Secondary Trajectories and the Saturation of Finite-Energy Sum Rules*

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A study is made of the constraints placed upon the couplings and resonance spectrum required to saturate finite-energy sum rules consistently in the region around $t=0$. It is argued that the probable existence of crossing trajectories and cuts places a fundamental limit on the usefulness of finite-energy sum rules. The demand that the spin of the resonances increase monotonically with mass implies that an infinite number of asymptotically parallel daughter trajectories in the resonance region are required for consistency. This result is independent of any statement about the energy dependence of the leading trajectory.

I. INTRODUCTION

THERE have been several attempts in the past year to use finite-energy sum rules (FESR) as the basis of dynamical schemes for bootstrapping Regge trajectories.[1-3] The basic idea is that the same Regge trajectory that controls the asymptotic behavior of the scattering amplitude also generates a sequence of resonances that dominates the amplitude at lower energies. Thus, the amplitude up to the cutoff energy is represented by a sum of resonances, usually in the narrow-width approximation; above the cutoff energy the amplitude is dominated by a few Regge poles. The FESR equations generated in this way are continuous functions of the momentum transfer. In addition, it is possible to generate additional equations by taking moments of the sum rules. The hope is that this large set of equations will determine both a set of Regge trajectories and residue functions and a unique spectrum of resonances and partial widths. In practice, a crucial part of the whole program is the choice of the t dependence of the leading Regge trajectory and its residue and the mass and spins of the narrow-width resonances used to saturate the sum rules. Mandelstam,[2] at one time, suggested that it might be possible to satisfy the equations approximately with a single leading resonance trajectory in each channel. Freund,[4] in fact, proposed just such a solution, but he used the FESR just at $t=0$. Dolen, Horn, and Schmid,[5] while expressing the same hope as Mandelstam, also indicate that in order to satisfy the t dependence of the equations, towers of resonance at each mass may be necessary. However, their argument is somewhat obscure and is based on a simple model containing a single leading Regge pole with a known residue function. Mandula and Slansky[6] arrived at a similar conclusion, but Goebel[7] showed that their arguments were incorrect. Ademollo, Rubinstein, Veneziano, and Virasoro[3] have shown that for $\pi\pi \to \pi\omega_s$ where the spin of ω_s is 1 or 3, the particles on the ρ trajectory may be accompanied by sets of particles on one or two parallel daughter trajectories. They used a linear ρ trajectory, a low cutoff energy, and required equality of two functions of t over a fairly large region. Veneziano[8] has written down a closed solution to the FESR equations for $\pi\pi \to \pi\omega$ which incorporates the linear-trajectory hypothesis in a fundamental way and contains an infinite number of parallel daughter trajectories. However, the Veneziano representation is not a unique solution to FESR.[9] Moreover, not only is it very difficult to reconcile with unitarity, but it also encounters certain fundamental difficulties when extended to pion-nucleon scattering.[10] In addition, the hypothesis of linear trajectories does not have a firm experimental foundation, particularly for bosons.[11] Kugler[12] has presented arguments for a square-root trajectory; he interprets as resonances the loops in an Argand diagram obtained from a partial-wave analysis of a t-channel Regge pole. Linear trajectories are also difficult to reconcile with traditional statements about the analytic structure of scattering amplitudes.[13]

The theoretical justification of linear trajectories seems to be stronger; however, the arguments are based on the properties of nonrelativistic trajectories.[2] In particular, the trajectory function $\alpha(s)$ is assumed to obey a dispersion relation with only a right-hand cut. This statement, coupled with the requirement that the narrow-width approximation be valid on the right-hand cut, strongly suggests that the trajectories are linear. On the other hand, in all models in which the Regge trajectories are known to have the assumed analyticity, $\mathrm{Re}\,\alpha(s)$ approaches some negative real number as s goes to infinity. In addition, there are both potential-theory

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[1] K. Igi, Phys. Rev. Letters **9**, 76 (1962); R. Dolen, D. Horn, and C. Schmid, *ibid.* **19**, 402 (1967).
[2] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).
[3] M. Ademollo, H. Rubinstein, G. Veneziano, and M. Virasoro, Phys. Rev. **176**, 1904 (1968).
[4] P. Freund, Phys. Rev. Letters **20**, 235 (1968).
[5] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).
[6] J. E. Mandula and R. C. Slansky, Phys. Rev. Letters **20**, 1402 (1968).
[7] C. Goebel, Phys. Rev. Letters **21**, 383 (1968).
[8] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[9] M. A. Virasoro, Phys. Rev. **177**, 2309 (1969). D. D. Coon, University of Washington (unpublished report) has generalized the Veneziano model to include nonlinear trajectories.
[10] K. Igi, Physics Letters **28B**, 330 (1968).
[11] The only candidate for a meson recurrence is the g meson which is supposed to lie on the ρ trajectory. In addition, K. V. L. Sarma and D. D. Reeder [Nuovo Cimento **51A**, 169 (1967)] obtain a ρ trajectory with curvature from a fit to scattering data.
[12] M. Kugler, Phys. Rev. Letters **21**, 570 (1968).
[13] N. N. Khuri, Phys. Rev. Letters **18**, 1094 (1967); C. E. Jones and V. Teplitz, *ibid.* **19**, 135 (1967).

