determinants,

$$\frac{\partial}{\partial\lambda_{N+1}}(\det_{N+1} M)\geq\det_N\left\{\delta_{ij}+\frac{\lambda_i\theta(-\lambda_i)}{a_i+a_j}\right\}\geq 0\,.$$

Thus the result is true for determinants of order $N+1$ and the induction proof is completed.

APPENDIX B

Here we will prove Theorem II; our proof is based upon that of Ostrowski and Taussky.[4]

We will show that if a real matrix $B=S+A$, where S is real symmetric and positive definite, and A is antisymmetric, then

$$\det B\geq\det S\,.$$

Let us first note that

$$\det B=(\det S)\,\det(1+S^{-1}A)\,.$$

Let us recall that if H is a Hermitian matrix, then $S^{-1}H$ can be diagonalized and has real eigenvalues[13]; thus the matrix $S^{-1}A$ can be diagonalized and has imaginary eigenvalues. In fact, since $S^{-1}A$ is a real matrix, the eigenvalues occur in complex conjugate pairs, so that

$$\det(1+S^{-1}A)\geq 1\,,$$

[13] See J. W. Dettman, *Mathematical Methods in Physics and Engineering* (McGraw-Hill Book Co., New York, 1962), p. 59.

or, equivalently,

$$\det B \geq \det S > 0.$$

The theorem is thus proved.

APPENDIX C

Here we will establish the relations (2.8) for the functions D_a and D_r, which are defined in Eq. (2.6) with the parameters subject to conditions (2.1) and (2.2), respectively. The argument will be presented only for the attractive case D_a; the result for D_r may be obtained analogously.

Let us rewrite Eq. (2.6) as a function of $x=-ik$,

$$D_a(-x^2)=1-g\sum_{m=1}^{M}\frac{\lambda_m}{a_m+x}D_a(-a_m^2). \qquad (2.6)$$

[Note that $D(-a_m^2)$ are real.] For sufficiently small g, D_a has no zeros in the region $\mathrm{Re}x>0$; thus $D_a(-a_m^2)$ are positive. In that case $D_a(-x^2)$ is a Herglotz function; it can vanish only for real values of x. D_a remains a Herglotz function so long as it has no zeros in the region $\mathrm{Re}x>0$; in fact, as g is increased, the first zero of D_a to enter the right-half x plane must do so at $x=0$.

Under condition (2.1) one can be sure that $D_a(0)$ is positive for $0\leq g\leq 1$. It follows that under this condition $D_a(-x^2)$ is a monotonically increasing function of x for positive real x; i.e., for $x>0$,

$$0\leq D_a(0)<D_a(-x^2)\leq D_a(-\infty)=1. \qquad \text{(C1)}$$

Now let us consider $\mathrm{Re}D_a(k^2)$ for k real and positive. From (2.6) one obtains

$$\mathrm{Re}D_a(k^2)=1-g\sum_{m=1}^{M}\frac{\lambda_m a_m}{a_m^2+k^2}D(-a_m^2). \qquad \text{(C2)}$$

Since under our conditions $D_a(-a_m^2)$ is positive, then

$$\mathrm{Re}D_a(k^2)\leq 1 \qquad \text{(C3)}$$

and

$$\mathrm{Re}D_a(k^2)\geq 1-g\sum_{m=1}^{M}\frac{\lambda_m}{a_m}D_a(-a_m^2)=D_a(0), \qquad \text{(C4)}$$

for real positive k.

The result (2.8) is thus established.

