some sense be as small as possible, for otherwise $\tilde{A}^+$ would be more than just a unitarity correction to A^+. Furthermore, as we want to consider unitarization only in connection with the problem of crossed-channel Regge asymptotic behavior, which requires only that $[\tilde{A}^+]^{-1}$ diverges for large $|\mathrm{Im}l|$, we will confine our discussions only to large $|l|$. Now if we take only elastic unitarity condition throughout, then (18) is replaced by $c_l(s')=-1$, in which case we may choose $k=1$. Since the integral is then independent of l for physical l, according to our previous point of view $E_l(s)$ must also be independent of l so that the correction terms on the right-hand side of (17) remain to be a minimum for all l. If we assume the condition of Carlson's theorem to be valid so that we can uniquely extend $c_l(s')$ and $E_l(s')$ in the l plane, then these correction terms must also be independent of l for $|\mathrm{Im}l|\rightarrow\infty$, so that $[\tilde{A}^+]^{-1}$ cannot diverge there. We conclude, therefore, that elastic unitarity alone is not sufficient to restore the crossed-channel Regge asymptotic behavior. If we are willing to take inelastic unitarity into account, then of course we may always find functions c_l and E_l so that they diverge as $|\mathrm{Im}l|\rightarrow\infty$. However, if we still want the conditions of Carlson's theorem to be valid, then all these functions are necessarily large for large real positive l too. The question is then whether we can still hope to regard the correction terms as small. The answer is yes, because we can see from (8) and (9) that $[A^+(s,l)]^{-1}$ grows exponentially in l as $\mathrm{Re}l\rightarrow\infty$, so that as long as the correction terms do not grow so fast, they may be considered as small.

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New Representation of the Scattering Amplitude*

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A new representation of the scattering amplitude that has good analytic properties, Regge asympotic behavior, and an arbitrary double-spectral boundary is proposed. The representation automatically yields partial waves with the correct threshold behavior for both their real and imaginary parts. The presence of the correct double-spectral boundary should be very important in decay problems, where an unstable external particle considerably modifies the analytic properties of the amplitude. This representation in its simplest version is applied to pion-pion scattering. Unitarity is enforced near threshold. By using the (degenerate) ρ- and f-meson trajectories, the ρ width, and the Adler self-consistency condition, the three isotopic scattering lengths, the f width, and the Regge-scale mass are predicted.

THE recent suggestion of Veneziano[1] for an explicit form for the scattering amplitude that satisfies crossing and Regge behavior has led to several interesting applications to meson-meson scattering[2] and three-meson decay[3] processes. However, when attempts have been made to satisfy unitarity and to include complex trajectory functions, severe difficulties with this representation have been pointed out and have led to several suggestions for modified building blocks.[4] All these proposed formulas have the disadvantage of being quite *ad hoc* and not being based on dynamical ideas. For example, while they satisfy cut-plane analyticity, they do not possess the correct double-spectral boundary. While this defect may be small in the asymptotic limit, it could be important in the low-energy regime and even crucial in three-particle decay applications where the presence of an unstable particle modifies the analytic properties.

In this paper we would like to present a new type of formula for the scattering amplitude which is motivated by dynamical considerations, but which has certain desired properties that do not seem to follow easily from simple calculations—for example, asymptotic behavior. The undetermined functions in the representation are then to be determined or restricted as far as possible by an appeal to detailed dynamics, especially unitarity.

The motivation for our representation is primarily the Mandelstam representation. For simplicity, let us consider the scattering of unit-mass spinless particles. The box diagram for scattering through a pair of mass

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] J. Yellin, Phys. Rev. **182**, 1482 (1969); M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969); D. Y. Wong, Phys. Rev. **181**, 1900 (1969); R. G. Roberts and F. Wagner, CERN Report No. Th.1014, 1969 (unpublished).
[3] C. Lovelace, Phys. Letters **28B**, 264 (1968); R. Jengo and E. Remiddi, Nuovo Cimento Letters **1**, 637 (1969).
[4] R. Roskies, Phys. Rev. Letters **21**, 1851 (1968); **22**, 265(E) (1969); M. A. Virasoro, Phys. Rev. **177**, 2309 (1969); R. Jengo, Phys. Letters **28B**, 261 (1968); S. Mandelstam, Phys. Rev. **183**, 1374 (1969); N. N. Khuri, *ibid.* **176**, 2026 (1968); E. Predazzi, *ibid.* (to be published); D. Coon, *ibid.* (to be published).