and field-theory models of intersecting Regge trajectories with left-hand cuts.[14,15] In fact, there is a theorem in potential scattering, which appears to be true in relativistic models, that states that no Regge trajectory can be real as it passes through a negative integer value of the angular momentum.[15]

If $\alpha(s)$ has a left-hand cut, $\mathrm{Im}\alpha(s)$ need not be small on the left, even though the narrow-width approximation is still valid on the right. The theoretical argument for linear trajectories then breaks down. We argue that not only is it possible that the trajectories intersect, but it is probable. In processes involving unequal-mass kinematics or high spins, daughter trajectories[16] are needed to cancel out unwanted singularities in the region near $s=0$. Since the coupling of these kinematic daughters to external particles differs greatly from the coupling of leading trajectories, it is unlikely in a unitary theory that they are parallel to the leading trajectory. Model calculations of these trajectories confirm this statement.[17] The parallel daughters in the Veneziano model[8] are apparently degenerate combinations of kinematic daughters or Lorentz poles, and dynamical daughters or satellite poles.[18] Since these two classes of daughters are distinguished by different couplings to external states, unitarity will break the degeneracy and probably generate intersecting trajectories. In any case, if there are both kinematic daughters and satellite trajectories, there are, almost certainly, crossing trajectories. In fact, a Regge-pole description of the region of negative s (or negative t in the crossed channel) is probably very complicated. In addition, cuts are almost certainly present in this same region,[19] although they are customarily ignored in applications of FESR. Thus, any procedure for solving FESR which is either sensitive to regions of large negative t, to contributions from secondary Regge poles, or to cuts near $t=0$ is questionable.

In this paper, we abandon the hypothesis of linear trajectories; we also abandon any statement about the t dependence of Regge residue functions, secondary trajectories, or the singularity structure for large negative t. Assuming only resonance saturation of FESR with monotonically rising trajectories, we address ourselves to the problem of determining the simplest spectrum of resonances that can be used to saturate the FESR equations. Our conclusion is that an infinite number of asymptotically parallel daughter trajectories is necessary to satisfy the constraints arising from the dependence on the momentum transfer t near $t=0$. This result means that the presence of an infinite number of daughters in the Veneziano representation[8] is not due to either the linear-trajectory hypothesis or the complete lack of unitarity, but rather it is a fundamental property of FESR. Whether the trajectories are linear in the resonance region is still an unsolved question. In addition, we conclude that, in the presence of an infinite set of daughter trajectories, finite-energy sum rules by themselves are insufficient to determine the parameters of both the resonances and the Regge poles. Additional information, such as provided by hypotheses like duality, linear trajectories, absence of cuts and intersecting trajectories, smoothness of appropriately parametrized residue functions, negligible background integral, absence of ghosts, or maximal simplicity of the resonance spectrum, is needed to obtain a unique solution. The theoretical basis of many of these assumptions is weaker than that of the FESR, so it is useful to explore the dynamical content of the FESR without them.

For simplicity, we consider the reaction $\pi\pi\rightarrow\pi\omega$, but the method can be generalized to arbitrary spins and masses, and our conclusions apply to baryon as well as boson trajectories. We start with the standard form for FESR derived from one-sided dispersion relations.[2,20] The resonance spectrum below the cutoff energy is saturated by a set of narrow-width resonances of $(\text{mass})^2=s(J)$ and spins J; the resonances lie on monotonically increasing trajectories. Although we use the zero-width limit, our method can be extended to finite-width resonances, at least in principle. Above the cutoff energy, the amplitude is represented by a complete sum over all t-channel Regge poles and cuts, plus the background integral. For $\pi\pi\rightarrow\pi\omega$, the ρ trajectory is the leading one in both channels. At $t=0$, the leading kinematic daughter is at $\alpha_\rho(0)-2$ and, according to Veneziano,[8] the leading satellite trajectory occurs at $\alpha_\rho(0)-1$. Our lack of knowledge of these secondary trajectories and residues, as well as the uncertainty in the parametrization of the residue of the leading pole, limits the information we are able to extract from the FESR. By successively increasing the cutoff energy to include a new leading resonance, or tower of resonances, we are able to derive an iterative equation relating the residues of successive resonances. The equation clearly displays the limitations mentioned above. When the spin J of the leading resonance is large and there are a fixed number of daughter trajectories, the equations can be differentiated with respect to t a finite number of times and evaluated at $t=0$. When the pole residues are eliminated from the resulting equations, a set of equations relating the mass and spin of the leading resonance

[14] N. F. Bali, S.-Y. Chu, R. W. Haymaker, and C.-I. Tan, Phys. Rev. **161**, 1450 (1967); A. R. Swift, *ibid.* **176**, 1848 (1968); R. E. Cutkosky and B. B. Deo, Phys. Rev. Letters **19**, 1256 (1967).

[15] P. Kaus, Nuovo Cimento **29**, 598 (1963); A. E. A. Warburton, *ibid.* **32**, 122 (1964); **37**, 267 (1965).

