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Interpolation of a Scattering Amplitude between Integral Values of an External Spin*

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It is shown that the π^* inverse-production amplitude in the reaction $\pi^*\pi \rightarrow \pi\pi$, where π^* is a pionlike particle of any spin, can be interpolated in a physically meaningful way between integral-spin values of the π^*. A unique interpolation is obtained by formally treating the π^* as though it had spacelike four-momentum, for which the spectrum of the angular momentum is continuous, and then analytically continuing the amplitude in the invariant square of the π^* four-momentum to timelike values. This procedure selects one of two equally "natural" interpolations. The selected interpolation is then shown to be compatible with the requirement of constructing partial-wave amplitudes, containing the ρ, that have factorizable singularities analytic in the spin of the π^*, while the other selection is forbidden by this requirement. An independent procedure, paralleling the usual Froissart-Gribov rule, also is shown to select the same favored interpolation of the partial-wave amplitude.

I. INTRODUCTION

In a $\pi\pi$ scattering bootstrap model recently proposed for studying the possibility of the spin continuation of an external particle, the problem of interpolating the scattering amplitude between integral-spin values was alluded to but remained unsettled.[1] In this model, the hypothetical self-consistent bootstrap of the ρ is exploited in the reaction $\pi^*\pi \rightarrow \pi\pi$, where π^* is a pionlike particle of arbitrary spin J and mass M.

More recently the Veneziano model,[2] which ensures crossing symmetry and Regge asymptotic behavior, was applied to the same reaction and thereby generated constraints for the external particle trajectory.[3] Although the models are strikingly different in content, they both suggest that an external mass-spin relation may be at least in part a consequence of internal consistency. In the Veneziano model, crossing is exact, in contrast to the possibly "complementary"[4] bootstrap model, where unitarity is emphasized[5] and plays a crucial role and where crossing is crudely approximated by bootstrap conditions in an N/D scheme. It remains to be seen whether or not Regge behavior can be easily incorporated into the bootstrap model.

The problem of interpolation between integral-spin values does not naturally arise in the Veneziano approach, since the analytic forms of the trajectories are assumed at the outset. In the bootstrap approach, on the other hand, the interpolation problem is unavoidable, since no assumption of any trajectory form is required.

The purpose of this paper is to formulate a definite rule for interpolating the scattering amplitude between integral-spin values of the π^* in the inverse-production reaction $\pi^*\pi \rightarrow \pi\pi$. We also present some additional theoretical evidence supporting the interpolation rule, as it manifests itself in a partial-wave projection, and relating it to other work.

The rule is derived from the work of Bargmann[6] and of Toller[7] on the irreducible representations of the three-dimensional Lorentz group or "little group" corresponding to a system of spacelike four-momentum. In such a representation the spectrum of the Casimir operator that is the analog of the square of angular momentum is continuous rather than discrete as it is in the three-dimensional rotation group O_3. Thus it is meaningful to speak of a continuous angular momentum for an elementary system of spacelike four-momentum. The central idea to which we appeal in formulating the interpolation rule is that the Born amplitude (input to the N/D scheme) for a *timelike* π^* but arbitrary J is defined by an analytic continuation from the amplitude for a *spacelike* π^* for which we make explicit use of the irreducible representation belonging to the value J.

For simplicity we carry out the present study in the framework of the earlier[1] one-channel, elastic unitary bootstrap model instead of the multichannel version,[5] and we adhere to a non-Reggeized, self-consistent ρ bootstrap in the reaction $\pi^*\pi \rightarrow \pi\pi$. It is hoped that these limitations do not substantially reduce the validity or generality of our results which are, after all, consequences of Lorentz invariance couched in the language of this particular model rather than of a peculiar choice of dynamical approximations.

The steps in our approach to the interpolation problem are as follows: (i) The ρ exchange amplitudes are computed as though the π^* had spacelike four-momentum; (ii) the amplitudes are continued to timelike four-momentum; (iii) comparison of these with the ρ-exchange amplitudes computed in a conventional manner yields a unique interpolation rule; (iv) the rule so obtained is shown to be compatible with the requirement that there exist combinations of the partial-wave amplitudes that have factorizable singularities

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that are analytic in J; (v) an obvious alternative interpolation rule is shown to be incompatible with this requirement; and (vi) an independent interpolation procedure, analogous to the Froissart-Gribov partial-wave projection rule in the usual Regge analysis, is shown to yield the same interpolated partial-wave amplitude that would be obtained from the conventional partial-wave projection of the ρ-exchange amplitude interpolated according to the rule derived from the spacelike π^*.

II. ρ-EXCHANGE AMPLITUDES FOR SPACELIKE π^*

We compute the ρ-exchange amplitude for π^* inverse production in the reaction $\pi^*\pi \to \pi\pi$, where the π^* is formally treated as if it had spacelike four-momentum. As an elementary system, the π^* is described by states making up irreducible representations of the Lorentz group belonging to a continuous spectrum of spin values. The method, used in Ref. 1, of coupling tensors comprised of products of momentum vectors to the spin tensors describing the spinning particles[8] does not, therefore, adequately handle the coupling of the ρ to the π^* and π. Such a coupling procedure is inherently discrete. Instead we introduce a set of elementary amplitudes describing the virtual decay $\pi \to \pi^* + \rho$ such that in the Lorentz frame where the fourth component of the π^* four-momentum is zero, and all three-momenta are collinear, the helicities have definite values. These are analogous to the elementary amplitudes describing the virtual decay of a timelike system in its rest frame into final particles of definite helicities moving along the spin quantization axis of the initial state.[9]

In the following constructions, angles are measured from the positive z axis in the clockwise direction about the positive y axis. Rotations are always carried out about the y axis and are indicated by $R(\theta)$. A Lorentz boost along the z axis is indicated by $L(\beta)$.

