

ρ - f^0 Double Bootstrap by the Balázs Method*

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We have studied the simultaneous bootstrap of the ρ and f^0 mesons, using the Balázs method. When we include the exchange of an f^0 with physical parameters in the input force, we find that we obtain an approximately self-consistent ρ with a mass of 710 MeV and full width of 100 MeV; the degree of self-consistency is somewhat improved if the width is reduced to 66 MeV. One can obtain a simultaneous bootstrap of both the ρ and f^0 with a ρ mass and full width of 710 and 46 MeV, and f^0 mass and full width of 1344 and 112 MeV, respectively. We believe that the double-bootstrap results are somewhat affected by our approximate treatment of inelastic effects, which involves a discontinuity in the inelasticity function R_I^I near the position of the f^0 . These results are based upon taking n , the number of poles approximating the left-hand cut, equal to 7. The use of a large number of poles is important, and it is not at all clear from the n dependence of our calculations that we have yet obtained a limiting form of the results as n becomes large, even with $n=7$.

I. INTRODUCTION

IN a previous paper¹ (hereafter referred to as AE) we studied the validity of the Balázs method²⁻⁴ and showed that it is a convenient method for doing bootstrap calculations and, coupled with the Williamson-Everett (WE) criterion,⁵ makes well-defined and reasonable predictions, at least in the ρ bootstrap problem. In this paper, we apply the Balázs method to the problem of the simultaneous ρ and f^0 bootstrap. We will draw heavily on the notation, terminology, and equations of the previous paper AE in order to avoid repetition.

In AE we found $n-1$ series of solutions for the bootstrap problem, where n is the number of poles approximating the left-hand cut. We numbered the solutions in order of increasing values of the ρ mass, and found that for all values of n tried, series No. I was the series chosen by the WE criterion.¹ In this paper, we will accordingly deal only with series No. I. Furthermore, the WE criterion is used all through this paper in choosing the optimum matching point ν_F and corresponding bootstrap values, even when not explicitly mentioned. A useful quantity connected with the WE criterion is the percentage difference between the n th derivatives of the two forms of the scattering amplitude evaluated at the matching point ν_F in an n -pole approximation. In the channel with isotopic spin I and angular momentum l , we define this quantity by

$$P_I^I(n, \nu_F) = \left| \frac{[A_I^I(\nu)]_{\nu=\nu_F}^{(n)} - \left[\frac{\nu^l}{(\nu - \nu_k)^{l-1}} \frac{N_I^I(\nu)}{D_I^I(\nu)} \right]_{\nu=\nu_F}^{(n)}}{[A_I^I(\nu)]_{\nu=\nu_F}^{(n)}} \right| \times 100, \quad (1)$$

where (n) indicates the n th derivative $N_I^I(\nu)$, $D_I^I(\nu)$, and $A_I^I(\nu)$ are defined by Eqs. (AE-3), (AE-4), and (AE-9), respectively. We choose the optimum matching point for a given partial wave by seeking the value of ν_F which makes the corresponding $P_I^I=0$, or at least minimizes it.

The ρ bootstrap problem has been the subject of intensive investigation; several examples of such calculations are given in Ref. 6. As is well known, one of the principal difficulties with most such calculations has been that while one could obtain, often with the help of an arbitrary cutoff parameter, a resonance at approximately the right position, the theoretical widths have generally been much too wide.^{1-4,6} One of the motivations for the present work is a calculation done recently by one of us,⁷ using another approximation method, also developed by Balázs,⁸ in which the scattering amplitude for the relativistic problem is obtained by the solution of an appropriate nonrelativistic Schrödinger equation with an energy-dependent effective potential. In Ref. 7 it was found that, if the exchange of the f^0 as well as of the ρ was included in the input force, one obtained a self-consistent ρ of the right mass and a width almost consistent with experiment, and, in addition, obtained an output f^0 of approximately the right mass and width, so that one had a simultaneous or double bootstrap. The relatively successful result of the calculation in Ref. 7 appears to be a consequence of the inclusion of a shorter-range attractive force, f^0 exchange, plus the fact that the effective potential method includes, albeit approximately, the contribution of the higher-order terms in the Mandelstam iteration.⁹ This latter fact has been emphasized by Finkelstein,¹⁰ and means that one can

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¹ A. F. Antippa and A. E. Everett, Phys. Rev. **178**, 2443 (1969).

² L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).

³ L. A. P. Balázs, Phys. Rev. **129**, 872 (1963).

⁴ L. A. P. Balázs, Phys. Rev. **132**, 867 (1963); **134**, AB1(E) (1964).

⁵ M. R. Williamson and A. E. Everett, Phys. Rev. **147**, 1074 (1966).

⁶ G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961); B. H. Brandtsen and J. W. Moffat, *ibid.* **21**, 505 (1961); M. Bander and G. L. Shaw, Phys. Rev. **135**, B267 (1964); J. R. Fulco, G. L. Shaw, and D. Y. Wong, *ibid.* **137**, B1242 (1965); P. D. B. Collins, *ibid.* **142**, 1163 (1966).

⁷ A. E. Everett, Phys. Rev. **173**, 1663 (1968).

⁸ L. A. P. Balázs, Phys. Rev. **137**, B1510 (1965).

⁹ S. Mandelstam, Phys. Rev. **112**, 1344 (1958); R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

¹⁰ J. Finkelstein, Phys. Rev. **145**, 1185 (1966).

at least hope that the effective potential method will provide a better treatment of the more distant singularities of the full amplitude in the t variable than, e.g., a conventional N/D calculation.

Although the results obtained in Ref. 7 are encouraging, there are theoretical questions concerning the result, which arise from two sources. The energy-dependent potential which yields the scattering amplitude in the channel where s is the square of the center-of-mass energy and t the four-momentum transfer squared has the form⁸

$$V(r, s) = \int dt v(t, s) \frac{e^{-r\sqrt{t}}}{r}.$$

In practical applications of the method, two approximations are made. The first is that one writes $v(t, s) = 2s^{-1/2}A_t(t, s)$, where A_t is the absorptive part of the amplitude in the t channel. This approximation is exact only for t less than the lowest t -channel inelastic threshold; for larger t , an exact calculation of v requires a Mandelstam-type iteration, with the usual attendant difficulties of divergent integrals, etc. Moreover, A_t , and therefore v itself, is in turn approximated by one or two low-lying resonances, and there is the question of the contributions from larger values of t . v is expected to vanish at t large enough that A_t is well represented by its Regge asymptotic form,⁸ but there might be significant contributions from intermediate energy regions. These considerations raise doubt particularly about whether the inclusion of the full f^0 contribution to A_t in the potential is correct. Chew¹¹ has suggested, on the basis of the new form of the strip approximation,¹² in which the potential is represented by the leading t -channel Regge trajectories at all values of s (and not just s very large), that the force due to f^0 exchange may be much smaller than its apparent value. It is not clear that the representation for the potential assumed by Chew is correct at small s (indeed, if taken completely literally, it leads to contradictions with t -channel unitarity¹¹). However, it would clearly be desirable to verify the conclusions of Ref. 7 that one can obtain a reasonably narrow-width ρ when higher-mass contributions, in addition to the ρ , to the crossed-channel absorptive part are included by using a different calculational procedure, with different approximations involved, than were there used.

In view of the foregoing, it seems interesting to study the double-bootstrap problem in the Balázs method. A start on this has already been made by Balázs in Ref. 3, but without the study of the effects of choice of number of poles and matching position which now seems to be necessary,^{1,5} and without the inclusion of the contributions of the large- t part of the crossed-channel amplitude as approximated by direct-

channel Regge poles; these contributions are included in the work of Refs. 4 and 1, and will be included here. Since the crossed-channel amplitude at large t is thus included approximately in the input, and since the poles which approximate the left-hand cut and whose positions and widths are determined by the method hopefully reproduce, in a rough way, the effects of both the near and distant singularities in s of the partial-wave amplitude, the Balázs method should share with the effective-potential method the advantage of giving a better treatment of distant singularities than the N/D method. Moreover, the Balázs method avoids the necessity of distinguishing between the potential and the t -channel absorptive part, since the absorptive part itself is the input into the calculation. In addition, the method allows the inclusion, in an approximate way, of inelastic effects in the direct channel; this cannot be done easily using the effective-potential procedure. The problem of the contributions from intermediate values of t in the crossed channel, where the amplitude may not be well represented either by resonances or by its asymptotic form, still remains. And, of course, there are additional approximations peculiar to the Balázs method itself.