[16] D. Z. Freedman and J.-M. Wang, Phys. Rev. Letters **17**, 596 (1966); Phys. Rev. **153**, 1596 (1967); G. Domokos, *ibid.* **159**, 1387 (1967).

[17] A. R. Swift, J. Math. Phys. **8**, 2420 (1967); Phys. Rev. **171**, 1466 (1968); M. Fontannaz, Nuovo Cimento **59A**, 215 (1969); R. Blankenbecler and R. L. Sugar, Phys. Rev. **166**, 1515 (1968); V. Chung and D. R. Snider, *ibid.* **162**, 1039 (1967).

[18] S. Mandelstam and K. Bardakci (unpublished).

[19] See, for example, F. Henyey, G. L. Kane, J. Pumplin, and M. Ross, Phys. Rev. Letters **21**, 946 (1968); Phys. Rev. **182**, 1577 (1969).

[20] G. Chew, *The Analytic S-Matrix* (W. A. Benjamin, Inc., New York, 1966).

is obtained. There is no solution to these equations when their number exceeds the number of daughter residues. If the number of daughter resonances increases with J, the procedure breaks down since the number of derivative equations must also depend on J. If the iterative equation is differentiated too many times, the approximation of neglecting high derivatives of $\alpha(t)$, the t dependence of the residue functions, and the presence of secondary poles or cuts, breaks down. In other words, if the number of daughters is proportional to J, we find that FESR are inadequate to fix completely the daughter residues without the additional assumptions mentioned above.

After this work was completed, a paper by Mohapatra[21] appeared which treats the same problem by similar techniques. He showed that the resonance-trajectory function must be of the form $c(s\ \ln s)^{1/2}$ if there are a finite number of daughter trajectories present. However, we show that if the t dependence of the equations is fully exploited, the constant c must be zero. He did not consider the case of an infinite set of secondary trajectories.

In the next section, we derive our basic equation relating the residues of resonances in successive intervals and discuss the approximations involved. In principle, there is no difficulty in extending the method to finite-width nonoverlapping resonances. We concentrate on the $\pi\pi\rightarrow\pi\omega$ problem and the ρ-meson trajectory. We demonstrate explicitly that the number of daughter resonances must be proportional to the spin of the leading pole. In Sec. III, we outline the extension of our approach to the problem of arbitrary external masses and spins. The conclusion is that all Regge trajectories (not just the ρ) must be accompanied by an infinite set of daughters.

II. SECONDARY TRAJECTORIES IN $\pi\pi\rightarrow\pi\omega$

A general t-channel scattering amplitude with helicities $\lambda_1+\lambda_2\rightarrow\lambda_3+\lambda_4$ obeys the following finite-energy sum rule[2]:

$$\int_0^N \frac{\mathrm{Im}A_{\lambda\mu}{}^t(s,t)ds}{(1-z_t)^{\frac{1}{2}|\lambda-\mu|}(1+z_t)^{\frac{1}{2}|\lambda+\mu|}} =\sum_i \beta_{\lambda\mu}{}^i(t)[z_t(N)]^{\alpha_i+1-M}, \quad (1)$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$, and $M=\max(|\lambda|,|\mu|)$.[22] $\mathrm{Im}A_{\lambda\mu}{}^t$ is the discontinuity in s of $A_{\lambda\mu}{}^t$; $z_t(N)$ is $\cos\theta$ in the t channel evaluated at $s=N$. The sum on the right-hand side of (1) is over *all* contributing Regge poles with trajectory functions $\alpha_i(t)$ and residues $\beta^i(t)$. The background integral and Regge-cut terms should also be included in the sum if the two sides are set identically equal. Additional equations can be generated[1,3] by multiplying $A_{\lambda\mu}{}^t(s,t)$ by s^m; however, since we are ultimately going to restrict ourselves to a narrow-width approximation, we limit ourselves to $m=0$. The higher moments distort the mass spectrum and enhance the contributions of resonance widths.

The discontinuity in s of the t-channel amplitude is related to the corresponding discontinuity of the s-channel amplitude by crossing. For arbitrary spins, the crossing matrix has the form $M_{\lambda\mu;\lambda'\mu'}(s,t)$, as may be found in the paper by Trueman and Wick.[23] If we consider $\pi\pi\rightarrow\pi\omega$, there is but one independent-helicity amplitude $A_{01}{}^t$,[24] and $M_{\lambda\mu;\lambda'\mu'}$ becomes the unit matrix. When the direct-channel amplitude is represented by a sum of narrow-width resonances, the imaginary part is a sum of δ functions multiplied by the appropriate coupling constants and angular factors. In this limit (1), $\pi\pi\rightarrow\pi\omega$ becomes

$$\sum_j \Gamma_j \frac{\sin\theta_s}{\sin\theta_t} P_j{}'(z_s)=\sum_i \beta^i(t)[z_t(N)]^{\alpha_i(t)}, \quad (2)$$