We first consider the transition amplitude $A(\lambda,\sigma)$ for the inverse virtual-decay process $\pi^*+\rho \to \pi$ where the π^* has spin J, spacelike four-momentum of invariant square $M^2 = -\mu^2 < 0$, and helicity λ in a Lorentz frame L_1 where the only nonzero component of its four-momentum is the z component μ; the π has mass $=1$, energy E in L_1 and is emitted in the xz plane; and the ρ, of invariant mass squared t, is moving at angle $\pi-\theta_\rho$ with helicity σ in L_1. We treat the ρ in a non-Regge manner as an off-shell state ($t<0$) of an elementary particle of timelike four-momentum and fixed spin $=1$ with three spin quantization states. The rest frame of the ρ is necessarily related to L_1 by a complex Lorentz transformation.

We prepare the state $|\rho_1\rangle$ of the ρ in L_1 by starting with a rest state $|-\sigma\rangle$ with spin projection $-\sigma$ along the z axis, giving it a complex Lorentz boost in the negative z direction and then rotating by angle $-\theta_\rho$. If we take the rest mass of the ρ to be $\epsilon i(-t)^{1/2}$, where $\epsilon = \pm 1$, then we find $|\rho_1\rangle = R(-\theta_\rho)L(-\beta_1)|-\sigma\rangle$, where $\beta_1 = (1-tE^{-2})^{1/2} > 1$. The correct root of $\gamma \equiv (1-\beta_1^2)^{-1/2} = -\epsilon_1 iE(-t)^{-1/2}$ is attained from the region $\beta_1 < 1$ by a counterclockwise (clockwise) continuation about the branch point at $\beta_1 = 1$ for $\epsilon_1 = +1$ (-1).

In a frame L_2, obtained by a real Lorentz transformation along the x axis, such that the x component of the π (and ρ) momentum vanishes, the state of the ρ is $|\rho_2\rangle = R(-\frac{1}{2}\pi)L(\beta_1 \sin\theta_\rho)R(\frac{1}{2}\pi)|\rho_1\rangle = L(-\beta_2)R(\alpha)|-\sigma\rangle$, where

$$\cos\alpha = \cos\theta_\rho (1-\beta_1^2 \sin^2\theta_\rho)^{-1/2},$$
$$\sin\alpha = -\epsilon_1 i \sin\theta_\rho\ (\beta_1^2-1)^{1/2}(1-\beta_1^2\sin^2\theta_\rho)^{-1/2},$$
$$\beta_2 = \beta_1 \cos\alpha > 1.$$

Thus the helicity composition of $|\rho_2\rangle$ is given by

$$|\rho_2\rangle = \sum_{\tau=-1}^{1} d_{\sigma\tau}{}^1(\alpha) L(-\beta_2)|-\tau\rangle,$$

where $L(-\beta_2)|-\tau\rangle$ is a state of helicity $+\tau$ moving along the negative z axis.

The π^* state in L_2 is given by

$$|\pi_2{}^*\rangle = R(-\tfrac{1}{2}\pi)L(\beta_1 \sin\theta_\rho)R(\tfrac{1}{2}\pi)|{}^*\lambda\rangle$$
$$= \sum_{\nu=-\infty}^{\infty} \bar{d}_{\nu\lambda}{}^J(-\zeta)|{}^*\nu\rangle,$$

where $|{}^*\lambda\rangle$ is the π^* state in L_1, and[7]

$$\bar{d}_{\nu\lambda}{}^J(-\zeta) = \frac{\Gamma(J+\nu+1)\Gamma(J-\nu+1)}{\Gamma(J+\lambda+1)\Gamma(J-\lambda+1)}(\cosh\zeta)^{\nu+\lambda}$$
$$\times[\sinh(-\zeta)]^{\nu-\lambda} P_{J-\nu}{}^{(\nu-\lambda,\,\nu+\lambda)}(\cosh 2\zeta) \quad \text{for } \nu \geq \lambda, \qquad (1)$$

$$\bar{d}_{\nu\lambda}{}^J(-\zeta) = \bar{d}_{-\nu-\lambda}{}^J(-\zeta), \qquad (2)$$

with

$$\zeta = \tfrac{1}{2}\tanh^{-1}(\beta_1 \sin\theta_\rho).$$

The connection between the normalization of the states $|{}^*\nu\rangle$ and the normalization of the corresponding states when the π^* is continued to timelike four-momentum is ignored at this point, but becomes important in Sec. III.

We may now write the amplitude for $\pi^*+\rho \to \pi$ in the form

$$A(\lambda,\sigma) = \langle \pi | T | |\pi_2{}^*\rangle,\quad |\rho_2\rangle\rangle$$
$$= a_1 \bar{d}_{1\lambda}{}^J(-\zeta) d_{\sigma 1}{}^1(\alpha) + a_0 \bar{d}_{0\lambda}{}^J(-\zeta) d_{\sigma 0}{}^1(\alpha)$$
$$+ a_{-1}\bar{d}_{-1\lambda}{}^J(-\zeta) d_{\sigma -1}{}^1(\alpha), \qquad (3)$$

where the elementary amplitudes a_ν are defined by

$$a_\nu \delta_{\nu\tau} = \langle \pi | T | |{}^*\nu\rangle,\quad L(-\beta_2)|-\tau\rangle\rangle. \qquad (4)$$

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[8] C. Zemach, Phys. Rev. **140**, B97 (1965).
[9] T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963)