The results are discussed in detail in Sec. IV. We find that it is important to go to larger numbers of Balázs poles than considered in AE. When this is done, we find that, if the f^0 with its physical mass and width is included in the input, one obtains a self-consistent ρ in good agreement with experiment. A simultaneous "bootstrap" of both the ρ and f^0 is possible; the resulting self-consistent values are reasonable, but not as close to the experimental values as the results obtained for the ρ with physical f^0 input. We believe that this is at least partially due to the fact that in our approximate calculation the value of the inelasticity parameter R_t^I is changed discontinuously from 1 (pure elastic) to 3 near the position of the f^0 . The predicted resonance widths are as narrow, and in most cases narrower, than the experimental values. The most unsatisfactory aspect of the present work is that it is not clear that the results would not be significantly changed by a further increase in the number of approximating poles. The amount of computer time involved made it impracticable to attempt to study this question by carrying out further computations. The results do, at least, establish, using different approximation techniques than used in Ref. 7, that a bootstrap calculation with a plausible treatment of more distant singularities, in addition to ρ exchange, can yield self-consistent results for both resonance masses and widths which are in reasonable (about 10–20%) agreement with experiment.

The values we have used for the experimental parameters of the ρ and f^0 are given in Table I, where ν_R is the center-of-mass momentum squared at resonance R , in units with $\hbar=c=m_\pi=1$, and is related to

¹¹ G. F. Chew, Phys. Rev. **140**, B1427 (1965).

¹² G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

TABLE I. Experimental values of the ρ and f^0 mass and width. ν_ρ and ν_f are in units of $\hbar = m_\pi c = 1$, Γ_ρ and Γ_f are dimensionless, m_ρ and m_f are the masses in MeV, and Δm_ρ and Δm_f are the full widths in MeV. Γ_f is computed with $\nu_K = -25$ in Eq. (3).

ν_ρ	m_ρ	Γ_ρ	Δm_ρ	ν_f	m_f	Γ_f	Δm_f
6.18	750	0.167	100	19.025	1253	0.279	140

the mass m_R (in MeV) by

$$m_R = 2m_\pi(\nu_R + 1)^{1/2}, \quad (2)$$

where m_π is the pion mass in MeV. Γ_R is the reduced half-width of the resonance R and is dimensionless. It is related to the full width in energy, expressed in MeV, by

$$\Delta m_R = 2\nu_R^{l_R+1/2}(\nu_R - \nu_K)^{1-l_R} m_\pi \Gamma_R / (\nu_R + 1), \quad (3)$$

where l_R is the spin of the resonance. The reader should note that, for reasons of convenience, when $l_R \neq 1$ the parameter Γ_R defined by Eq. (3) differs slightly from the usual reduced half-width. Measurements of the mass and width of the resonances show some variations, but these are not significant for the present work, considering the probable accuracy of the approximations involved.

II. PROCEDURES AND CHOICE OF PARAMETERS

Two modifications are introduced in this paper over AE, namely, the treatment of the ρ trajectory and the form of the Pomeranchuk residue. The ρ trajectory is given by

$$\alpha_\rho(s) = 1 + \epsilon_\rho(\nu - \nu_\rho) \quad (4a)$$

or

$$\alpha_\rho(s) = \alpha_\rho(0) + \frac{1}{4}\epsilon_\rho s, \quad (4b)$$

where $s = 4(\nu + 1)$, as usual. In AE, the slope was fixed and set at $\epsilon_\rho = 0.05$. In the present paper, we have fixed the intercept $\alpha_\rho(0)$ instead, since experimentally it seems somewhat better determined. The slope is now given by

$$\epsilon_\rho = [1 - \alpha_\rho(0)] / (\nu_\rho + 1). \quad (4c)$$

Also, in AE a Pomeranchuk residue function of the form

$$\beta_P(s) = -\beta_P(0)\nu^{\alpha_P(s)} \quad (5a)$$

was used, with $\beta_P(0)$ given by the optical theorem as

$$\beta_P(0) = -\sigma_{\text{tot}}^I / 4\pi^2, \quad (5b)$$

where

$$\sigma_{\text{tot}}^I = 0.75 \text{ (15 mb)}. \quad (5c)$$

$\alpha_P(s)$, the Pomeranchuk trajectory function, was taken as

$$\alpha_P(s) = \alpha_P(0) + \frac{1}{4}\epsilon_P s, \quad (6a)$$

with

$$\alpha_P(0) = 1, \quad \text{and} \quad \epsilon_P = 0.05. \quad (6b)$$

In fact, the Pomeranchuk trajectory seems to have a

much smaller slope and a much more rapidly varying residue function.¹³ In this paper, we use a Pomeranchuk residue function of the form

$$\beta_P(s) = (\sigma_{\text{tot}}^I / 4\pi^2)^{\alpha_P(s)} e^{c_P s}. \quad (7a)$$

Rarita *et al.*¹³ give the coupled values for ϵ_P and c_P in our units as

$$\epsilon_P = 0.00856 \quad \text{and} \quad c_P = 0.008173. \quad (7b)$$

The change in the form of the Pomeranchuk residue function affects the form of the input amplitude in the $I=0$ channel, and the estimate of $R_l^I(\nu)$, the ratio of total to elastic partial-wave cross sections. The input amplitude is discussed in Sec. III, and $R_l^I(\nu)$ is evaluated in Appendix A. In AE, we found an estimate of $R_l^I(\nu)$ following a procedure given by Balázs in Ref. 4 (hereafter referred to as B III). The results are isospin independent and are given in Fig. 7 of AE for $l=0-3$. In calculating $R_l^I(\nu)$, we had used Eqs. (5a)–(6b) and argued that the large slope will compensate for the slowly varying residue function. This is confirmed by Fig. 1 which shows the value of $R_l^I(\nu)$ calculated by using Eqs. (7a) and (7b); it is seen that $R_l^I(\nu) \approx 3$, the value used in AE, is a reasonable average value of

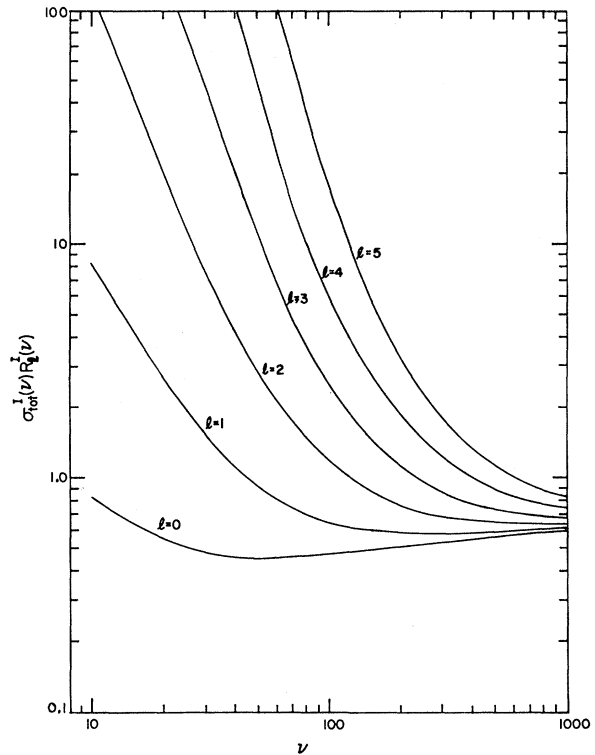


FIG. 1. $R_l^I(\nu)\sigma_{\text{tot}}^I(\nu)$ as a function of ν for several values of l . $R_l^I(\nu)$ = ratio of total to elastic partial-wave cross section. The Pomeranchuk residue function $\beta_P(s)$ is given by Eq. (7a) with $\epsilon_P = 0.00856$ and $c_P = 0.008173$.

¹³ W. Rarita, R. Riddell, Jr., C. Chiu, and R. Phillips, Phys. Rev. 165, 1615 (1968).

$R_l^I(\nu)$ which is almost independent of l , as well as of I . The asymptotic form of $R_l^I(\nu)$ as given by Eq. (AE-7) remains unchanged.

As discussed in AE, the results are independent of the subtraction point ν_0 , as defined by Eqs. (AE-4) and (AE-5), if the exact form of the output reduced half-width given by Eq. (AE-32) is used, and depend only weakly on ν_0 , even if one uses the approximate form given by Eq. (AE-34) in which a linear output denominator function $D_l^I(\nu)$ is assumed. $D_l^I(\nu)$ is defined by Eqs. (AE-1) and (AE-2) and its linear approximation is given by Eq. (AE-33). Repeating Eqs. (AE-32) and (AE-34) in their general form, we have for the exact output width the equation

$$(\Gamma_R)_{\text{out}} = (\Gamma_l^I)_{\text{out}} = -\frac{N_l^I(\nu_R)}{\partial \text{Re} D_l^I(\nu)/\partial \nu|_{\nu=\nu_R}}, \quad (8a)$$

and for the approximate output width, the equation

$$(\Gamma_R)_{\text{out}} = (\Gamma_l^I)_{\text{out}} = (\nu_R - \nu_0) N_l^I(\nu_R). \quad (8b)$$

In the following, we refer to the use of Eqs. (8a) and (8b) as the exact- and approximate-width cases, respectively. In the present work, we investigate the difference in results due to the difference in the two forms of the output width and find that using the exact form of the output width can change the results appreciably, as will be seen in Sec. IV.