with

$$2p_tq_tz_t(N)=N+\tfrac{1}{2}(t-\Sigma), \quad \Sigma=3m_\pi{}^2+m_\omega{}^2,$$

and p_t and q_t are the t-channel momenta of the $\pi\pi$ and $\pi\omega$ systems, respectively. The sum over j includes all resonances with spin j and (mass)^2 less than N. The angular part of the residue function $d_{01}{}^j(z)$ has been written as $c(j)\sin\theta_sP_j{}'(z_s)$ and $c(j)$ absorbed into the definition of Γ_j. Wherever s occurs on the left of (2), it is evaluated at $s=s(j)$, the (mass)^2 of the jth resonance. The resonance sum on the left-hand side of (2) could be smoothed out by replacing the sum by an integral.[21] However, the subtraction procedure we adopt below in going from (3) to (4) would effectively replace the integral by its integrand and lead to the same final equation. In addition, we assume there is no non-resonant background that should be included in (2).[25]

If Λ_J is the sum over all those resonances in (2) with (mass)^2 nearly equal to $s(J)$, the (mass)^2 of the leading resonance of spin J, Eq. (2) becomes

$$\sum_1^J \Lambda_{J'}=\beta(t)[z(N(J))]^\alpha+\beta'[z(N(J))]^{\alpha'}+\cdots. \quad (3)$$

The cutoff parameter $N(J)$ lies between $s(J)$ and $s(J+2)$, since in $\pi\pi\rightarrow\pi\omega$ the intermediate states can have only odd angular momentum. The requirement that $N(J)$ lie between $s(J)$ and $s(J+2)$ implies that, up to an additive constant, $N(J)$ and $s(J)$ are the same smooth functions of J for large J. On the right of (3), only the leading Regge pole and the first secondary pole

[21] R. N. Mohapatra, Phys. Rev. Letters 22, 735 (1969).

[22] For convenience we have omitted the individual helicity indices in (1), but it should be remembered that a number of different λ_1, λ_2 can combine to give the same set of net helicity λ.

[23] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).

[24] Conservation of parity tells us that $A_{0-1}=+A_{01}$ and $A_{00}=0$.

[25] H. Harari, Phys. Rev. Letters 20, 1395 (1968).

are written explicitly. For $\pi\pi\to\pi\omega$, the first satellite trajectory appears at $\alpha'=\alpha-1$. In $\pi\omega\to\pi\omega$, there is a kinematic daughter at $\alpha'=\alpha-1$; through unitarity, it may contribute to (3), although with a nonsingular residue. When there is more than one leading pole, we know only that $\alpha>\alpha'$. We subtract from (3) the corresponding equation with $N=N(J-2)$ and obtain

$$\Lambda_J=\beta(t)\{[z(J)]^\alpha-[z(J-2)]^\alpha\}+\beta'(t)\{[z(J)]^{\alpha'}-[z(J-2)]^{\alpha'}\}. \quad (4)$$

When (4) is divided into the similar equation for Λ_{J+2}, we find that

$$\Lambda_{J+2}=F\Lambda_J, \quad (5)$$

where

$$F=\frac{[z(J+2)]^\alpha-[z(J)]^\alpha+(\beta'/\beta)\{[z(J+2)]^{\alpha'}-[z(J)]^{\alpha'}\}+\cdots}{[z(J)]^\alpha-[z(J-2)]^\alpha+(\beta'/\beta)\{[z(J)]^{\alpha'}-[z(J-2)]^{\alpha'}\}+\cdots}.$$

It should be emphasized that, insofar as the narrow-width approximation with monotonically rising trajectories and no nonresonant background is exact, Eq. (5) is exact when all secondary Regge terms are taken into account.

In (5), F depends on J through the dependence of $z(J)$ on $N(J)$. If $N(J-2)$ and $N(J+2)$ are related to $N(J)$ by power-series expansions in J, F is given by

$$F-1=2\left[\frac{N''}{N'}+(\alpha-1)\frac{N'}{N}\right]-\frac{N'(t-\Sigma)}{N^2}(\alpha-1)+2\left[\frac{N''}{N'}+(\alpha-1)\frac{N'}{N}\right]^2+2\frac{\beta'}{\beta}(\alpha'-\alpha)\frac{N'}{N}N^{\alpha'-\alpha}+\cdots, \quad (6)$$

where N' and N'' are the first and second derivatives of N with respect to J. The first term on the right of (6) is of order J^{-1}. The third term is of order J^{-2}, while the second and fourth terms are of order J^{-1-n} and $J^{-1-n(\alpha-\alpha')}$ if $s(J)=aJ^n$. If the most important secondary trajectory has $\alpha'=\alpha-1$, then F is known only to order J^{-1}. Without further information about secondary parameters, there is clearly a limitation to the accuracy with which we know F. Note that F to order J^{-1} depends on t only through $\alpha(t)$. Fits to experimental cross sections tell us the value and slope of $\alpha_\rho(t)$ at zero, but tell us very little about the higher derivatives of $\alpha_\rho(t)$.