Conservation of parity tells us that the elementary amplitudes are not all independent. Let Y be the reflection in the xz plane. Then we have[10]

$$YL(-\beta)|-\tau\rangle=(-)^{1+\tau}\eta_\rho L(-\beta)|\tau\rangle, \quad (5)$$

where η_ρ is the parity of the ρ. However, for the π^* states we have

$$Y|{}^*\nu\rangle=y|{}^*-\nu\rangle, \quad (6)$$

where the phase y is *independent* of ν. This difference is attributed to the fact that the O_3 generators obey $YJ_+=-J_-Y$, while the corresponding generators of the three-dimensional Lorentz group obey[11] $YL_+=L_-Y$. Applying Eqs. (5) and (6) to Eq. (4), we find $a_\nu=\eta_\pi\eta_\rho y(-)^{1+\nu}a_{-\nu}$, which reduces to $\eta_\pi y\eta_\rho=-1$ when we set $\nu=0$. Hence $a_1=-a_{-1}$ and Eq. (3) reduces to

$$A(\lambda,\sigma)=a_0\bar{d}_{0\lambda}{}^J(-\zeta)d_{\sigma 0}{}^1(\alpha)+a_1[\bar{d}_{1\lambda}{}^J(-\zeta)d_{\sigma 1}{}^1(\alpha) -\bar{d}_{-1\lambda}{}^J(-\zeta)d_{\sigma -1}{}^1(\alpha)]. \quad (7)$$

We then consider the transition amplitude $A(\sigma)$ for the other half of the over-all reaction $\pi^*\pi\to\pi\pi$, the virtual decay $\pi\to\rho+\pi$. The incoming π is moving in the negative z direction at speed $\beta_\pi=(s+\mu^2-1)/S$ in L_1, where $s=$square of c.m. energy of the over-all reaction and $S\equiv[s^2-2s(-\mu^2+1)+(\mu^2+1)]^{1/2}$. The state of the ρ, emitted in the xz plane at angle $\pi-\theta_\rho$ with helicity σ, has already been worked out. We omit the details of this more familiar computation. The result is

$$A(\sigma)=-a'd_{0\sigma}{}^1(\pi-\delta), \quad (8)$$

where $a'\delta_{\tau 0}=\langle\tau|L(-\beta_3)T|\pi\rangle$ defines the elementary amplitude a' and where

$$\cos\delta=(\beta_\pi\cos\theta_\rho-\beta_1)/d,$$
$$\sin\delta=-\epsilon_1 i\beta_\pi(\beta_1{}^2-1)^{1/2}\sin\theta_\rho/d,$$
$$\beta_3=d(1-\beta_1\beta_\pi\cos\theta_\rho)^{-1},$$
$$d=(\beta_1{}^2+\beta_\pi{}^2-2\beta_1\beta_\pi\cos\theta_\rho-\beta_1{}^2\beta_\pi{}^2\sin^2\theta_\rho)^{1/2}.$$

The amplitude for the over-all reaction is then obtained by combining Eqs. (7) and (8) to give

$$A_\lambda\sim\sum_{\sigma=-1}^{1}A(\lambda,\sigma)A(\sigma)=a_0'\bar{d}_{0\lambda}{}^J(-\zeta)d_{00}{}^1(\alpha+\pi-\delta) +a_1'[\bar{d}_{1\lambda}{}^J(-\zeta)d_{01}{}^1(\alpha+\pi-\delta) -\bar{d}_{-1\lambda}{}^J(-\zeta)d_{0-1}{}^1(\alpha+\pi-\delta)], \quad (9)$$

where a_0' and a_1' depend in an unknown way on t and J. Omitting some trivial but lengthy kinematical reductions, we may express Eq. (9) in terms of the Mandelstam invariants s, t, and $u=3-\mu^2-s-t$, using

$$d_{00}{}^1(\alpha+\pi-\delta)=(-t)^{1/2}(-t+4)^{-1/2}\times(2s+t+\mu^2-3)T^{-1}, \quad (10)$$

$$d_{0\pm1}{}^1(\alpha+\pi-\delta)=\mp i\epsilon_1\sqrt{2}[stu-(\mu^2+1)^2]^{1/2}\times(4-t)^{1/2}T^{-1}, \quad (11)$$

$$z\equiv-\cosh2\zeta$$
$$=-\tfrac{1}{2}[S^2-(t-u)(s-\mu^2-1)]S^{-1}T^{-1},$$
$$T=[t^2-2t(1-\mu^2)+(1+\mu^2)^2]^{1/2}, \quad (12)$$

and, from Eqs. (1) and (2) and identities[12] relating Jacobi and Legendre functions,

$$\bar{d}_{00}{}^J(-\zeta)=P_J(-z), \quad (13)$$

$$\bar{d}_{10}{}^J(-\zeta)+\bar{d}_{-10}{}^J(-\zeta)=ST\mu^{-1}[stu-(\mu^2+1)^2]^{-1/2}\times[P_{J-1}(-z)-zP_J(-z)], \quad (14)$$

$$\bar{d}_{01}{}^J(-\zeta)=-2(J+1)^{-1}\mu\times[stu-(\mu^2+1)^2]^{1/2}S^{-1}T^{-1}P_J'(-z), \quad (15)$$

$$\bar{d}_{11}{}^J(-\zeta)+\bar{d}_{-11}{}^J(-\zeta)=2[P_J(-z)+zJ^{-1}(J+1)^{-1}P_J'(-z)]. \quad (16)$$

III. AMPLITUDES FOR TIMELIKE π^*

The analytic continuation of amplitude (9) from spacelike to timelike π^* is carried out by analytically continuing the functions on the right-hand sides of Eqs. (10)–(16) from negative $\nu\equiv-\mu^2$ to positive $\nu=M^2$, where M is the mass of the timelike π^*. Equivalently, we continue μ to the imaginary value $i\epsilon_2M$, where $\epsilon_2=\pm1$. In this continuation we hold J, s, and $x=\cos\theta$ fixed, where θ is the c.m. scattering angle in the s channel.