The choice of parameters is essentially the same as in AE although some changes have been introduced. Table II gives the pole positions (w_i , $i=1, n$), as defined by Eqs. (AE-3) and (AE-24)–(AE-29), as a function of n . The values of w_i were evaluated for $\nu = -2$, $\nu_R = 5.5$ from figures similar to Fig. 6 of AE; in general, the pole positions are very well determined.

The value of ν_D is slightly increased in this paper as compared to AE because of the introduction of the f^0 resonance as input into the low-momentum-transfer amplitude given by Eq. (AE-10). ν_D is the point of separation between low- and high-energy regions and is the point at which Regge behavior sets in. Thus ν_D must have a value which is larger than the c.m. momentum squared corresponding to the highest resonance which is considered explicitly. In our calculation this is the f^0 with an experimental value of $\nu_f = 19.025$ and a self-consistent value as large as $\nu_f = 24$ in some cases; hence we set $\nu_D = 25$. In AE, where only the ρ resonance was considered explicitly, we had set $\nu_D = 20$.

TABLE II. Variation of the pole positions $w_i(n, \nu, \nu_R)$ ($i=1, n$) with n for $\nu_R = 5.5$ and $\nu = -2$.

n	w_1	w_2	w_3	w_4	w_5	w_6	w_7
2	8	35
3	7.3	13	60
4	6.8	9.5	19	105
5	6.6	8.4	13	28	170
6	6.6	7.7	10	17	39	260	...
7	6.6	7.3	9.1	13	22	52	370

A new variable, which did not play any role in AE, appears in this paper. This is ν_K defined by Eq. (AE-1). As mentioned there, the factor $(\nu - \nu_K)^{1-l}$ is inserted to factor the asymptotic behavior of the scattering amplitude $A_l^I(\nu)$ out of the N/D equations. As seen from Eq. (AE-1), ν_K needs to be large enough not to destroy the threshold behavior ν^l for small ν , and small enough to allow the asymptotic behavior to set in at a point which is consistent with the assumed starting point of asymptotic behavior in the rest of the equations, specifically the point at which Regge behavior becomes dominant. Thus, it seems reasonable to set $\nu_K = -\nu_D$.

Experimentally, the values obtained for the intercept of the ρ trajectory, $\alpha_\rho(0)$, as defined by Eqs. (4b) and (4c), vary from 0.483 to 0.58. We found that varying $\alpha_\rho(0)$ from 0.58 to 0.483 hardly changes the results of the single ρ bootstrap. Thus for the rest of the computations, as suggested by Ref. 13, we set

$$\alpha_\rho(0) = 0.58. \quad (9a)$$

Based on Fig. 1 and a $\sigma_{\text{tot}}^I = 0.75$ (15 mb) (as justified in AE), we chose the central average values $R_1^I = R_2^0 = 3$. The subtraction point we set at $\nu_0 = -2$, and, for the Pomeranchuk trajectory and residue, ϵ_P and c_P are given by Eq. (7b). The form of the ρ residue function is the same as in AE. That is, we take the ρ residue function to have the same form as that of the Pomeranchuk residue in Eq. (7a), but with

$$c_\rho = 0. \quad (9b)$$

III. INPUT AMPLITUDE

In performing a simultaneous ρ and f^0 bootstrap, two low-energy resonance terms are retained in Eq. (AE-10). Each resonance is defined by a set $(\nu_i, \Gamma_i, l_i, I_i)$. One term is due to the ρ resonance with $(\nu_\rho, \Gamma_\rho, 1, 1)$, and the other is due to the f^0 resonance with $(\nu_f, \Gamma_f, 2, 0)$. Thus, we have

$$A_l^{I(L)}(\nu) = [A_l^{I(L)}(\nu)]_\rho + [A_l^{I(L)}(\nu)]_f, \quad (10a)$$

where

$$[A_l^{I(L)}(\nu)]_\rho = \frac{12\beta_{I1}}{\nu} \Gamma_\rho \nu_\rho P_1 \left(1 + \frac{2(\nu+1)}{\nu_\rho} \right) Q_l \left(1 + \frac{2(\nu_\rho+1)}{\nu} \right) \quad (10b)$$

and

$$[A_l^{I(L)}(\nu)]_f = \frac{20\beta_{I0}}{\nu} \frac{\Gamma_f \nu_f^2}{(\nu_f - \nu_k)} \times P_2 \left(1 + \frac{2(\nu+1)}{\nu_f} \right) Q_l \left(1 + \frac{2(\nu_f+1)}{\nu} \right). \quad (10c)$$

As for the Regge term given by Eq. (AE-11), we have already mentioned the fact that the Pomeranchuk residue function varies more rapidly than implied by an approximation of the form of Eq. (BIII-35), and it

was precisely such an approximation that was used in obtaining Eq. (AE-11). Now we slightly modify the procedure of Balázs to obtain an equation similar to Eq. (AE-11), but one which allows a more realistic variation of the residue function of the form

$$\beta(s) = \text{const} \times \nu^{\alpha(s)} e^{cs}. \quad (11a)$$

This can be written as

$$\beta_j(s) = (\beta_j(s_i) / \nu_i^{\alpha_j(s_i)}) \nu_i^{\alpha_j(s)} e^{c_j(s-s_i)}. \quad (11b)$$

Starting from Eq. (BIII-20), we have

$$A_I^{I(H)}(\nu) = -\nu^l (2\alpha+1) \frac{\beta(s) c_1(\alpha) c_2(l)}{\nu^\alpha \alpha - l} (2\nu_D)^{\alpha-l}, \quad (12a)$$

with

$$c_1(\alpha) = 2^\alpha \Gamma(\alpha + \frac{1}{2}) / \pi^{1/2} \Gamma(\alpha + 1) \quad (12b)$$

and

$$c_2(l) = \pi^{1/2} \Gamma(l+1) / 2^{l+1} \Gamma(l + \frac{3}{2}). \quad (12c)$$

Now instead of approximation (BIII-35) [which is inconsistent with Eq. (11b) except in the case $c_j=0$], we make the following approximation:

$$(2\alpha+1)c_1(\alpha) \approx \text{const}, \quad (13a)$$

$$\text{Re}\alpha \approx l_R + \epsilon(\nu - \nu_R). \quad (13b)$$

Then, evaluating Eq. (13) at $\nu = \nu_R$ ($\alpha = l_R$) and substituting it back in Eq. (12a), summing over all trajectories that contribute to the scattering amplitude $A_I^{I(H)}(\nu)$, and using the definition (AE-12) of $K_j(l)$, we have

$$A_I^{I(H)}(\nu) = \sum_j K_j(l) \frac{\beta_j(s_j) e^{A_j c_j(\nu - \nu_j)} \nu^l (2\nu_D)^{l_j - l + \epsilon_j(\nu - \nu_j)}}{\nu_j^{l_j} [l_j - l + \epsilon_j(\nu - \nu_j)]}. \quad (14)$$

In doing the ρ and f^0 double bootstrap, only one term in Eq. (14) was retained, corresponding to the ρ trajectory in the isospin $I=1$ channel and the Pomernanchuk trajectory in the $I=0$ channel. The ρ and Pomernanchuk trajectories are given by

$$\text{Re}\alpha_\rho(s) = 1 + \epsilon_\rho(\nu - \nu_\rho), \quad \text{Im}\alpha_\rho(s) \approx 0, \quad (15a)$$

and

$$\text{Re}\alpha_P(s) = 1 + \epsilon_P(\nu + 1), \quad \text{Im}\alpha_P(s) \approx 0. \quad (15b)$$

Thus, from Eqs. (14) and (15) we have

$$A_1^{1(H)}(\nu) = K_\rho(1) \frac{\beta_\rho(s_\rho) e^{A_\rho c_\rho(\nu - \nu_\rho)} (2\nu_D)^{\epsilon_\rho(\nu - \nu_\rho)}}{\nu_\rho \epsilon_\rho(\nu - \nu_\rho)} \nu \quad (16a)$$

and

$$A_2^{0(H)}(\nu) = K_P(2) \frac{\beta_P(0) e^{A_P c_P(\nu + 1)} (2\nu_D)^{\epsilon_P(\nu + 1) - 1}}{1 - \epsilon_P(\nu + 1)} \nu^2. \quad (16b)$$

From Eq. (AE-12), we then have

$$K_\rho(1) = -1.0 \quad \text{and} \quad K_P(2) = -0.4, \quad (17)$$

while from (BIII-25), we have

$$\beta(s_R) = \nu_R^{\alpha(s_R)} (\nu_R - \nu_K)^{1 - \alpha(s_R)} \Gamma_{\alpha(s_R)}^I \times [d\alpha(s)/d\nu]_{s=s_R}. \quad (18)$$

Thus, since $\alpha_\rho(s_\rho) = 1$, we have

$$\beta_\rho(s_\rho) = \nu_\rho \Gamma_\rho \epsilon_\rho. \quad (19)$$

Putting all the above together, Eq. (16a) reduces to

$$A_1^{1(H)}(\nu) = -\Gamma_\rho [\nu / (\nu - \nu_\rho)] (2\nu_D)^{[1 - \alpha_\rho(0)](\nu - \nu_\rho) / (1 + \nu_\rho)}. \quad (20)$$

For the Pomernanchuk trajectory term, $\beta_P(0)$, c_P , and ϵ_P are given by Eqs. (5b), (5c), and (7), while the ρ parameters are given by Eqs. (9) and (4c).