$2M_1$ in the t channel and a pair $2M_2$ in the s channel can be written in the form

$$B_2(s,t)=\frac{1}{\pi}\int_{4M_2^2}^{\infty} ds'(s'-s)^{-1}\,\mathrm{Im}B_2(s',t). \quad (1)$$

The absorptive part is easily calculated and turns out to be a function of s and t, but the important point is that t always occurs multiplied by a simple function of s. In fact, one finds

$$\mathrm{Im}B_2(s,t)=A_2(s,\tau)=g^4(1-4M_2^2/s)^{1/2} \times\frac{1}{\pi}\int_4^{\infty} d\tau'\frac{(s+s_1)^{-1}}{(\tau'-\tau)[\tau'(\tau'-4)]^{1/2}}, \quad (2)$$

where

$$\tau=tC(s)=t(s-4M_2^2)/M_1^2(s+s_1),$$
$$M_1^2s_1=(M_1^2+M_2^2-1)^2-4M_2^2M_1^2. \quad (3)$$

The Mandelstam boundary for this diagram is given by the curve $\tau=4$. The function $C(s)$ corresponding to elastic scattering in the t channel is then

$$C_t=(s-4M^2)/(s+M^4-4M^2),$$

and the elastic s-channel function is

$$C_s=(s-4)/M^2(s+M^2-4),$$

where the C's vanish below their respective thresholds. The full Mandelstam boundary is then given by the curve $\tau=tC(s)=4$, where

$$C(s)=\max(C_t,C_s). \quad (4)$$

The important behavior of $C(s)$ is that in general it vanishes as $q^2(s)$ near threshold and approaches 1 at infinity. If one studies the manner in which this model develops Regge behavior when the elastic ladders are summed in the t direction, then to lowest order[5]

$$\alpha_2(t)=-1+\frac{g^2}{\pi}\int_4^{\infty} d\tau'(\tau'-t)^{-1}[\tau'(\tau'-4)]^{-1/2} \quad (5)$$

and

$$\alpha_2(-\infty)=-1.$$

One of the effects of the elastic box in the s channel is to add to α an inelastic cut starting at $4M^2$.

Motivated by a desire to preserve the analytic structure exhibited in Eq. (2), and to generalize it to all orders so that B will have a Regge behavior for large s, we will assume that the total absorptive part has the form

$$A(s,\tau)=\frac{q}{\sqrt{s}}\beta\left(1-\frac{\alpha(\tau)}{\alpha(-\infty)}\right)\left(\frac{s+s_1}{s_0}\right)^{\alpha(\tau)}. \quad (6)$$

This ansatz reduces to Eq. (2) if α is replaced by its lowest-order terms and if the constants β, s_0, and s_1 are chosen appropriately.

[5] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962); T. W. B. Kibble, *ibid.* **117**, 1159 (1960).

One might expect, on the basis of Eq. (3), that the parameter s_1 is of order M^2-4 if $\alpha(t)$ arises primarily from the binding of states of mass $4M^2$. However, if α is dominated by the 2π state, then s_1 is of order $M^2(M^2-4)$, where M is the mass of the particle giving rise to the force.

One of the main advantages, and disadvantages, of this type of function is the fact that the integral over s is complicated by the s dependence in τ. It is precisely this dependence, however, that guarantees the correct spectral boundary. It has not been explicitly shown that this simple ansatz, which attempts to interpolate between the low-order graphs and the asymptotic behavior, accurately reproduces the large-s behavior of the three-particle exchange graphs, but it reduces upon simplification to the multiperipheral type of expansion.[6] Our ansatz, however, does not collapse the spectral boundaries due to multiparticle exchange to straight lines, but approximates them by the exact box-diagram boundary.