For s real and greater than 4, S has a finite square-root cut extending along the real ν axis from ν_- to ν_+, where $1<\nu_-<\nu_+$. Hence S may be continued from any negative ν to a positive value in the neighborhood of $\nu=1$ without encountering zeros or singularities. Since $[stu-(\nu-1)^2]^{1/2}=\frac{1}{2}(s-4)^{1/2}S\sin\theta$, the same statement is true for this function.

The functions $-t$ and T^2 have the same analytic structure as S for fixed s and x. They are both positive and real for any real $\nu<\nu_-$, except that they develop second-order zeros at $\nu=1$ when $x=1$. These zeros are a minor nuisance when we wish to continue the amplitude to values of $\nu>1$, as we now explain. It is easily seen that T^{-1} is regular in the neighborhood of $\nu=1$ for $-1\le x<1$. The problem of extending this interval to $x=1$ arises because, as an analytic function of x, T (or T^{-1}) has a square-root cut over the interval $x_-\le x\le x_+$ where $1\le x_-$. An expansion of x_- in the variable $\nu-1$,

$$x_-=1+\tfrac{1}{4}(s-4)^{-1}(\nu-1)^2+\cdots,$$

reveals that the left end of the cut in x just touches the right end of the physical region ($x=1$) when $\nu=1$, and this contact destroys the analyticity of T^{-1} in ν at $x=1$.

[10] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[11] L. Sertorio and M. Toller, Nuovo Cimento 33, 413 (1964).
[12] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. II, p. 173.

An example of this nonanalyticity is exhibited by the function

$$z \equiv -\cosh 2\zeta = -[S-(s-4)^{1/2}s^{-1/2}(s+\nu-1)x]T^{-1},$$

which, at $x=1$, is discontinuous in ν; $z=-1$, 0, $+1$ when $\nu<1$, $\nu=1$, $\nu>1$, respectively. (At $x=-1$, z always equals -1). The correct prescription for the continuation of z from negative ν to positive values such that $1<\nu<\nu_-$ must therefore be to continue ν into this region first for $-1\leq x<1$ and then continue in x to $x=1$, so that the full physical region is covered.

We discover that z, under the above continuation, is the cosine of the scattering angle in the π^* rest frame, i.e.,

$$z=\cos\psi,$$

where ψ is defined in Ref. 1.

The two independent amplitudes continued to positive ν are then

$$A^0=\{a_0'(-t)^{1/2}(4-t)^{-1/2}T^{-1}\} \times(2s+t-M^2-3)P_J(-\cos\psi)-\{\epsilon_1\epsilon_2\sqrt{2}a_1'M^{-1}(4-t)^{-1/2}\} \times S[P_{J-1}(-\cos\psi)+\cos\psi\, P_J(-\cos\psi)], \quad (17)$$

and

$$A_1=-\{2i\epsilon_2 a_0'M(-t)^{1/2}(4-t)^{-1/2}T^{-2}(J+1)^{-1}\} \times(2s+t-M^2-3)S^{-1}[stu-(M^2-1)^2]^{1/2}P_J'(-\cos\psi) -\{2\sqrt{2}i\epsilon_1 a_1'T^{-1}(4-t)^{-1/2}\}[stu-(M^2-1)^2]^{1/2} \times[P_J(-\cos\psi)+J^{-1}(J+1)^{-1}\cos\psi\, P_J'(-\cos\psi)]. \quad (18)$$

The corresponding ρ-exchange amplitudes for a timelike π^* computed by the covariant-tensor-coupling method [Eq. (18) of Ref. 1], may be brought into the similar forms

$$B_0'=\{(t-m^2)^{-1}T^{J-2}[aT^2+bJ(t+M^2-1)]\} \times(2s+t-M^2-3)P_J(\cos\psi)+\{-2bJ(t-m^2)^{-1}T^{J-1}\} \times S[-P_{J-1}(\cos\psi)+\cos\psi P_J(\cos\psi)] \quad (19)$$

and

$$B_1'=\{-2J^{-1/2}(J+1)^{-1/2}(t-m^2)^{-1}MT^{J-3} \times[aT^2+bJ(t+M^2-1)]\}(2s+t-M^2-3)S^{-1} \times[stu-(M^2-1)^2]^{1/2}P_J'(\cos\psi) +\{4bJ^{1/2}(J+1)^{1/2}(t-m^2)^{-1}MT^{J-2}\} \times[stu-(M^2-1)^2]^{1/2}[P_J(\cos\psi)-J^{-1}(J+1)^{-1} \times\cos\psi\, P_J'(\cos\psi)], \quad (20)$$

where m is the mass of the ρ, and a and b are J-dependent coupling constants. We have replaced the coupling constant b of Ref. 1 by $b\rightarrow(2J)^{1/2}(J+1)^{1/2}Mb$.

We note that in Eqs. (17)–(20), the expressions in curly brackets are dependent solely on J and t, while in the remaining parts of the right-hand sides of these equations, the dependence on t and J is mixed with s in an explicit and inseparable way. The mixed parts of the corresponding amplitudes are identical only at even integral values of J. These are, in fact, the only values of J for which Eqs. (19) and (20) are physical, if we assume that π^* has even signature. If, on the right-hand sides of (19) and (20), we then make the replacements

$$P_J(\cos\psi)=P_J(-\cos\psi), \quad (21)$$

$$JP_{J-1}(\cos\psi)=-JP_{J-1}(-\cos\psi), \quad (22)$$

$$P_J'(\cos\psi)=-P_J'(-\cos\psi), \quad (23)$$

which are valid at the even non-negative integral values of J, the mixed parts of the corresponding amplitudes are identical for all J, and a definite interpolation rule is established. The interpolation rule conjectured in Ref. 1, i.e., Eqs. (19) and (20) as they stand, is accordingly incorrect since, for example, $P_J(\cos\psi)$ and $P_J(-\cos\psi)$ are distinctly different analytic functions of J.