To solve (5) for the dependence of Λ_J on J, we note that $\sin\theta_s/\sin\theta_t$ can be written as a product of two terms:

$$\sin\theta_s/\sin\theta_t=(p_t^2q_t^2t)^{1/2}(p_s^2q_s^2s)^{-1/2}.$$

The t-dependent factor is independent of J and can be cancelled from both sides of (5). The $s(J)$-dependent factor is absorbed into the definition of Γ_j. Then (5) becomes

$$\Sigma_{J+2}=F\Sigma_J, \quad (7)$$

where

$$\sum_{j=1}^{J}=\sum^{J}\tilde{\Gamma}_jP_j'(z_s)$$

and

$$\tilde{\Gamma}_j=\Gamma_j(p_s^2q_s^2s)^{-1/2}.$$

Equation (7) is a difference equation for Σ_J. If only the J^{-1} terms in F are retained, it can be converted into a differential equation and solved to give

$$\Sigma_J(t)=g(t)s'(J)s(J)^{\alpha(t)-1}, \quad (8)$$

where $g(t)$ is an arbitrary function of t. A convenient choice for $N(J)$ is

$$N(J)=\tfrac{1}{2}[s(J)+s(J+2)]=s(J)(1+s'/s+s''/s+\cdots) \quad (9)$$

and the difference between $N(J)$ and $s(J)$ does not appear in the J^{-1} part of F or in (8).

Equation (8) coupled with a statement about the dependence of $s(J)$ on J gives us the asymptotic J dependence of the sum of the residues of resonances with masses in the region of $s(J)$. If the number of resonances included in Σ_J is small and fixed, the continuous dependence on t can be used to explore the J dependence of the individual resonance functions. For example, if there is only the leading resonance, its residue for large J is obtained by evaluating (8) at $t=0$:

$$\tilde{\Gamma}_J=2g(0)s's^{\alpha(0)-1}/J^2. \quad (10)$$

The expression for the mth derivative of a Legendre polynomial with $z=1$ has been used in (10).

$$\left.\frac{d^m}{dz^m}P_J(z)\right|_{z=1}=\frac{(J+m)!}{(J-m)!}\frac{1}{2^m m!}\approx\frac{J^{2m}}{2^m m!}. \quad (11)$$

For the particular case of $\pi\pi\to\pi\omega$ kinematics, z_s at $t=0$ is given by

$$z_s(0)=\frac{(s-\Sigma)}{4p_sq_s}=1+\frac{2m_\pi^2(m_\omega^2-m_\pi^2)^2}{s^3}.$$

The pion mass is sufficiently small to set $z(0)=1$. Since (8) is a continuous function of t near $t=0$, it can be differentiated m times and evaluated at $t=0$. If there is just a single resonance, its residue is eliminated by dividing by (10). The resulting equation for the trajectory is

$$\frac{1}{s}\frac{J^2}{2!}=\frac{\dot{g}(0)}{g(0)}+\dot{\alpha}\ln s, \quad (12)$$

where $\dot{z}=dx/dt\approx 2/s$ and the 2! comes from the $m!$ in the denominator of (11). For the ρ trajectory, $\dot{\alpha}(0)$ is known. Equation (12) says that the Regge trajectory satisfies a nonlinear equation of the form

$$J^2=as+2!\dot{\alpha}s\ln s. \qquad (13)$$

The second-derivative equation also relates the various trajectory parameters:

$$\frac{1}{3!}\frac{J^4}{s^2}=\frac{\ddot{g}(0)}{g(0)}+\left(2\frac{\dot{g}(0)}{g(0)}\dot{\alpha}(0)+\ddot{\alpha}(0)\right)\times\ln s+[\dot{\alpha}(0)\ln s]^2. \qquad (14)$$

Although $\ddot{g}(0)$, $\dot{g}(0)$, and $\ddot{\alpha}$ are all unknown, the dominant term on the right side of (14) for large s is $(\dot{\alpha}\ln s)^2$. Since $3!\neq(2!)^2$, (14) is incompatible with (13). The mth derivative equation is of the form

$$\frac{1}{(m+1)!}\left(\frac{J^2}{s}\right)^m\approx(\dot{\alpha}\ln s)^m. \qquad (15)$$

Since the mass-versus-spin relation for the leading resonance does not depend on how many times (8) is differentiated, more than one resonance is required to saturate the FESR leading to (8).

At this point, it is amusing to note that if (4) is used for $J=1$ and $J=3$ with z^α replaced by N^α and no secondary terms, the unknown residue function $\beta(t)$ can be replaced by the residue of the ρ pole ($J=1$) multiplied by $[N(1)]^{-\alpha(t)}$. When the first derivative of the $J=3$ equation is used to eliminate the pole residue, an equation for the mass of the $J=3$ pole is obtained:

$$s(3)-\Sigma=\frac{5(1-R^{-\alpha(0)})}{\dot{\alpha}(0)\ln R},$$

where $R=N(3)/N(1)$. If we choose $N(3)=s(3)+[s(3)-s(1)]/2$, $N(1)=\frac{1}{2}[s(3)+s(1)]$, $\alpha(0)=\frac{1}{2}$, $\dot{\alpha}(0)=1$, and $s(1)=m_\rho^2=0.6$ GeV^2, we can solve the equation for $s(3)$. The solution is $s(3)=(1.640)^2$ GeV^2. The agreement with the suggested[26] recurrence of the ρ at 1.650 GeV should not be taken seriously. However, the fact that the ρ, its recurrence, and $\alpha(0)$ from scattering data provide three points on a straight line is not evidence for the linearity of the ρ trajectory function. In particular, this solution for $s(3)$ assumes no secondary trajectories and uses just the first derivative equation. These same conditions lead to (12) in the asymptotic limit.