APPENDIX D

Here we will show $\mathrm{Re}D(k^2)>0$ for $k^2>0$ under the assumption of (2.1) and (2.2). One may use Eqs. (2.5) to obtain the relation

$$\mathfrak{D}\,\mathrm{Re}D(k^2)=1-\sum_{m=1}^{M}\frac{\lambda_m a_m}{k^2+a_m^2}D(-a_m^2)+\sum_{n=1}^{N}\frac{\mu_n b_n}{k^2+b_n^2}D(-b_n^2). \qquad \text{(D1)}$$

One may eliminate the auxiliary quantities to write this in matrix form,

$$\mathfrak{D}\,\mathrm{Re}D(k^2)=\det_{M+N+1}\begin{vmatrix} 1 & g\dfrac{\lambda_j a_j}{(k^2+a_j^2)} & -h\dfrac{\mu_l b_l}{(k^2+b_l^2)} \\ 1 & \delta_{jm}+\dfrac{g\lambda_j}{(a_j+a_m)} & -\dfrac{h\mu_l}{(b_l+a_m)} \\ 1 & \dfrac{g\lambda_j}{(a_i+b_n)} & \delta_{ln}-\dfrac{h\mu_l}{(b_l+b_n)} \end{vmatrix}. \qquad \text{(D2)}$$

One may eliminate the first column and then manipulate the determinant into the following simpler form:

$$\mathfrak{D}\,\mathrm{Re}D(k^2)=\det_{M+N}\begin{vmatrix} \delta_{jm}+\dfrac{g\lambda_j}{a_j+a_m}\dfrac{k^2-a_ja_m}{(k^2+a_j^2)^{1/2}(k^2+a_m^2)^{1/2}} & -\dfrac{h\mu_l}{b+a_m}\dfrac{k^2-a_mb_l}{(k^2+a_m^2)^{1/2}(k^2+b_l^2)^{1/2}} \\ \dfrac{g\lambda_j}{a_j+b_n}\dfrac{k^2-a_jb_n}{(k^2+a_j^2)^{1/2}(k^2+b_n^2)^{1/2}} & \delta_{ln}-\dfrac{h\mu_l}{b_l+b_n}\dfrac{k^2-b_lb_n}{(k^2+b_l^2)^{1/2}(k^2+b_n^2)^{1/2}} \end{vmatrix}. \qquad \text{(D3)}$$

The diagonal submatrices in (D3) are positive definite, as will be shown, so that the conditions of Theorem II are met. One can see, in fact, that the determinants of these submatrices are simply $\mathfrak{D}_a\,\mathrm{Re}D_a(k^2)$ and $\mathfrak{D}_r\,\mathrm{Re}D_r(k^2)$, where $\mathfrak{D}_a$ and D_r are defined in Eq. (2.6). As a result of the inequalities (2.8) it follows that $\mathfrak{D}\,\mathrm{Re}D(k^2)\geq 0$.

We will now establish that the diagonal submatrices in (D3) are positive. The principal subdeterminants of the top matrix are positive, since they are equal to $\mathfrak{D}_a\,\mathrm{Re}D_a(k^2)$ with some λ_m set equal to zero. Similarly, the principal subdeterminants of the bottom matrix are also positive.

The desired result is thus obtained.

APPENDIX E

Here we will show that B and C, defined by Eqs. (4.7) and (4.8), respectively, obey the relation $B+C<1$ under the assumption $F<1$, where F is defined by Eq. (4.9). We will show this by establishing that $B+C<F$.

First, we bound C. It is simple to show that for p^2 and q^2 real and negative, the quantity H defined in (4.5) is bounded by

$$|H(p^2,q^2)|\leq\int_0^\infty \frac{ds}{m}\frac{1}{s^2-p^2}=\frac{\pi}{2m\surd(-p^2)}.$$

This relation immediately leads to the following bound on C:

$$C\leq\frac{1}{\pi m}\int_{-\infty}^{-\mu^2}\frac{dp}{\surd(-p^2)}|\Delta T(p^2)|. \tag{E1}$$

Now B must be bounded. One can show that for q^2 real and negative and $k^2\mp i\epsilon$ real and positive, the quantity $H(p^2,k^2)$ is given as follows:

$$\mathrm{Im}H(p^2,k^2)=\pm[\pi/2(k^2+m^2)^{1/2}][k/(k^2+p^2)], \tag{E2}$$

$$\mathrm{Re}H(p^2,k^2)=E+G, \tag{E3}$$

where

$$E=\frac{1}{k^2-p^2}\int_0^\infty\frac{dq}{(q^2+m^2)^{1/2}}\frac{-p^2}{q^2-p^2},$$

$$G=\frac{1}{k^2-p^2}\mathrm{p.v.}\int_0^\infty\frac{dq}{(q^2+m^2)^{1/2}}\frac{k^2}{q^2-k^2}.$$