IV. RESULTS

Because of the complexity and length of the equations involved in a bootstrap self-consistent calculation, complete reliance on systematic methods of iteration is not possible. The situation is even more complicated by the fact that simultaneous self-consistency in two variables (four variables in the case of a double-bootstrap calculation), the c.m. momentum squared and the reduced half-width, is required. We have found that we are forced to rely on familiarity and experience with the specific problem at hand coupled with guess work, intuition, and a set of "rules of thumb" that develop along the way. This makes it necessary to weigh the cost and effort needed to obtain a specific "bootstrap" for a given value of n and ν_F against the usefulness of that result. Thus, in Figs. 2-5, we have plotted the results of bootstrap calculations for a certain range of ν_F , which is different for different values of n . We believe that self-consistent solutions of (ν_ρ, Γ_ρ) exist for ν_F beyond both ends of the range plotted but that they are difficult to obtain, while they do not serve any specially useful purpose. In special cases where we thought the extension of the range into a "difficult" region would shed more light on the problem, we extended it. There are three main factors that make a self-consistent solution difficult to obtain. First, as n increases, the output ν_ρ becomes very sensitive to the input ν_ρ , thus necessitating excellent initial guesses. This is especially true for small ν_F , and also for odd n cases. The second situation is that $d\nu_\rho/d\nu_F$ sometimes becomes very large, thus necessitating an expensive adiabatic extension of the range of solutions in ν_F in order to be able to make good initial guesses. This is also true for $d\Gamma_\rho/d\nu_F$. The third hazard, which is the most troublesome, occurs when Γ_ρ becomes excessively large. The reason for this is that Γ_ρ enters the low-energy part of the input amplitude $A_I^{I(L)}(\nu)$ defined by Eq. (10) as a multiplicative factor, and thus any sensitivity this amplitude may have to variations in the input value of ν_ρ grows linearly with Γ_ρ . For Γ_ρ of the order of 1.5, the initial guess on ν_ρ must be correct to about 4 or 5 significant figures before a successful

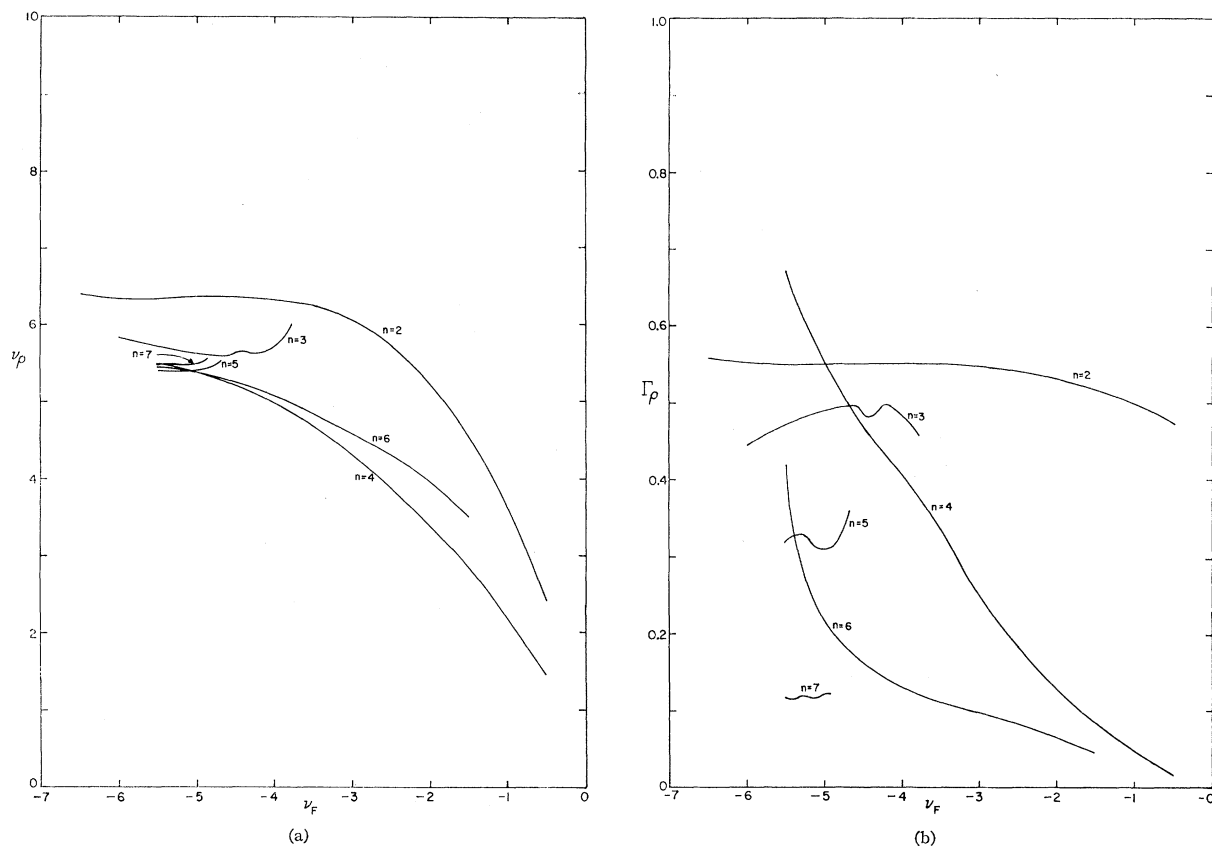


FIG. 2. Self-consistent values, in the approximate-width case, Eq. (8b), of ν_ρ and Γ_ρ in a ρ bootstrap calculation with ρ and f^0 input, as a function of the matching point ν_F and the number of poles n . The mass and width of the input f^0 are fixed at their experimental values, as given by Table I, while those of the ρ are determined self-consistently. (a) ν_ρ ; (b) Γ_ρ .

bootstrap can be accomplished. The full range of variation of ν_F is $-(\nu_\rho+1) < \nu_F < 0$. A careful study of Figs. 2-5 will show that when the solutions extend through a limited range of ν_F which is considerably smaller than the allowable range, it is because one of the three above-mentioned difficult bootstrap conditions exists. In any event, it appears that the optimum value of ν_F always lies in the ranges of ν_F for which results are given, since, in general, the quantity $P_I I$, defined in Eq. (1), grows large near the ends of these ranges.

In presenting the results, we retain three significant

figures in general and sometimes four. On the other hand, as already mentioned, in doing bootstrap calculations the output values are in some cases sensitive to the fourth and fifth significant figures of the input values. This should be kept in mind in any attempt to reproduce the results quoted here.

We first studied a ρ bootstrap calculation in which f^0 exchange, with parameters equal to those of the physical f^0 , was included in the input, and the ρ parameters were determined self consistently. Figure 2 gives the results for the self-consistent values of the ρ parameters as functions of ν_F and n in the approximate-width case; Fig. 3 gives the corresponding results for the exact-width case. The two cases differ markedly for small n but become essentially indistinguishable for large n . In Tables III and IV, we give the self-consistent values of ν_ρ and Γ_ρ at the optimum values of the matching points, as a function of n , for the approximate- and exact-width cases of Figs. 2 and 3. In order to attempt to answer the question as to whether the results really approach a limiting value as n becomes large, we have carried the calculations out through the case $n=7$. As a practical matter, for some n it proved difficult to obtain the self-consistent values in the exact-width case at what would evidently be the optimum value of ν_F , so that some lines of Table IV are

TABLE III. Self-consistent values of (ν_ρ, Γ_ρ) taken from Fig. 2 at the optimum matching point $(\nu_F)_{\text{opt}}^{II}$, as judged by the WE criterion, as a function of n , the number of poles. $(\nu_F)_{\text{opt}}^{II}$ and ν_ρ are in units with $\hbar=c=m_\pi=1$, m_ρ and Δm_ρ are in MeV, and Γ_ρ is dimensionless. The value of Δm_ρ for $n=7$ marked by an asterisk is self-consistent to about 6%, the other values to about 1%, as discussed in the text.

n	$(\nu_F)_{\text{opt}}^{II}$	m_ρ	ν_ρ	Δm_ρ	Γ_ρ
2	-5.50	757	6.36	334	0.550
3	-4.60	717	5.60	278	0.497
4	-5.30	710	5.46	344	0.624
5	-5.20	706	5.40	184	0.336
6	-5.29	708	5.43	162	0.294
7	-5.30	711	5.49	66	0.120
7	-5.30	711	5.49	100*	0.167