We could adopt another point of view and simply regard the left- and right-hand sides of Eqs. (21)–(23) as two equally "natural" interpolations in the absence of any evidence favoring one side over the other. We have seen that the right-hand sides, irrespective of naturalness, are the required interpolations when the amplitudes are continued from a spacelike π^*. In Secs. IV–VI we demonstrate that, given the choice, the right-hand sides are *compatible* with other requirements of the model while the left-hand sides are expressly *forbidden*.

We conclude this section by demonstrating that the two methods of computing the ρ-exchange amplitudes can be brought into complete agreement, although the required equalities seem overdetermined at first glance. Complete agreement is achieved by requiring $A_0=B_0'$ and $A_1=nB_1'$, where the factor n is introduced to accommodate a discrepancy between the timelike and spacelike helicity-state normalizations. The mixed parts of these equalities cancel, leaving

$$a_0'(-t)^{1/2}(4-t)^{-1/2}T^{-1} =(t-m^2)^{-1}T^{J-2}[aT^2+bJ(t+M^2-1)],$$

$$-\epsilon_1\epsilon_2 2^{1/2}a_1'M^{-1}(4-t)^{-1/2} =-2^{1/2}b(t-m^2)^{-1}T^{J-1}M^{-1}J^{1/2}(J+1)^{-1/2},$$

$$-2i\epsilon_2 a_0'(J+1)^{-1}MT^{-2}(-t)^{1/2}(4-t)^{-1/2} =2n(t-m^2)^{-1}J^{-1/2}(J+1)^{-1/2}MT^{J-3} +[aT^2+bJ(t+M^2-1)],$$

$$-2^{3/2}i\epsilon_1 a_1'T^{-1}(4-t)^{-1/2}=2^{3/2}nb(t-m^2)T^{J-2},$$

which are identically satisfied if we choose a relative normalization factor

$$n=-i\epsilon_2 J^{1/2}(J+1)^{-1/2},$$

and relate the elementary amplitudes a_λ' to the coupling constants by

$$a_0'=(t-m^2)^{-1}(-t)^{-1/2}(4-t)^{1/2}T^{J-1} \times[aT^2+bJ(t+M^2-1)],$$

$$a_1'=\epsilon_1\epsilon_2 b(t-m^2)^{-1}(4-t)^{1/2}T^{J-1}J^{1/2}(J+1)^{-1/2}.$$

IV. FACTORIZABILITY OF KINEMATIC ZEROS

It has been shown[1,5] that there exist linear combinations of the helicity amplitudes f_λ describing the direct formation and decay of a ρ that have threshold singularities and zeros that are factorizable and analytic in J. The combinations are f_1 and f_0-cf_1, where $c=c(s;J,M)=(s+M^2-1)(2s)^{-1/2}M^{-1}J^{1/2}(J+1)^{-1/2}$. Such a uniform factorization is required in order to carry out a unitarization procedure using dispersion relations. The pure helicity amplitudes are unsatisfactory under the requirement of factorizability of zeros because of the existence of a nonsense channel at $J=0$. The general rule[5] for obtaining the correct combination is to construct out of the helicity amplitudes those amplitudes having natural zeros[13] in J.

The corresponding factorizability of the same combinations of the angular momentum $=1$ projection of the *ρ-exchange* helicity amplitudes is not obvious, but in fact must be true for all J if the bootstrap models are to exist. In this section, we exhibit the factorizability property of the ρ-exchange amplitudes for all J when $M=1$. For nonintegral J, we show that the factorizability of zeros is sensitive to the interpolation rule.

The angular momentum $=1$ projection of the ρ-exchange amplitudes (19) and (20) is

$$B_\lambda=\frac{3}{2}\int_0^\pi d_{\lambda 0}{}^1(\theta)B_\lambda{}'\sin\theta\,d\theta. \tag{24}$$

In order to investigate the behavior of $\bar{B}_\lambda$ in the neighborhood of the threshold $s=4$ when $M=1$, we change the integration variable to $z=\cos\psi$, let $\sigma=s-4$, and introduce the symbol $N=N(\sigma,z)$ to represent *any* functions of z and σ, regular in σ in the neighborhood of $\sigma=0$, of the form

$$N(\sigma,z)=1+\sum_{n=1}^{\infty}\sigma^n f_n(z),$$

where $f_n(z)$ is any polynomial in z. It is legitimate, for example, to write $N^2=N$. For equal-mass kinematics ($M=1$) we have, in the neighborhood of $\sigma=0$, $d(\cos\theta)=-4Nzdz$, $S=2\sigma^{1/2}N$, $t=-z^2\sigma N$, etc.; and Eq. (24), with the right-hand sides of Eqs. (21)–(23) and with $\bar{z}=-z$, becomes

$$B_0=-3\times 2^{J+1}\sigma^{J/2}m^{-2}\int_0^1 d\bar{z}(1-2\bar{z}^2)N(\sigma,\bar{z})\times[(4a+Jb)\bar{z}^{J+1}P_J(\bar{z})-2bJ\bar{z}^JP_{J-1}(\bar{z})] \tag{25}$$

and

$$\bar{B}_1=3\times 2^{J+1}\sigma^{J/2}m^{-2}(2J)^{1/2}(J+1)^{1/2}\int_0^1 d\bar{z}(1-\bar{z}^2)N(\sigma,\bar{z})\times\{J^{-1}(J+1)^{-1}[4a-(J+2)b]\times\bar{z}^{J+2}P_J{}'(\bar{z})+2b\bar{z}^{J+1}P_J(\bar{z})\}. \tag{26}$$

These integrals may be evaluated using the formula[14]

$$\int_0^1 \bar{z}^{J+n}P_J(\bar{z})d\bar{z}=\frac{\pi^{1/2}2^{-J-n-1}\Gamma(J+n+1)}{\Gamma(1+\frac{1}{2}n)\Gamma(J+\frac{1}{2}n+\frac{3}{2})}.$$

We obtain

$$\bar{B}_0=\sigma^{J/2}m^{-2}\{a[4J+O(\sigma)]+Jb[4-3J+O(\sigma)]\} \tag{27}$$

and

$$\bar{B}_1=\sigma^{J/2}m^{-2}(2J)^{-1/2}(J+1)^{1/2}\times\{Ja[4+O(\sigma)]+Jb[4-3J+O(\sigma)]\}, \tag{28}$$

where the symbols $O(\sigma)$ do not vanish when $J\to 0$.