One can easily show that

$$0\leq E\leq\pi/2m\surd(-p^2). \tag{E4}$$

The principal-value integral for G may be evaluated explicitly,

$$G=-\frac{k}{(k^2+m^2)^{1/2}}\frac{1}{k^2-p^2}\ln\frac{m}{(m^2+k^2)^{1/2}-m}.$$

It is straightforward to show that

$$0\leq\ln[m/(m^2+k^2)^{1/2}-k]\leq\ln[(m+k)/m]\leq k/m.$$

In other words,

$$0\leq=G\leq1/m\surd(-p^2). \tag{E5}$$

Since E is positive and subject to (E4), whereas G is negative and subject to (E5), therefore

$$|\mathrm{Re}H(p^2,k^2)|\leq\pi/2m\surd(-p^2).$$

From (E2), one can show that

$$|\mathrm{Im}H(p^2,k^2)|\leq\pi/2m\surd(-p^2),$$

for $k^2\mp i\epsilon>0$ and $p^2<0$. One is thus led to the following bound upon B:

$$B\leq\frac{\surd2}{\pi m}\int_{-\infty}^{-\mu^2}dp^2\frac{|\Delta T(p^2)|}{\surd(-p^2)}. \tag{E6}$$

As a result of (E1) and (E6), the relation $B+C\leq F$ is valid, and the desired result is established.

APPENDIX F

We wish to show that B_λ and C_λ defined by Eqs. (5.5) and (5.6) are subject to the constraints $B_\lambda+C_\lambda<1$ under the restriction (5.7). We will establish this by bounding them separately.

It is straightforward to show that for q^2 and p^2 real and negative,

$$H_\lambda(q^2,p^2)\leq\int_0^\infty dr\frac{1+gr}{(r^2+m^2)^{1/2}}\frac{1}{r^2-q^2}$$

$$\leq\frac{1+gm}{m}\frac{\pi}{2\surd(-g^2)}. \tag{F1}$$

Thus

$$C_\lambda\leq\frac{1+gm}{\pi m}\int_{-\infty}^{-\mu^2}dq^2\frac{|\Delta T(q^2)|}{\surd(-q^2)}. \tag{F2}$$

The object now is to bound B_λ. Let us note that for $p^2<0$ and $k^2\pm i\epsilon>0$, $H_\lambda(p^2,k^2)$ is given by

$$\mathrm{Im}H_\lambda(p^2k^2)=\pm\frac{\lambda(k^2)}{(k^2+m^2)^{1/2}}\frac{\pi k}{2(k^2-p^2)}, \tag{F3}$$

$$\mathrm{Re}H_\lambda(p^2,k^2)=\mathrm{p.v.}\int_0^\infty\frac{dq^2\lambda(q)}{(q^2+m^2)^{1/2}}\frac{q^2}{(q^2-k^2)(q^2-p^2)}. \tag{F4}$$

One may easily show that

$$|\mathrm{Im}H_\lambda(p^2,k^2)|\leq\frac{1+gm}{m}\frac{\pi}{2\surd(-p^2)}.$$

One can rewrite $\mathrm{Re}H_\lambda$ as the sum of two terms:

$$\mathrm{Re}H_\lambda=P_1+P_2,$$

where

$$P_1=\lambda(k)\mathrm{p.v.}\int_0^\infty\frac{dq}{(q^2+m^2)^{1/2}}\frac{q^2}{(q^2-k^2)(q^2-p^2)} \tag{F5}$$

and

$$P_2=\int_0^\infty\frac{dq}{(q^2+m^2)^{1/2}}\frac{\lambda(q)-\lambda(k)}{q^2-k^2}\frac{q^2}{q^2-p^2}. \tag{F6}$$