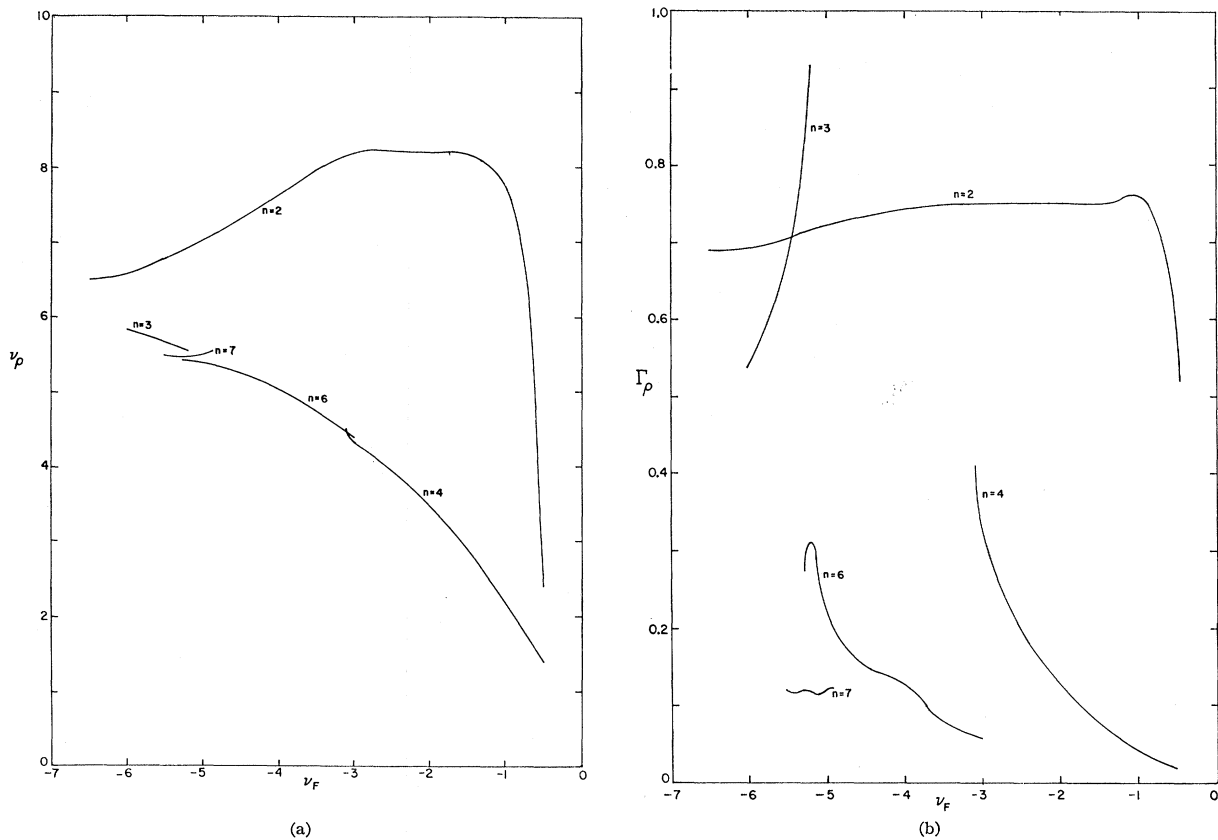


FIG. 3. Same as Fig. 2, except that the exact form of the output width, Eq. (8a), is used. (a) ν_ρ ; (b) Γ_ρ .

left blank. We feel certain that solutions do exist for all n . However, it appears that the minimum percentage difference will, for $n=4$ and 5, occur at a value of ν_F for which the width will be very large, probably greater than 1.5, and as already mentioned, self-consistency is difficult to achieve under these circumstances because of the extreme sensitivity of output to input.

It is clear from Tables III and IV that one has a well-defined theoretical prediction for the ρ mass, about 710 MeV, in the presence of physical f^0 exchange; this differs from the experimental value by about 5%. The $n=2$ and 3 results differ somewhat from this value, but all the results for $n \geq 4$ yield essentially the same result, so that the method appears to yield a well-defined limiting value for m_ρ .

The situation with the width is much less clear. It seems evident that if one wishes to obtain a reliable result for the width, one must go to values of n at least as large as those considered here. Using the approximate form of the output width, one finds the bootstrap value for the width drops very sharply between $n=4$ and $n=5$. The values for $n=5$ and 6 are roughly constant, and slightly larger than the experimental width. However, between $n=6$ and $n=7$ there is again a rather sharp change, and the theoretical value of the width drops to about $\frac{2}{3}$ of the experimental value. The behavior in the exact-width case is roughly the same, except that apparently the behavior is more

erratic for $n=4$ and 5, where we believe that the theoretical widths are very large. The preceding results, which correspond to the first six lines of Tables III and IV, were obtained by requiring the input and output widths to agree to about 1%. In view of the approximations involved in the method, which presumably affect the input and output widths in different ways, there is probably no reason to demand this degree of self-consistency. It is not clear how much one might relax this requirement. As shown in the last lines of Tables III and IV, if one agrees to accept the weaker requirement $|\Gamma_{in} - \Gamma_{out}| \leq 0.01$, or self-consistency to within about 6%, which seems not unreasonable, then $\Delta m_\rho = 100$ MeV, which is essentially the experimental value, is also a self-consistent solution in the 7-pole case.

Thus, from the foregoing, we see that we have an acceptable bootstrap of the ρ , in the 7-pole case, with the exchange of an f^0 with the physical parameters, and with a ρ mass about 5% less than the observed value and a width consistent with experiment. The degree to which one is encouraged by this result depends on one's interpretation of the variation of the width with n . One can take an optimistic view, particularly if one takes the second of the two solutions for $n=7$ in Table V, and notes that the values of the width for $n=4$ and 5 are probably very large, so that the change

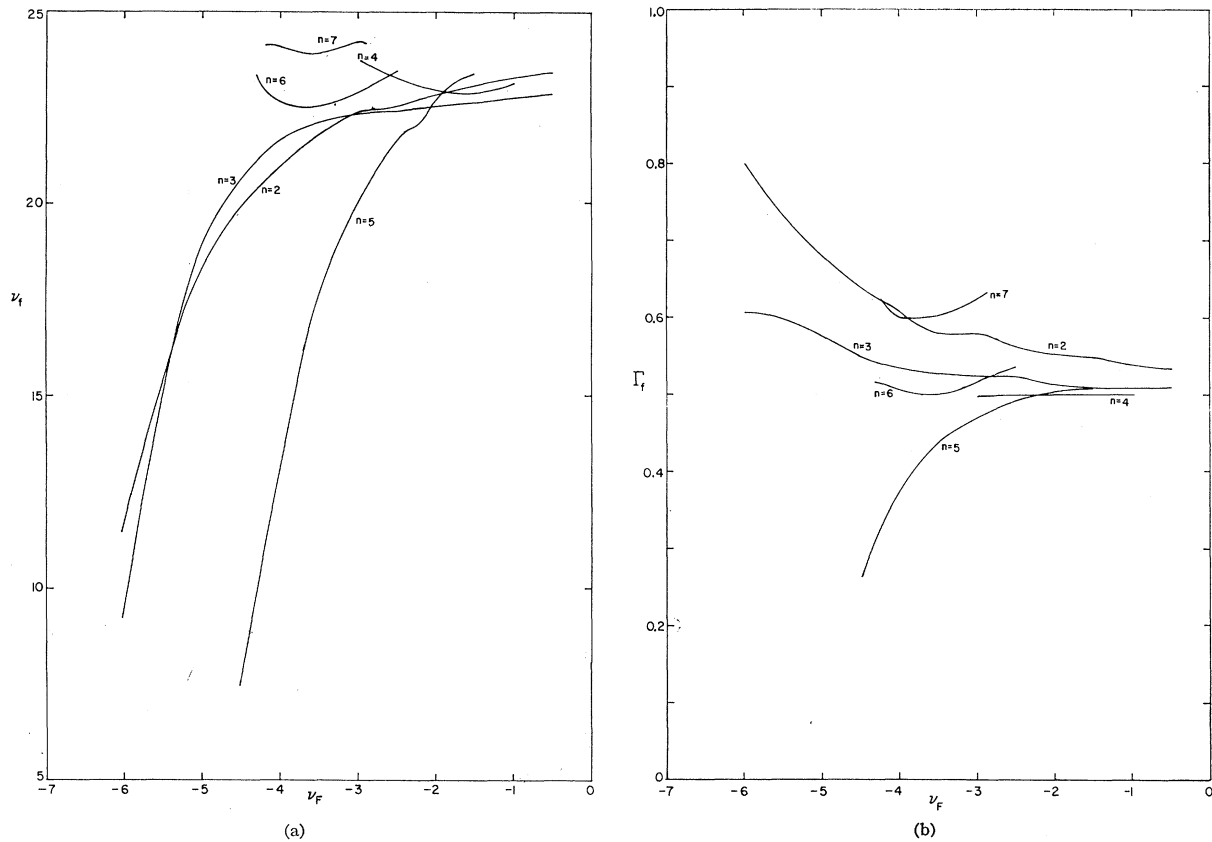


FIG. 4. Self-consistent values, in the approximate-width case, of (ν_f, Γ_f) in an f^0 bootstrap calculation with ρ and f^0 input, as a function of (ν_F, n) . The mass and width of the input ρ are fixed at the experimental values as given by Table I, while those of the f^0 are determined self-consistently. (a) ν_f ; (b) Γ_f .