It is interesting to note that the correct threshold conditions in s for all partial waves for both the real and imaginary parts of B are satisfied by the general form of our type of ansatz because of the presence of the boundary curve $C(s)$ in the t dependence. At this point it should be stressed that the choice of Eq. (6) is one of the simplest possible consistent with our constraints. If one were to embark on an ambitious calculational program, it should probably be modified by introducing a more general ansatz with additional parameters, which would then be determined by unitarity. Also, note that the asymptotic behavior of B for large s comes from the large-s behavior of A. For the leading term one may set $\tau=t$ and a standard Regge behavior is then achieved. The nonleading terms can be quite complicated and they depend in detail on the ansatz for A.[7] Finally, the poles in these building blocks occur only in their second argument.

Now let us try to apply this form to the physical case of pion-pion scattering. We will follow the notation of Chew and Mandelstam,[8] except for normalization. The amplitude $A(s,t,u)$ can be written in terms of ρ_s and ρ, where ρ_s is symmetric and ρ has no definite symmetry. We define the three functions F, G, and H:

$$G(s,t)+F(t,s)=\frac{1}{\pi^2}\int\int\frac{ds'dt'\rho(s',t')}{(s'-s)(t'-t)}, \quad (7a)$$

[6] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962), and references herein; G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968); T. W. B. Kibble, *ibid.* **131**, 2282 (1963).

[7] For the choice of Eq. (6), the daughters turn out to be multiple poles. These can be canceled easily and turned into simple poles if desired. However, the solution of the Bethe-Salpeter equation does yield an increasing multiplicity of secondary trajectories, which in our model are made degenerate. For a discussion of this multiplicity, see A. R. Swift, J. Math. Phys. **8**, 2420 (1967); E. Halliday and P. Landshoff, Nuovo Cimento **56A**, 983 (1968); L. Caneschi, Nuovo Cimento Letters **1**, 70 (1969); R. Haymaker and R. Blankenbecler, Phys. Rev. (to be published).

[8] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

$$H(s,t)+H(t,s)=\frac{1}{\pi^2}\int\int\frac{ds'dt'\rho_s(s',t')}{(s'-s)(t'-t)};\qquad(7b)$$

then

$$A(s,t,u)=G(s,t)+F(t,s)+G(s,u)+F(u,s)+H(u,t)+H(t,u).\qquad(8)$$

The amplitudes B and C follow from A by permutations of s, t, and u. It is the functions F, G, and H that we assume have the structure of the basic building blocks discussed previously. In order to prevent $I=2$ poles in any channel, we set $H(x,y)=-G(x,y)$. The $I=1$ poles arise in $[G(t,s)-G(u,s)]$, and the $I=0$ poles arise in $[F(t,s)+F(u,s)]$; thus we find a natural separation of the even- and odd-signature Regge-pole contributions. For the purposes of this example, we will assume that only the ρ and f trajectories contribute to this process and, in addition, that these trajectories are degenerate. Since the f is an even-signature trajectory with a negative zero intercept, F has been subtracted once to eliminate a ghost at $\alpha_f=0$. In addition, we will assume that $F=\gamma G$, which implies that the ρ chooses nonsense at $\alpha_\rho=0$. The parameter γ will be a measure of the breaking of strict exchange degeneracy between the f and ρ. Finally, since trajectories seem to be linear over a large range, we will set $\alpha(\tau)/\alpha(-\infty)\ll 1$ so that the square brackets in Eq. (6) can be approximated by unity. Our ansatz can thus be written

$$G(s,t)=g+\frac{s-4}{\pi}\int_4^\infty\frac{ds'}{(s'-s)(s'-4)}\frac{q'}{\sqrt{s'}}\beta\left(\frac{s'+s_1}{s_0}\right)^{\alpha(\tau')}\equiv g+I(s,t),\qquad(9)$$

where g is a constant and $\tau=tC(s)$:

$$C(s)=\tfrac{1}{4}(1-4/s)\quad\text{for}\quad 4<s<20$$
$$=1-16/s\quad\text{for}\quad 20<s<\infty.\qquad(10)$$

In order to determine the parameters, we will enforce unitarity near threshold for the lowest partial wave in each isotopic amplitude. The Adler self-consistency condition will also be required. This latter condition requires a zero-mass extrapolation of one of the external legs. Since this is not a very long extrapolation, the dependence of the spectral function will be neglected. This is, however, in contrast to the three-pion decay problem, where such an extrapolation has to be done with more care.