In analogy to work in Appendix E, one can show that

$$|P_1|\leq\frac{\pi}{2\surd(-p^2)}\frac{1+gm}{m}. \tag{F7}$$

One can bound $|P_2|$ by writing it as

$$\int_0^{2k}dq+\int_{2k}^\infty dq$$

and bounding each term separately, using the condition (5.4) in the process. We merely state the result here,

$$|P_2|\leq[g/\sqrt{(-p^2)}](\tfrac{1}{2}\pi+\alpha^{-1}). \quad \text{(F8)}$$

Thus we have shown that

$$|\mathrm{Re}H_\lambda(p^2,k^2)|\leq\left(\frac{1+2gm}{m}\frac{\pi}{2}+\frac{g}{\alpha}\right)\frac{1}{\sqrt{(-p^2)}}. \quad \text{(F9)}$$

One can now employ bounds (F3) and (F9) to establish that B_λ is subject to the bound

$$B_\lambda\leq\left(\frac{1}{\pi m}+\frac{2g}{\pi}+\frac{2g}{\pi^2\alpha}\right)\int_{-\infty}^{-\mu^2}\frac{dp^2}{\sqrt{(-p^2)}}|\Delta T(p^2)| +\frac{g}{m}\int_{-\infty}^{-\mu^2}dp^2|\Delta T(p^2)|. \quad \text{(F10)}$$

From the inequalities (F2) and (F10), it follows that condition (5.7) is sufficient to guarantee that $B_\lambda+C_\lambda<1$.

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Experimental Aspects of Dual Theories for Baryons*

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Formulations of baryon-exchange-degenerate schemes incorporating $SU(3)$ symmetry are presented. We discuss the relevance of the $SU(3)$ solutions to the baryon spectrum and backward-scattering data and present a more empirical broken-$SU(3)$ approach to reconcile the duality constraints with experiment.

I. INTRODUCTION

THE duality of direct-channel resonances and Regge exchange poles[1] has led in some instances to a considerable simplification in dynamical models. The combination of duality and $SU(3)$ theories gives further severe restrictions on trajectories and residue functions, but unfortunately we find that some of these constraints appear to be at variance with empirical observation. Since $SU(3)$ is a broken symmetry, it is an interesting question whether a broken exchange-degenerate $SU(3)$ theory can exist. In this paper, we investigate duality constraints for s, t, and u channels in meson-baryon scattering, seeking a solution consistent with the baryon spectra and meson-baryon scattering data.

In those pseudoscalar-meson-baryon scattering reactions for which the t channel involves exotic meson-exchange quantum numbers, the absence of forward peaks is one of the more striking empirical observations in strong-interaction scattering; similarly, backward peaks are absent to a very low level for reactions requiring the exchange of exotic baryons.[2,3] The absence of exotic exchanges, as inferred from scattering data, provides valuable insight into the dynamics of these two-body reactions through the duality principle. The duality constraints connected with t-channel exotic may be considered separately from those connected with u-channel exotic. There is already some evidence on the validity of the u-channel exotic constraints from K^+p backward-scattering data.[4] On the other hand, the empirical absence of t-channel peaks should place even more stringent requirements on the dynamics, since the allowed forward cross-section peaks are generally an order of magnitude larger than the allowed backward cross-section peaks. We consider it to be most reasonable, in view of these facts, to impose both t- and u-channel exotic duality conditions simultaneously. This we do in Secs. II and III.

A further complication in applying duality for baryon exchanges concerns the MacDowell symmetry.[3] Analytic requirements on fermion exchange amplitudes force a conspiracy at $u=0$ between exchanges of opposite parity. If we first restrict our attention to positive u (the mass region), then duality conditions can be applied to a u-channel parity-conserving helicity amplitude of definite τP quantum number[5] and we ob-

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[5] At asymptotic energies ($s\to\infty$), fermion Regge-pole exchange contributions can be isolated according to τP quantum number through the u-channel parity-conserving helicity-amplitude formalism (cf. Ref. 3). By this separation, we can deal independently with the dominant and reflected (in $\sqrt{u}$) branches of the fermion exchange amplitudes. If instead, we were to consider directly the invariant amplitudes, A and B, then both τP branches would be involved simultaneously.