in width between $n=6$ and $n=7$ is, relatively speaking, small. Then one may conclude that the theoretical value of Γ_ρ is approaching a limiting value of about 100 MeV, although perhaps with a fairly large uncertainty because of our lack of knowledge as to the degree of self-consistency to be demanded. On the other hand, one could argue that the continued decrease in the width with n , and its somewhat erratic behavior, at least in the approximate-width case, in going from $n=6$ to $n=7$, means that no strong conclusion can be drawn from the present results. Unfortunately, it did not seem feasible, in view of the amount of computer time involved, to attempt to extend the results to still higher values of n in an un-

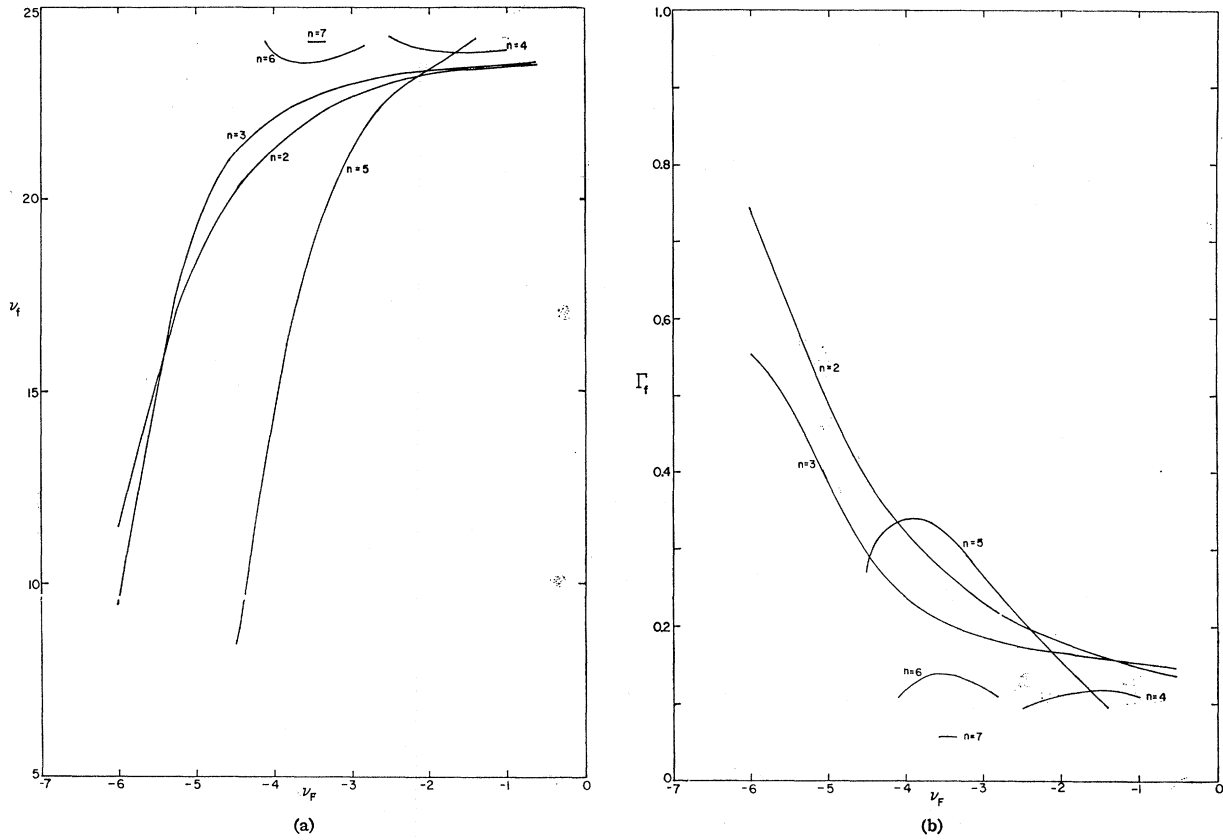
certain effort to clarify the situation. It does seem likely that the eventual value of the width will be narrow, but one may be confronted with the opposite of the usual situation, namely, one may have a theoretical width which is much too narrow. Moreover, one might again take a pessimistic viewpoint and argue that, in view of the almost monotonic decrease of the theoretical width with n , that there is a systematic effect in the Balázs method which overestimates the attraction and leads to narrow values of the width for large n , so that the observed results are a result of the method rather than of physics. We do not see why this should be so, but it is difficult to put the approximations involved in the method on a precise mathematical footing and so exclude the possibility of pathological behavior for large n .

We have also made a similar calculation for the f^0 , fixing the parameters of the exchanged ρ at the physical values, and determining the f^0 parameters self-consistently. The resulting f^0 parameters are given as functions of ν_F and n in Figs. 4 and 5 for the approximate- and exact-width cases, respectively; Tables V and VI give the values of the parameters at the optimum values of the matching point for the cases presented, respectively, in Figs. 4 and 5.

We observe here that it is important to use the exact

TABLE IV. Self-consistent values of (ν_ρ, Γ_ρ) taken from Fig. 3 at the optimum matching point $(\nu_F)_{\text{opt}}^{11}$, as judged by the WE criterion, as a function of n , the number of poles. The units and the meaning of the asterisk are the same as in Table III.

n	$(\nu_F)_{\text{opt}}^{11}$	m_ρ	ν_ρ	Δm_ρ	Γ_ρ
2	-2.50	848	8.23	536	0.750
3	-5.50	721	5.68	386	0.681
4
5
6	-5.20	707	5.42	172	0.312
7	-5.30	711	5.49	66	0.120
7	-5.30	711	5.49	100*	0.167

FIG. 5. Same as Fig. 4 except that the exact form of the output width is used. (a) ν_f ; (b) Γ_f .

width, since doing so yields a markedly different value of the width for all values of n studied. As will be seen from Table VI, the predicted value of the f^0 mass is nearly independent of n , and is about 1390 MeV, which is too large by about 11%. The predicted width is about 90 MeV, about 65% of the experimental best value; for $n=7$, the predicted value of the width dips to 39 MeV.

In evaluating the results of the f^0 bootstrap, it is important to bear in mind that, because of its larger mass, the f^0 may be more sensitive than the ρ to the necessarily crude approximations used in treating the effects of inelasticity in our calculation. The reader is reminded that our procedure, following that of BIII, is to take the factor R_I^I (the ratio of total to elastic cross section) = 1 for $\nu < \nu_D$, and $R_I^I = 3$ for $\nu > \nu_D$. The value 3 is obtained by taking a rough energy average of R_I^I as calculated from the Pomeranchuk contribution only, as given in Fig. 1. We have chosen, rather arbitrarily, to take $\nu_D = 25$, somewhat above the experimental position of the f^0 . It is clear that the use of the Pomeranchuk contribution to calculate R_I^I in the vicinity of $\nu = \nu_D$ is simply a substitute for ignorance, and also that the discontinuity in our approximate form of R_I^I at $\nu = \nu_D$ is likely to distort the amplitude at nearby values of ν . In view of this, the results given in Table VI seem reasonably satisfactory. In particular,

it may well be that the sudden change in the value of the width for $n=7$ is due to the fact that the self-consistent value of ν in that case happens to be particularly close to the discontinuity at ν_D .