Since we are enforcing unitarity only at threshold, and neglecting other inelastic channels which are important in the determination of the meson resonance parameters, we will assume that the masses of the ρ and f, and the width of the ρ, are given.

The Adler condition requires that $F(1,1)=0$ and, hence, $g=-I(1,1)$. In general, there is no similar condition on G, but it also vanishes because of our proportionality assumption. To enforce unitarity near threshold, the isospin amplitudes in the forward direction will be written

$$A_s^0\to a_0+ia_0^2q,$$
$$A_s^1\to 3q^2(a_1+ia_1^2q^3),\qquad(11)$$
$$A_s^2\to a_2+ia_2^2q.$$

We now expand the isospin amplitudes near threshold in terms of our building blocks, keeping in mind that $\mathrm{Im}\alpha\to\mathrm{const}\times(s-4)^{\alpha(4)+1/2}$, so that it does not contribute to the lowest-order unitarity cut. By comparison with Eq. (11), we obtain[9]

$$a_0=10g\gamma+6\gamma I(0,4)+2(\gamma-2)I(0,0),\qquad(12a)$$

$$a_0^2=(\gamma+2)\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)},\qquad(12b)$$

$$a_2=4g\gamma+2(1+\gamma)I(0,0),\qquad(12c)$$

$$a_2^2=(\gamma-1)\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)},\qquad(12d)$$

$$a_1=\frac{2\beta}{3\pi}\int_4^\infty\frac{ds'}{s'^2}\left(\frac{s'-4}{s'}\right)^{1/2}\left[2\left(\frac{s'+s_1}{s_0}\right)^{\alpha[4C(s')]}+(1-\gamma)\left(\frac{s'+s_1}{s_0}\right)^{\alpha(0)}+O(4\alpha'(1-\gamma))\right],\qquad(12e)$$

$$a_1^2=\frac{\gamma+1}{6}\alpha'\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)}\ln\left(\frac{s_1+4}{s_0}\right),\qquad(12f)$$

where $\alpha(s)=\alpha(0)+s\alpha'$ for $s\sim 0$. Unfortunately, except for $I(0,0)$, we cannot perform the required integrals in Eq. (12) in closed form because of the s' dependence in the argument of α. However, since most of the integral comes from large values of s', $C(s')$ can be set equal to 1 in computing the integrals. It is crucial, however, that one does not make this approximation when computing the imaginary parts of the amplitudes. We can then evaluate $I(0,t)$:

$$I(0,t)=-\frac{\beta}{2\pi}\left(\frac{4}{s_0}\right)^{\alpha(t)}\frac{\Gamma(1-\alpha(t))\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\alpha(t))}\times F(-\alpha(t),\,1-\alpha(t);\,\tfrac{3}{2}-\alpha(t);\,-\tfrac{1}{4}s_1).\qquad(13a)$$

As we will see later, $\frac{1}{4}s_1\gg 1$, so that we must analytically continue the hypergeometric function. For $\alpha(t)$ near $\frac{1}{2}$, we can, to a good approximation, keep only the first-order term of the expansion in powers of $4/s_1$:

$$I(0,t)\simeq-\frac{\beta}{\pi}\left(\frac{s_1}{s_0}\right)^{\alpha(t)}.\qquad(13b)$$

[9] If we set $I(0,0)=I(0,4)$, $\gamma=1$, and $g=-\frac{3}{4}I(0,0)$, we obtain from Eqs. (12a) and (12c), $a_2=-(2/7)a_0=I(0,0)$, which is the result obtained by Weinberg [Phys. Rev. Letters **17**, 616 (1966)]. See also D. I. Fivel and P. K. Mitter, Phys. Rev. **183**, 1240 (1969); M. G. Olsson and L. Turner, *ibid.* **181**, 2141 (1969); J. Fulco and D. Wong, Phys. Rev. Letters **19**, 1399 (1967); E. P. Tryon, Columbia University Report (unpublished).