The next question is whether some finite set of resonances lying on daughter trajectories can satisfy (8). These secondary trajectories must be asymptotically parallel to the leading trajectory or there will be a series of values of J for which Σ_J will contain only the leading resonance. However, we have shown that Σ_J must contain more than one resonance. If there are n such parallel daughter trajectories, then n derivative equations can be used to determine the $n+1$ residue functions, and any further derivative equations determine the trajectory. Again, it is found that the trajectory depends on the number of derivatives. The explicit proof of this fact is given in the Appendix. We have, therefore, proved our fundamental result—no finite, fixed number of daughter trajectories satisfies the FESR equation. This proof does not involve any assumptions about the shape of the leading trajectory, the residue functions, or the nature of secondary Regge poles.

There is a limit to the number of derivative equations. In particular, if the number is of order J (i.e., $m=aJ+b$), then the approximation for F breaks down. Thus, if the number of daughters is proportional to J, we cannot determine all the residues and do not generate any constraints on the trajectory function. Even if we specify linear trajectories, we cannot determine the residues by this approach. One way to see that the information contained in the t dependence of (7) is limited is to note that if definite partial waves are projected out, there is an integration over z_s from -1 to 1. This integration in turn samples a region of t between $-s(J)$ and 0. If $|t|$ is as large as $s(J)$, the neglected terms in F [see (6)] are the same order of magnitude as those we have retained, if not larger. In addition, for large negative t the singularity structure in the complex angular-momentum plane is very complex and the identification of the dominant term is difficult.[19]

While it can be argued that the derivative procedure samples the function in detail near $t=0$ and should be valid even if there are singularities for negative t, the approximation for F still fails. For large J, the function F has the form

$$F=1+(2\delta/J)+2(\delta^2+\epsilon)/J^2+\cdots,$$

where $\delta=n-1+n(\alpha-1)$, if $s=aJ^n$. The unknown terms are represented by $\epsilon(t)$. The new solution for Σ_J is

$$\Sigma_J=g(t)e^{-(\epsilon+\delta)/J}J^\delta. \qquad (16)$$

When we take m derivatives of (16), we find that

$$\frac{1}{\Sigma_J}\frac{d^m}{dt^m}\Sigma_J=(\dot{\delta}\ln J)^m-m(\dot{\delta}\ln J)^{m-1}\frac{(\dot{\epsilon}+\dot{\delta})}{J}+\cdots$$
$$+\tfrac{1}{2}m(m+1)(\dot{\delta}\ln J)^{m-2}\left(\ddot{\delta}\ln J-\frac{\ddot{\epsilon}+\ddot{\delta}}{J}+\cdots\right)$$
$$+m\frac{\dot{g}}{g}\Big[\cdots+\tfrac{1}{2}(m-1)(m+2)(\dot{\delta}\ln J)^{m-2}$$
$$\times\left(\ddot{\delta}\ln J-\frac{\ddot{\epsilon}+\ddot{\delta}}{J}\right)+\cdots\Big]+\cdots. \qquad (17)$$

If $m=aJ+b$, $(\dot{\delta}\ln J)^m$ is no longer the dominant term.

[26] N. Barash-Schmidt *et al.*, Rev. Mod. Phys. 41, 109 (1968).

The dominant term will involve higher derivatives of $\alpha(t)$ and those parts of F of order J^{-2}. In particular, if $\alpha(t)$ is a linear function of t while $\epsilon(t)$ is quadratic, the $\epsilon(t)$ contribution dominates the known terms in F. An expansion of F in inverse powers of J becomes meaningless in this limit.

The FESR does provide a great deal of information about the J dependence of resonance residues. However, without additional assumptions about unknown functions or some other physical information, the FESR's do not determine the parameters of the resonances and trajectories used to satisfy them. The constraints generated by (8) must, of course, be satisfied by any solution.

The approximation for F breaks down for low values of J. If secondary trajectories are known to occur at $\alpha'=\alpha-2$, then the fact that the J^{-2} terms in F are also known helps to extend the region of validity. However, there is no point in using $P_\alpha(-z)$ instead of $(-z)^\alpha$ in deriving (1), since the extra terms are the same order as those that are unknown. To show how the secondary terms become more important, we mention that with a scale mass of 1 GeV and a cutoff N of 1 GeV, the fourth term in the expression for F in (6) is the same order as the first (unless β'/β is unusually small). The success of Ademollo *et al.*[3] might be interpreted as a statement that secondary effects are indeed small.