Amplitude $\bar{B}_0$, just like f_0, exhibits the nonuniform factorizability of zeros, since $\bar{B}_0\sigma^{-J/2}$ is free of threshold zeros and singularities for $J>0$, while $\bar{B}_0\sigma^{-J/2-1}$ is free of zeros and singularities for $J=0$. We now may observe, from the structures of the right-hand sides of Eqs. (27) and (28), that the combination $\bar{B}_0-c\bar{B}_1$, where $c=c(s;J,1)=(2J)^{1/2}(J+1)^{-1/2}[1+O(\sigma)]$, behaves like $\sigma^{J/2+1}$ for $J\geq 0$, and therefore satisfies the requirement of uniform factorizability of threshold zeros.

On the other hand, let us now take the other choice of "natural" interpolation mentioned in Sec. III and suppose that the *left-hand* sides of Eqs. (21)–(23) are substituted into the appropriate places in Eqs. (25) and (26). It is sufficient to consider values of J in the infinitesimal neighborhood of some integral values, say, $J=0$. We can evaluate the resulting integrals $\int_0^1\bar{z}^{J'+n}\times P_{J'}(-\bar{z})d\bar{z}$ in the neighborhoods of $J'=0$, -1 by using the expansion

$$P_{J-1/2\pm 1/2}(-\bar{z})=1\pm J\ln(\tfrac{1}{2}-\tfrac{1}{2}\bar{z})+O(J^2),$$

which follows from the representation[15]

$$P_{J'}(1-2x)=\sum_{m=0}^{\infty}\frac{\Gamma(m-J')\Gamma(J'+m+1)x^m}{\Gamma(-J')\Gamma(J'+1)(m!)^2}.$$

In this way, we obtain

$$\bar{B}_0{}'=\sigma^{J/2}m^{-2}\{4a[-J+O(J^2)+O(\sigma)]+Jb[-4+O(J)+O(\sigma)]\} \tag{29}$$

and

$$\bar{B}_1{}'=\sigma^{J/2}m^{-2}(2J)^{-1/2}(J+1)^{1/2}\times\{Ja[-28+O(J)+O(\sigma)]+Jb[20+O(J)+O(\sigma)]\}, \tag{30}$$

where the primes on $\bar{B}_\lambda{}'$ serve to distinguish these interpolated amplitudes from the first choice Eqs. (27) and (28).

Again $\bar{B}_0{}'$ exhibits the nonuniform factorizability of zeros for $J>0$ and $J=0$. This time, however, the non-

[13] H. F. Jones and M. D. Scadron, Phys. Rev. **171**, 1809 (1968).

[14] See Ref. 12, Vol. I, p. 171

[15] See Ref. 14, p. 125.

uniform factorizability persists in the combination $\bar{B}_0'-c\bar{B}_1'$, with $c=c(s;J,1)$. Moreover, even if we try to find some new c (an oscillating interpolation of the old c, for example, since we know that the old c works satisfactorily for integral J), we find that the factor c_a needed for the parts of (29) and (30) that are proportional to a is different from the factor c_b needed for the parts proportional to b. Hence, it is impossible to find a new c such that $\bar{B}_0'-c\bar{B}_1'$ is uniformly free of kinematic zeros; we conclude that the model collapses under this choice of interpolation.

V. FROISSART-GRIBOV RULE

In the usual Regge analysis of spinless-particle scattering, the correct interpolation in angular momentum of the partial-wave projection $A_l(s)=\frac{1}{2}\int_{-1}^{1}dx \times P_l(x)A(s,x)$ is found[16] by (i) making the replacement $P_l(x)=i\pi^{-1}[Q_l(x+i\epsilon)-Q_l(x-i\epsilon)]$, valid in the interval $-1\leq x\leq 1$, (ii) rewriting the projection as a contour integral $-\frac{1}{2}i\pi^{-1}\oint dx\, Q_l(x)A(s,x)$, where the contour encloses the cut in Q_l over the interval $-1\leq x \leq 1$, and (iii) deforming the contour of integration to enclose the singularities of $A(s,x)$ instead of the finite cut in $Q_l(x)$. In this way, we obtain the representation $A_l(s)=\pi^{-1}\int_c dy\, Q_l(y)D(y,s)$ which now yields the desired interpolation in l through the standard continuation of the Legendre function Q_l. The function $D(y,s)$ is the weight function when the full amplitude $A(s,x)$ is dispersed in the cos$x$ of the c.m. scattering angle; i.e., $D(y,s)$ is such that $A(s,x)=\pi^{-1}\int_c dy\, D(y,s)(y-x)^{-1}$.

In this section, we carry out the continuation in spin of the external π^* by following steps closely paralleling the above Froissart-Gribov prescription. Thus we express the partial-wave amplitude for integral J as a contour integral over the Q_J function in the cosψ plane (rather than the x plane), where ψ is the scattering angle in the π^* rest frame, expand the contour to infinity and thereby pick up contributions from outlying singularities, and finally assume the standard continuation in Q_J in the resulting representation. Working only in the limit $M\to 1$, we find that the resulting interpolated partial-wave amplitude is identical to the representation (24) in which B_λ' is interpolated according to the right-hand sides of Eqs. (21)–(23).