TABLE V. Self-consistent values of (ν_f, Γ_f) taken from Fig. 4 at the optimum matching point $(\nu_F)_{\text{opt}}^{20}$ as judged by the WE criterion, as a function of n , the number of poles. $(\nu_F)_{\text{opt}}^{20}$ and ν_f are in units with $\hbar = c = m_\pi = 1$, m_f and Δm_f in MeV, and Γ_f is dimensionless.

n	$(\nu_F)_{\text{opt}}^{20}$	m_f	ν_f	Δm_f	Γ_f
1	-0.50	1377	23.33	334	0.535
3	-1.50	1354	22.51	306	0.510
4	-1.50	1363	22.85	306	0.500
5	-2.10	1351	22.42	300	0.500
6	-3.80	1354	22.51	300	0.500
7	-3.50	1392	23.86	384	0.600

TABLE VI. Self-consistent values of (ν_f, Γ_f) taken from Fig. 5 at the optimum matching point $(\nu_F)_{\text{opt}}^{20}$ as judged by the WE criterion, as a function of n , the number of poles. The units are the same as in Table V.

n	$(\nu_F)_{\text{opt}}^{20}$	m_f	ν_f	Δm_f	Γ_f
2	-0.50	1384	23.58	90	0.142
3	-0.50	1383	23.53	92	0.145
4	-1.50	1392	23.85	76	0.119
5	-1.90	1382	23.50	92	0.147
6	-3.60	1383	23.54	88	0.140
7	-3.50	1410	24.51	39	0.059

TABLE VII. Self-consistent values of $(\nu_\rho, \Gamma_\rho, \nu_f, \Gamma_f)$ for a simultaneous ρ and f^0 double bootstrap. $(\nu_F)_{\text{opt}}^{11}$ and $(\nu_F)_{\text{opt}}^{20}$ are the optimum matching points for the $l=1$, $l=1$ and $l=0$, $l=2$ partial waves, respectively. The meaning of the other quantities and their units are the same as in Tables III–VI.

n	$(\nu_F)_{\text{opt}}^{11}$	m_ρ (MeV)	ν_ρ	Δm_ρ (MeV)	Γ_ρ	$(\nu_F)_{\text{opt}}^{20}$	m_f (MeV)	ν_f	Δm_f (MeV)	Γ_f
6	−5.40	708	5.44	118	0.216	−3.401	1410	24.50	42	0.0632
7	−5.30	711	5.49	46	0.0831	−3.601	1344	22.17	112	0.189

Finally, for the cases $n=6$ and $n=7$, we have carried out a simultaneous bootstrap of both the ρ and f^0 . We have adopted the following procedure in performing the double bootstrap. A ρ bootstrap, with ρ and fixed f^0 input, was performed for different values of the matching point ν_F , and the values of (ν_ρ, Γ_ρ) corresponding to the optimum matching point, $(\nu_F)_{\text{opt}}^{11}$, are taken to be the output. This constitutes the first iteration. In the second iteration we perform an f^0 bootstrap, with ρ input taken from the first iteration, for various values of ν_F . The values of (ν_f, Γ_f) corresponding to $(\nu_F)_{\text{opt}}^{20}$ are taken as the output and then used as input into the next ρ bootstrap calculation, and so on. The iterations are continued until the output values of the ρ and f^0 parameters are equal to the input values. The exact form of the output width is used throughout.

The results are given in Table VII, and are quite similar to those given in Tables IV and VI. For $n=7$, the f^0 mass is somewhat decreased and the width becomes somewhat larger in the double-bootstrap case as compared with the case when the ρ parameters are held at their physical values, so that the double bootstrap f^0 is in somewhat better agreement with experiment. The ρ mass is unchanged, but the width, for $n=7$, is narrower and in worse agreement with experiment than when physical f^0 exchange is used. It might be added that, unlike the results given in Table IV, the value of Γ_ρ here is quite well determined even if the degree of self-consistency required is relaxed; it appears that in the 7-pole case Γ_ρ must lie in the range 0.083 ± 0.01 if one is to obtain self-consistency to within 10%.

From Table VII, we see that, in the 6- and 7-pole cases, the Balázs method does allow a double bootstrap of the ρ and f^0 with results in fair agreement with experiment; in contrast to the usual calculation, the resonance widths turn out to be somewhat too narrow. Again, one has the troublesome problem that there are significant changes, especially in the widths, when n is changed from 6 to 7, so that it is difficult to know whether the results we have obtained are close to what might be obtained as a limiting value if n were made very large. We find it somewhat comforting that, in the double-bootstrap case, the f^0 width increases between $n=6$ and $n=7$, giving us additional confidence that the method does not lead to a continuing systematic decrease to negative widths as n is increased. We should add that, as far as the values of the ρ parameters are concerned, we consider the results given in Table V,

obtained by fixing the f^0 input parameters at their physical values, as more significant than the double-bootstrap results. As we have already said, we feel that the f^0 parameters obtained from the method are somewhat unreliable because of our method of treating inelasticity, and in a double-bootstrap calculation inaccuracies in the f^0 parameters are in turn reflected in the results obtained for the ρ .

It is perhaps worth pointing out that our double-bootstrap results differ quite significantly, especially for the f^0 mass, from those obtained by Balázs³ in his original treatment of the problem, in which he found masses of 685 and 892 MeV for the ρ and f^0 , and widths of about 160 MeV for each. This merely emphasizes the necessity of worrying about such things as the choice of optimum matching points, the use of a reasonable number of approximating poles, and the use of the exact form of the output width, if one is going to employ this method of doing bootstrap calculations with any hope of success.

Finally, in discussing the results, it is perhaps worthwhile to point out that the effects on the output values of the parameters of changes in the input values do not always accord with one's intuitive expectations based on the idea that the forces we are dealing with are "attractive." For example, one would expect that an increase in the input ρ width, and hence in the strength of the attractive ρ -exchange force, would lower the output ρ mass. Again, it would be expected that if the f^0 input width were increased, one would have to decrease the input ρ width to keep the output ρ mass unchanged, since f^0 exchange is also attractive. While there is a strong tendency for these and similar expectations to be correct, one finds that for some values of the parameters exactly the opposite effects occur; e.g., in some cases the ρ output mass increases when the input width increases. These effects result from the complicated nonlinear nature of the equations being dealt with. It is true, e.g., that increasing the input ρ width gives a positive contribution to the amplitudes $A_I^I(\nu)$ in the nearby unphysical region. This, in turn, tends to cause $N_I^I(\nu)$ to be more positive for physical values of ν , which in turn causes the zero in the denominator function, i.e., the output mass, to occur at a lower value of ν . The foregoing is, in fact, exactly what we mean when we say ρ exchange is attractive. However, the ρ contributions to some of the derivatives of the $A_I^I(\nu)$ turn out to be negative, and in some cases these negative changes in the derivatives turn out to mean that increasing the strength of

the ρ exchange force actually decreases the value of $N_I^I(\nu)$, on the average, in the physical region so that the zero of the denominator function is actually moved to a larger value of ν . One can look at the matter somewhat differently by noting that the Balázs method automatically includes the effects of short-range forces which are required to produce a distant left-hand cut which is reasonably approximated by the Balázs poles. These short-range forces (which are determined by the requirement that the two forms of the partial-wave amplitude, and of $n-1$ of its derivatives, agree at the point $\nu=\nu_P$) may turn out to be repulsive, and evidently for some values of the parameters the change due to an increase, e.g., in the attractive ρ -exchange force is more than compensated for by an increased effective short-range repulsion. It is not clear whether this apparent short-range repulsion should be taken literally, or whether it results from the approximate form of the t -channel absorptive part; it is possible that the difference between the form of the amplitude used and that which would be obtained from ρ and f^0 exchange by the Mandelstam iteration procedure could correspond to an effective short-range repulsion.

V. CONCLUSIONS

The results of our calculations are in at least fair agreement with experimental information on the ρ and f^0 . Indeed, when the input f^0 parameters are taken from experiment, thus minimizing somewhat the uncertainties due to our lack of knowledge of inelasticity, one obtains satisfactory self-consistency for a ρ mass differing from experiment by about 5%, and a ρ width in agreement with the experimental values. The results show that, when contributions to the left-hand cut other than just ρ exchange are taken into account, one can obtain narrow resonances in a bootstrap calculation, and, to this extent, support the conclusions of Ref. 7. Indeed, there is some indication in the results that one may, in fact, be troubled with theoretical resonance widths which are too narrow.

As far as the usefulness of the Balázs method is concerned, the present results are not terribly reassuring, principally because one finds the results, in some cases, still changing significantly with the number of approximating poles, even when that number is made as large as 7. In this respect, the results of Ref. 1, based on the behavior for $n=2, 3$, and 4, were somewhat misleading in suggesting that the n dependence was not severe. It should also be pointed out that, when calculations are done with the Balázs method with the care which seems to be required, they can be quite time-consuming. The work reported in this paper required the equivalent of about two hours of computing time on a CDC 6600. Against these factors must be weighed the theoretical advantages of the method, its freedom from cutoff parameters, and its

ability to take into account contributions from distant singularities.