III. EXTENSION TO ARBITRARY EXTERNAL SPINS

In this section we show how the results of the previous section can be extended to processes involving external particles of arbitrary masses and spins. When $\mathrm{Im}A_{\lambda\mu}{}^t$ in (1) is crossed to the s channel and saturated by resonances, Eq. (2) is replaced by

$$\sum_j \frac{M_{\lambda\mu;\lambda'\mu'}(s(j),t)\Gamma_{\lambda'\mu'}{}^j d_{\lambda'\mu'}{}^j(\theta_s)}{(1-z_t)^{\frac{1}{2}|\lambda-\mu|}(1+z_t)^{\frac{1}{2}|\lambda+\mu|}} =\sum_i \beta_{\lambda\mu}{}^i(t)[z_t(N)]^{\alpha_i+1-M}. \quad (18)$$

The crossing matrix $M_{\lambda\mu\,\lambda'\mu'}(s(j),t)$ is given by a sum over products of rotation functions $d_{\lambda_i\lambda_i'}{}^{S_i}(\chi_i)$, S_i being the spin of the external particles. The crossing angles χ_i and the explicit form of $M_{\lambda\mu;\lambda'\mu'}$ are found in the paper by Trueman and Wick.[23] The sum over all resonances in (18) with masses nearly equal to $s(J)$ is represented by $\Lambda_J{}^{\lambda\mu}$. We then proceed as before and subtract successive towers of resonances to obtain

$$\Lambda_{J+1}{}^{\lambda\mu}=F_{\lambda\mu}\Lambda_J{}^{\lambda\mu}, \quad (19)$$

where

$$F_{\lambda\mu}=1+(s''/s'+(\alpha-M)(s'/s))+\cdots \quad (20)$$

and $M=\max(|\lambda|,|\mu|)$. If the masses of the resonances in $\Lambda_J{}^{\lambda\mu}$ are assumed equal, the crossing matrix and other factors depending on t and s can be factored out of the sum to define $\Sigma_J{}^{\lambda\mu}$ by

$$\Lambda_J{}^{\lambda\mu}=\frac{(1+z_s)^{\frac{1}{2}|\lambda'+\mu'|}(1-z_s)^{\frac{1}{2}|\lambda'-\mu'|}}{(1+z_t)^{\frac{1}{2}|\lambda+\mu|}(1-z_t)^{\frac{1}{2}|\lambda-\mu|}}M_{\lambda\mu;\lambda'\mu'}\Sigma_J{}^{\lambda'\mu'} =\tilde{M}_{\lambda\mu;\lambda'\mu'}\Sigma_J{}^{\lambda'\mu'}$$

and

$$\Sigma_J{}^{\lambda\mu}=\Sigma_j\Gamma_j{}^{\lambda\mu}P_{j-M}{}^{|\lambda-\mu|,|\lambda+\mu|}(z_j).$$

$P_n{}^{\alpha,\beta}(z)$ is a Jacobi polynomial. When (20) is multiplied by $\tilde{M}_{\lambda\mu;\lambda'\mu'}{}^{-1}(s(J+1),t)$, we obtain

$$\Sigma_{J+1}{}^{\lambda\mu}=\delta_{\lambda\lambda'}\delta_{\mu\mu'}\left\{1+\left[\frac{s''}{s'}+\frac{s'}{s}\times\left(\alpha+\frac{|\lambda-\mu|}{2}\right)\right]\right\}\Sigma_J{}^{\lambda',\mu'}, \quad (21)$$

where

$$\tilde{M}_{\lambda\mu;\sigma\tau}{}^{-1}(s(J+1))F_{\sigma\tau}\tilde{M}_{\sigma\tau;\lambda'\mu'}(s(J))\equiv Y_{\lambda\mu;\lambda'\mu'} =\delta_{\lambda\lambda'}\delta_{\mu\mu'}\left\{1+\left[\frac{s''}{s'}+\frac{s'}{s}(\alpha+\tfrac{1}{2}|\lambda-\mu|)\right]\right\}+\cdots. \quad (22)$$

To order J^{-1}, Eq. (21) is diagonal in helicities. The omitted terms in (22) include off-diagonal terms that can be explicitly calculated. For example, for $\pi\pi\to\pi\omega_s$, where ω_s has arbitrary spin s, the off-diagonal terms have the form

$$Y_{\lambda0;\lambda+n0}=\left(\frac{s'}{s^{5/2}}\right)^n a_n{}^+(\lambda,t) \quad (23a)$$

and

$$Y_{\lambda0;\lambda-n0}=\left(\frac{s'}{s^{3/2}}\right)^n a_n{}^-(\lambda,t), \quad (23b)$$

where $a_n{}^\pm(\lambda,t)$ are known functions of λ and t. The solution of (21) is

$$\Sigma_J{}^{\lambda\mu}=g^{\lambda\mu}(t)s'(J)s(J)^{\alpha+\frac{1}{2}|\lambda-\mu|}. \quad (24)$$

When this is substituted into (23), we find that it is consistent to neglect the off-diagonal elements of (22). Equation (24) contains (8) as a special case if the additional factor of $(sp^2q^2)^{-1/2}$ in Σ_J for $\pi\pi\to\pi\omega$ is taken into account.