We demonstrate this last statement explicitly for that part of the angular momentum $=1$ projection of B_0' that is proportional to a and which we call x_0. Thus, with J initially assumed to be an even non-negative integer, we have

$$x_0=\frac{3}{2}\int_{-1}^{1}(t-m^2)^{-1}T^J(2s+t-M^2-3) \times P_J(\cos\psi)x\,dx. \quad (31)$$

[16] M. Froissart (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1962); **41**, 1962 (1962) [English transls.: Soviet Phys.—JETP **14**, 478 (1962); **14**, 1395 (1962)].

When we change the integration variable to $z=\cos\psi$, and take $M=1$, we find

$$z=-[s(1-x)]^{1/2}[s+4-(s-4)x]^{-1/2},$$

and so the physical range of z is $-1\leq z\leq 0$. The contour integral therefore would not appear to enclose the entire cut of $Q_J(z)$, and correspondingly it is not clear at the moment what is meant by an expansion of the contour away from the cut.

To understand the source of this difficulty, we take $M>1$ in which case the physical range of z is normal, i.e., $-1\leq z\leq 1$, and we consider the behavior of the integrand of x_0 in the z complex plane as $M\to 1$. We find that all factors making up the integrand have nonfactorizable square-root branch points at $z=z_\pm\equiv \pm i\alpha(k^2-1)^{1/2}$, where $\alpha\equiv(s+M^2-1)S^{-1}$ and $k\equiv sM(s+M^2-1)^{-1}$, and hence the integrand is a two-sheeted function of z with these same branch points. If we define the *physical* sheet by extending the cuts to $\pm i\infty$ along the imaginary axis with a gap on the finite segment between z_+ and z_-, then the only other singularities in the integrand are the simple pole of dynamical origin on the real axis (if $m>M+1$) at $z=z_m\equiv -\alpha(h+k)(h^2-1)^{-1/2}$, where $h\equiv(m^2-M^2-1) \times(2M)^{-1}$ and a pole of order $J+3$ at $-\alpha$. On the *unphysical* sheet these poles are found at $-z_m$ and $+\alpha$. If the branch points are joined instead by a straight cut along the finite segment of the imaginary axis between z_+ and z_-, then the integrand defined on this sheet is a real-analytic *even* function of z, for any integral value of J.

For s above threshold and M in the infinitesimal neighborhood of $M=1$, it is clear that $\alpha>0$ and that $k>1$, $k=1$, $k<1$ for $M>1$, $M=1$, $M<1$, respectively. Therefore as $M\to 1$, the branch points $z_\pm\to 0$ and the function $T(z)$, in particular, is pinched into two distinct analytic functions on either sheet. On the *physical* sheet, $T(z)\to -4\alpha z(\alpha^2-z^2)^{-1}$ for Re$z<0$ while $T(z)\to 0$ for Re$z>0$. Hence, in light of this structure of T, we could write the integral over the normal range of z from -1 to $+1$ even when $M=1$ because the vanishing of T in the right half-plane cancels that part of the integral from 0 to $+1$.

For $M>1$, Eq. (31) may be rewritten in the form

$$x_0=-\frac{3}{2}i\pi^{-1}\oint(t-m^2)^{-1} \times T^J(2s+t-M^2-3)Q_J(z)x\frac{dx}{dz}dz, \quad (32)$$

where the contour encircles the cut of Q_J. Since we find that the integrand of Eq. (32) behaves like $|z|^{-2J-4}$ as $|z|\to\infty$, the contour may be opened up in the left

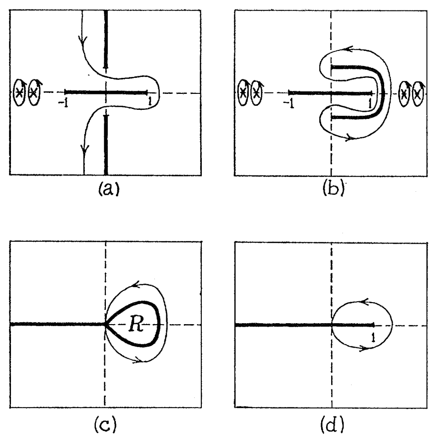

FIG. 1. Sequence of contour deformations in the z plane for the interpolation of the partial-wave amplitude between integral values of J.

half-plane, for $J \geq -1$, as shown in Fig. 1(a), leaving loops around the poles indicated by crosses.

The integral becomes simpler in structure if the square-root cuts and the infinite branches of the path of integration are joined in the right half-plane as shown in Fig. 1(b). This procedure exposes the poles on the *unphysical* sheet. Then, because of the *evenness* of the function so exposed, the residues of the poles in the right and left half-planes cancel and there only remains the loop around the finite square-root cut. At this point the interpolation in J for $M>1$ follows from the standard continuation in Q_J which happens to develop a cut along the entire negative real axis for nonintegral J.

If we now take the limit $M \to 1$ of this picture, the square-root points pinch the Q_J cut at the origin, as in Fig. 1(c), and the contribution from the remnant domain R of the right half of the *physical* sheet vanishes. The integral is now

$$x_0 = i\beta\pi^{-1} \int g(z) Q_J(z) dz,$$

where

$$\beta \equiv 48s^2[16s(s-4)]^{J/2},$$

$$g(z) = z^{J+1}[s-2-(s-4)z^2][s-(s+4)z^2] \times [s-(s-4)z^2]^{-J-3}[m^2 s-(m^2-4)(s-4)z^2]^{-1},$$

and the path of integration is shown in Fig. 1(c). A factorizable Jth-root cut at the origin develops when $M=1$ because of the factor z^J in $g(z)$. Analyticity in M is unaffected if we take this cut along the negative real axis.