APPENDIX A

In the π - π problem the ratio of total to elastic partial-wave cross section $R_I^I(\nu)$ is given by

$$R_I^I(\nu) = -\frac{1}{2}(s/\nu)^{1/2} \text{Im}[1/A_I^I(\nu)] \quad (\text{A1})$$

and the optical theorem takes the form

$$\text{Im}A_I^I(s,0) = \{[s(s-4)]^{1/2}/16\pi\} \sigma_{\text{tot}}^I(s), \quad (\text{A2})$$

where

$$s=4(\nu+1), \quad m_\pi=\hbar=c=1. \quad (\text{A3})$$

For $\nu > \nu_D$ we evaluate $R_I^I(\nu)$ through equation (A1) with $A_I^I(\nu)$ given by Eq. (BIII-22) as

$$A_I^I(\nu) = \sum_{I'} \beta_{II'} \left\{ \int_{-1}^{+1} d(\cos\theta) P_I(\cos\theta) \right. \\ \times A^{I'}(-2\nu(1-\cos\theta), 4(\nu+1)) \\ \left. - \frac{2 \sin\pi l}{\pi} \int_{-\infty}^{-1} d(\cos\theta) Q_I(-\cos\theta) \right. \\ \left. \times A^{I'}(-2\nu(1-\cos\theta), 4(\nu+1)) \right\}, \quad (\text{A4})$$

where

$$\cos\theta = 1 + 2t/(s-4).$$

We perform a Regge expansion, in terms of t -channel Regge poles, of $A^I(t,s)$ and for large s , which we assume throughout, we can neglect the background integral and are left with only the Regge-pole contributions

$$A^I(t,s) = \sum_{\alpha_I} -\frac{1}{2}\pi \frac{[2\alpha_I(t)+1]}{\sin\pi\alpha_I(t)} \beta_{\alpha_I}^I(t) \\ \times \left[P_{\alpha_I(t)}\left(-1-\frac{2s}{t-4}\right) + (-1)^I P_{\alpha_I(t)}\left(1+\frac{2s}{t-4}\right) \right] \quad (\text{A5})$$

or, from BIII, Eq. (A1), and keeping only the highest trajectory

$$A^I(t,s) = -\frac{1}{2}\pi \frac{[2\alpha_I(t)+1]}{\sin\pi\alpha_I(t)} \beta_{\alpha_I}^I(t) c_1(\alpha_I(t)) \\ \times \left[e^{-i\pi\alpha_I(t)} + (-1)^I \right] \left(\frac{2s}{t-4} \right)^{\alpha_I(t)}, \quad (\text{A6})$$

where

$$c_1(\alpha) = 2^\alpha \Gamma(\alpha + \frac{1}{2}) / \pi^{1/2} \Gamma(\alpha + 1). \quad (\text{A7})$$

We will neglect everything except the Pomeranchuk trajectory, and so retain only the $I=0$ term.

The Pomeranchuk trajectory is given by

$$\text{Re}\alpha_P(t) = 1 + \frac{1}{4}\epsilon_P t, \quad \text{Im}\alpha_P(t) \approx 0, \quad (\text{A8})$$

and the Pomeranchuk residue is given by

$$\beta_P(t) = -\beta_P(0) \left(\frac{1}{4}t - 1 \right)^{\alpha_P(t)} e^{\epsilon_P t}. \quad (\text{A9})$$

From Eqs. (A6)–(A9), we have

$$A^0(t, s) = -\frac{3}{2}\pi\beta_P(0)e^{c_P t}[\tan(\frac{1}{8}\pi\epsilon_P t) + i](\frac{1}{2}s)^{\alpha_P(t)}. \quad (\text{A10})$$

In obtaining Eq. (A10) we made the approximation that $c_1(\alpha_P(t))[2\alpha_P(t) + 1]$ is constant and evaluated it at $t=0$, to get

$$c_1(\alpha_P(t))[2\alpha_P(t) + 1] \approx 3c_1(1) = 3. \quad (\text{A11})$$

We also used the trigonometric identity

$$(1 + \cos x)/\sin x = \cot \frac{1}{2}x. \quad (\text{A12})$$

Using the optical theorem as expressed by Eq. (A2), the crossing matrix, and Eq. (A10), we obtain Eq. (5b) which we repeat here:

$$\beta_P(0) = -\sigma_{\text{tot}}^I(\infty)/4\pi^2. \quad (\text{A13})$$

From the discussion in the Appendix of BIII, we have

$$\tan(\frac{1}{8}\pi\epsilon_P t) \approx \frac{1}{8}\pi\epsilon_P t \quad (\text{A14})$$

to about 10%. Also

$$\frac{1}{2}s \approx 2\nu, \quad t = -2\nu(1 - \cos\theta), \quad (\text{A15})$$

and hence,

$$(\frac{1}{2}s)^{\alpha_P(t)} = (2\nu)e^{-\epsilon_P \nu \ln(2\nu)/2} e^{\epsilon_P \nu \ln(2\nu) \cos\theta/2}. \quad (\text{A16})$$

Collecting the above equations, we obtain

$$A^0(-2\nu(1 - \cos\theta), 4(\nu + 1)) = \frac{3}{2}K(\nu)[b(\nu)(\cos\theta - 1) + i]e^{g(\nu) \cos\theta}, \quad (\text{A17})$$

where we have set

$$\begin{aligned} K(\nu) &= \frac{\nu\sigma_{\text{tot}}^I(\infty)}{2\pi} e^{-g(\nu)}, \\ b(\nu) &= \frac{1}{4}\pi\epsilon_P \nu, \\ g(\nu) &= [\frac{1}{2}\epsilon_P \ln(2\nu) + 2c_P]. \end{aligned} \quad (\text{A18})$$

Combining Eqs. (A17) and (A4), we obtain

$$\begin{aligned} A_I^I(\nu) &= \frac{1}{2}K(\nu) \left\{ \int_{-1}^{+1} d(\cos\theta) P_l(\cos\theta) \right. \\ &\quad \times [b(\nu)(\cos\theta - 1) + i] e^{g(\nu) \cos\theta} \\ &\quad - \frac{2 \sin \pi l}{\pi} \int_{-\infty}^{-1} d(\cos\theta) Q_l(-\cos\theta) \\ &\quad \left. \times [b(\nu)(\cos\theta - 1) + i] e^{g(\nu) \cos\theta} \right\}. \quad (\text{A19}) \end{aligned}$$

Integrating Eq. (A19) gives us the real and imaginary parts of $A_I^I(\nu)$ from which we can calculate $\text{Im}(A_I^I(\nu))^{-1}$ and, consequently, $R_I^I(\nu)$ through Eq. (A1).

We will now explicitly evaluate Eq. (A19) for the case of integer l . In this case since $\sin \pi l = 0$, we have

$$A_I^I(\nu) = \frac{1}{2}K(\nu) \int_{-1}^{+1} [dx P_l(x) [b(\nu)(x - 1) + i] e^{g(\nu)x}]. \quad (\text{A20})$$

$P_l(x)$ is a polynomial in x of order l , and can be written as

$$P_l(x) = \sum_{k=0}^l e_{lk} x^k. \quad (\text{A21})$$

We also define the integral

$$I_k(g) = \int_{-1}^{+1} dx x^k e^{gx}. \quad (\text{A22})$$

Then (A20) can be written as

$$A_I^I(\nu) = \frac{1}{2}K(\nu) \sum_{k=0}^l e_{lk} \{ b(\nu) [I_{k+1}(g(\nu)) - I_k(g(\nu))] + i I_k(g(\nu)) \}, \quad (\text{A23})$$

and $R_I^I(\nu)$ is given by

$$R_I^I(\nu) = \left(\frac{\nu+1}{\nu} \right)^{1/2} \frac{2}{K(\nu)} \frac{a_{\text{Im}}^{II}(\nu)}{[a_{\text{Re}}^{II}(\nu)]^2 + [a_{\text{Im}}^{II}(\nu)]^2}, \quad (\text{A24})$$

where

$$a_{\text{Re}}^{II}(\nu) = b(\nu) \sum_{k=0}^l e_{lk} [I_{k+1}(g(\nu)) - I_k(g(\nu))] \quad (\text{A25})$$

and

$$a_{\text{Im}}^{II}(\nu) = \sum_{k=0}^l e_{lk} I_k(g(\nu)). \quad (\text{A26})$$

By partial integration a recursion formula for $I_k(g)$ can be obtained from which the following formula can be proved by mathematical induction:

$$I_k(g) = \frac{(-1)^k k!}{g^k} \sum_{n=0}^k \left(\frac{g^n}{n!} \frac{[(-1)^n e^g - e^{-g}]}{g} \right). \quad (\text{A27})$$

Asymptotically,

$$I_k(g) \underset{g \rightarrow \infty}{\sim} \frac{1}{g} [e^g + (-1)^{k+1} e^{-g}]; \quad (\text{A28})$$

hence

$$a_{\text{Re}}^{II}(\nu) \xrightarrow{\nu \rightarrow \infty} 0, \quad (\text{A29})$$

$$a_{\text{Im}}^{II}(\nu) \xrightarrow{\nu \rightarrow \infty} \frac{e^g}{g}, \quad (\text{A30})$$

since

$$\sum_{k=0}^l e_{lk} = P_l(1) = 1. \quad (\text{A31})$$

Substituting Eqs. (A29), (A30), and definition (A18) in (A24), we have

$$R_I^I(\nu) \underset{\nu \rightarrow \infty}{\sim} \frac{2\pi\epsilon_P \ln(2\nu)}{\sigma_{\text{tot}}^I(\infty)}. \quad (\text{A32})$$