The complete analysis of the previous section can be carried out with (24) replacing (8). The same conclusions follow, subject to the same conditions. In general, there will be trajectories with $\alpha'=\alpha-x$, where x is less than 1. However, (24) is still the asymptotic solution, and, except for the coefficient $g^{\lambda\mu}(t)$, it depends only on the helicity and leading cross-channel trajectory. The only delicate point is that $z_s(t=0)$ is not sufficiently near 1,

$$z_s(0)=1+2[(m_1{}^2-m_3{}^2)(m_2{}^2-m_4{}^2)/s^2]+\cdots,$$

to set the Jacobi polynomial equal to its value at $z=1$. For linear trajectories, the J^{-2} dependence of $z-1$ is compensated by the J^{2m} dependence of the mth derivative of the Jacobi polynomials. This difficulty is avoided either by working at $t=0$ and taking into account this extra J dependence or by choosing t so that $z=1$ for $s=s(J)$. Then, $z(0)$ differs from unity for $s=s(J+2)$ by terms of order s^{-3} and these are handled explicitly in the derivative equations. The conclusions are unchanged. No fixed number of daughter trajectories satisfies the FESR equations; on the other hand, the equations by themselves are insufficient to determine the resonance parameters if the number of resonances increases with J. Since (21) is valid for both baryons and mesons, the conclusions should also apply to baryon trajectories.

APPENDIX

In this appendix it is proved explicitly that no fixed number of daughter trajectories can be used to satisfy (8). If Σ_J is given by

$$\Sigma_J=\sum_n \Gamma_{J-2n}\frac{P_{J-2n}'(z)}{P_{J-2n}'(1)}, \tag{A1}$$

where the sum over n includes all the daughter resonances, then the mth derivative of Σ_J evaluated at $z=1$ is[27]

$$\Sigma_J^{(m)}=\left(\frac{J^2}{s}\right)^m \frac{1}{(m+1)!} \times\sum_n \sum_{r=0}^{2m} \frac{(2n)^r(-1)^r(2m)!}{r!(2m-r)!}\frac{\Gamma_{J-2n}}{J^r}. \tag{A2}$$

We have redefined Γ_{J-2n} to contain an explicit factor of $[P_{J-2n}'(1)]^{-1}$, and set $dz/dt=2/s$. When (A2) is divided by the zero-derivative equation, it can be written in the form

$$\sum_{r=0}^{2m} \frac{(2m)!}{(2m-r)!}\frac{B_r}{B_0}=(m+1)!\left(\frac{s}{J^2}\right)^m\frac{\Sigma_J^{(m)}}{\Sigma_J}, \tag{A3}$$

with

$$B_r=\frac{(-1)2^r}{J^r r!}\sum_n n^r\Gamma_{J-2n}.$$

[27] We neglect terms proportional to n^rJ^{-r-m}, $m\geq 1$, since they are higher-order corrections to B_r.

If there are N daughter resonances, there are N independent functions of J labeled by B_r. The first $N+1$ derivative equations are sufficient to eliminate the B_r and obtain an equation for the trajectory. The problem is one of solving a set of linear equations. If the right-hand side of (A3) is labeled by $Y_m{}^0$, then the coefficients of B_1 to B_n are zero in $Y_m{}^n$, $m>n$, which is generated by

$$Y_m{}^n=\frac{m!}{n!(m-n)!}Y_n{}^{n-1}-Y_m{}^{n-1}. \tag{A4}$$

Thus, $Y_{N+1}{}^N$ is independent of all the pole residues and is equal[28] to 1. Equation (A4) enables us to write $Y_{N+1}{}^N$ in terms of $Y_m{}^0$. The result is

$$1=-\sum_{r=0}^{N}\frac{(-1)^{N+1-r}(N+1)!}{(N+1-r)!}\frac{1}{r!}Y_{N+1-r}{}^0. \tag{A5}$$

The leading term on the right-hand side of (A3) is, from Eq. (15),

$$Y_m{}^0\approx(m+1)![(s/J^2)\dot{\alpha}\ln s]^m=(m+1)!X^m. \tag{A6}$$

Thus, X satisfies the equation

$$\sum_{r=0}^{N+1}\frac{(N+2-r)(-X)^{N+1-r}}{r!}=0. \tag{A7}$$

Although (A7) was derived under the assumption that there were N resonances with mass $s(J)$, it holds to the same approximation as (A2), even if there are only $N'<N$. In other words, X must satisfy (A7) for all $N>N'$ ($N\ll J$), but it cannot depend on N itself. If (A7) is written as $F(N,X)=0$, then

$$G(N,X)=F(N,X)+XF(N-1,X)=\sum_{0}^{N+1}\frac{(-X)^{N+1-r}}{r!}=0$$

and

$$G(N+1,X)+XG(N,X)=1/(N+2)!=0. \tag{A8}$$

Therefore, there is no solution to (A7) independent of N, and the number of secondary resonances must exceed the number of derivative equations.

[28] Consistent with earlier approximations, we have set B_r for $r>N$ equal to zero.