The function $g(z)Q_J(z)$ in the domain of integration may now be analytically continued through the cut surrounding R so as to leave only the finite part of the Q_J cut extending from the origin to $z=1$ as shown in Fig. 1(d). The path of integration finally encircles just this portion of the cut, and we may write

$$\begin{aligned} x_0 &= i\beta\pi^{-1} \oint g(z) Q_J(z) dz \\ &= -\beta \int_0^1 g(z) P_J(z) dz \\ &= \beta \int_{-1}^0 g(-z) P_J(-z) dz \\ &= \lim_{M\to 1} \frac{3}{2} \int_{-1}^1 (t-m^2)^{-1} T^J (2s+t-M^2-3) \\ &\qquad \times P_J(-\cos\psi) x\, dx. \end{aligned} \quad (33)$$

The last line of Eq. (33) is thus the same as the right-hand side of Eq. (31) but with $P_J(\cos\psi)$ replaced by $P_J(-\cos\psi)$. A similar treatment of the partial-wave projections of the other parts of B_λ' confirms that the right-hand sides of Eqs. (21)–(23) emerge in the integrand of Eq. (24).

VI. CONCLUSION

It follows almost directly from the structure of Eqs. (17) and (18) or, equivalently, from the structure of Eqs. (19) and (20), with the substitutions indicated in Eqs. (21)–(23), that we may summarize our knowledge of the ρ-exchange amplitudes for $\pi^*\pi \to \pi\pi$ in the form

$$B_\lambda' = \phi_\lambda F_\lambda + \gamma_\lambda G_\lambda,$$

with $\lambda = 0, 1$. Here F_λ and G_λ are explicitly known functions of M, J, t, and s such that the dependence on M, J, and t is inseparately mixed with s. The quantities ϕ_λ and γ_λ, functions only of M, J, and t, are partially unknown in the sense that we only have explicit knowledge of the ratios

$$\phi_0/\phi_1 = \tfrac{1}{2} J^{1/2}(J+1)^{1/2} T M^{-1},$$
$$\gamma_0/\gamma_1 = -\tfrac{1}{2} J^{1/2}(J+1)^{-1/2} T M^{-1}.$$

If ϕ_λ and γ_λ are assumed to carry the correct t-channel threshold singularities and zeros and the ρ pole, and if the quantities a and b of Eqs. (19) and (20) are interpreted in the customary way as t-independent coupling constants, the dependence of ϕ_λ and γ_λ on M, J, and t separates according to

$$\phi_0 = [aT^2 + bJ(t+M^2-1)] T^{J-2}(t-m^2)^{-1}, \quad (34)$$

$$\gamma_0 = -2bJT^{J-1}(t-m^2)^{-1}, \quad (35)$$

$$\phi_1 = -2[aT^2 + bJ(t+M^2-1)] \times J^{-1/2}(J+1)^{-1/2} M T^{J-3}(t-m^2)^{-1}, \quad (36)$$

$$\gamma_1 = 4bJ^{1/2}(J+1)^{1/2} M T^{J-2}(t-m^2)^{-1}, \quad (37)$$

where now the only unknown functional dependence is that of a and b on the variable J. We have tacitly assumed that the right-hand sides of Eqs. (34)–(37)

already exhibit the proper interpolated dependence on J.

The functions

$$F_0=(2s+t-M^2-3)P_J(-\cos\psi), \quad (38)$$

$$G_0=S[P_{J-1}(-\cos\psi)+\cos\psi P_J(-\cos\psi)], \quad (39)$$

$$F_1=(2s+t-M^2-3)S^{-1}[stu-(M^2-1)^2]^{1/2} \times P_J'(-\cos\psi), \quad (40)$$

$$G_1=[stu-(M^2-1)^2]^{1/2}[P_J(-\cos\psi) +J^{-1}(J+1)^{-1}\cos\psi\, P_J'(-\cos\psi)], \quad (41)$$

are attributed largely to the transformation properties of the spin states involved in the reaction and depend in an explicit way on the scattering angle as seen in the rest frame of the π^* and the unphysical rest frame of the ρ. (When either of these particles has spacelike four-momentum, the corresponding scattering angle is purely imaginary.) The investigation of the interpolation problem has focused on the J dependence of these functions. The transformation properties of the spacelike π^* spin states uniquely lead to amplitudes which, after analytic continuation to timelike π^*, contain the functions listed in Eqs. (38)–(41), where the Legendre functions have the standard continuation in J.

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Superconvergent Dispersion Relations and Pion-Pion Scattering Sum Rule

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An alternative derivation based on the technique of asymptotic symmetry is given for the pion-pion scattering sum rule. We propose a new asymptotic symmetry hypothesis in terms of the proper amplitudes, which has the merit of narrowing the discrepancy between experiment and the Adler sum rule.

1. INTRODUCTION

INTENSIVE use of Gell-Mann's chiral current algebra[1] and the hypothesis of partially conserved axial-vector current (PCAC) has been made in the past to derive a number of sum rules in strong-interaction physics. The earliest of these was the sum rule derived by Adler[2] and Weisberger[3] (AW), relating the axial-vector neutron β-decay coupling constant to an integral over the pion-nucleon scattering cross sections. The AW sum rule is, in fact, now regarded as a direct confirmation of the validity of the chiral current algebra, in view of its excellent agreement with experiment. It is thus of great interest to look at alternative methods of deriving such sum rules which do not make explicit use of current commutation relations.

It has been recently demonstrated by Fayyazuddin and Hussain[4] that it is possible to derive the AW sum rule as a consequence of the hypothesis of asymptotic $SU(2)\times SU(2)$ symmetry and PCAC without the explicit need of current commutation relations. This derivation leads to the AW sum rule for zero-mass pions (i.e., $q^2=0$). Besides giving new relations between form factors for q^2 away from zero, this derivation brings to light some new features of current-algebra sum rules: (1) One avoids the tricky ambiguities of the soft-pion limit, i.e., the vanishing of the four-momentum of the pion, which is usually assumed in deriving current-algebra sum rules; (2) the assumption of a symmetry

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