Finally, to simplify the calculation, we will assume that $I(0,4)\simeq I(0,0)$. Equation (12) then becomes

$$a_0=(\tfrac{3}{4}\gamma+4)\frac{\beta}{\pi}\left(\frac{s_1}{s_0}\right)^{\alpha(0)}, \tag{14a}$$

$$a_0{}^2=(\gamma+2)\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)}, \tag{14b}$$

$$a_2=(\tfrac{3}{2}\gamma-2)\frac{\beta}{\pi}\left(\frac{s_1}{s_0}\right)^{\alpha(0)}, \tag{14c}$$

$$a_2{}^2=(\gamma-1)\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)}, \tag{14d}$$

$$a_1=(3-\gamma)\frac{\beta}{9\pi}\left(\frac{s_1}{s_0}\right)^{\alpha(0)}, \tag{14e}$$

$$a_1{}^2=\frac{\gamma+1}{6}\alpha'\beta\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)}\ln\left(\frac{s_1+4}{s_0}\right). \tag{14f}$$

From Eqs. (14a)–(14d), we have a cubic for γ that has the solution $\gamma=1.028$. Then, using Eqs. (14a) and (14b), we obtain

$$\beta=1.313\left(\frac{s_1+4}{s_0}\right)^{\alpha(0)}\left(\frac{s_0}{s_1}\right)^{2\alpha(0)}; \tag{15}$$

thus

$$a_0=2.0\left(\frac{s_1+4}{s_1}\right)^{\alpha(0)}, \tag{16}$$

$$a_2/a_0=-1/10.4\,, \tag{17}$$

$$a_1/a_0=1/21.8\,. \tag{18}$$

Equation (14f) then gives

$$\alpha'\ln\left(\frac{s_1+4}{s_0}\right)=0.96\ \text{GeV}^{-2}. \tag{19}$$

Now, in order to proceed, we assume a straight-line trajectory that passes through the ρ and f:

$$\alpha_0=0.415\,,\quad \alpha'=1.0\ \text{GeV}^{-2}. \tag{20}$$

Then from Eq. (19) we obtain

$$(s_1+4)/s_0=2.62\,. \tag{21}$$

Near $\text{Re}\alpha_s=1$, we set $\alpha_s=1+\alpha'(s-m_\rho{}^2)+i\,\text{Im}\alpha$ and, neglecting nonresonant terms, we find

$$A^1{}_{\text{res}}=\frac{\beta q^2}{3\pi\alpha'}\frac{4}{s_0}\frac{3z}{m_\rho{}^2-s-i(\text{Im}\alpha)/\alpha'},\quad z=\cos\theta. \tag{22}$$

If we require that this saturate unitarity, we obtain

$$\Gamma_\rho=\frac{2\beta q^3}{3m_\rho{}^2\pi\alpha'}\frac{4}{s_0}\bigg|_{\sqrt{s}=m_\rho}. \tag{23}$$

Similar considerations near $\text{Re}\alpha_s=2$ will yield

$$\Gamma_f=\frac{\beta q^5}{5m_f{}^2\pi\alpha'}\left(\frac{4}{s_0}\right)^2\bigg|_{\sqrt{s}=m_f}. \tag{24}$$

The experimental width of the ρ is 120 MeV and, using Eq. (23), we find $s_0=26.7$ and, from Eq. (21), $s_1=74$. The width of the f is then determined to be 160 MeV, which compares very well with the experimental value of 145 MeV.

The scattering lengths are given by Eqs. (16)–(18):

$$a_0=2.04\,,\quad a_1=0.0935\,,\quad a_2=-0.196\,,$$

where, for comparison, we note that Weinberg's value of a_0 is 0.20.[9]

Finally, we look at the high-energy behavior. In the limit $s\to\infty$, $t\lesssim 0$, the leading term in the scattering amplitudes is proportional to $G(s,t)$ as long as $\text{Im}\alpha$ is sufficiently large (see, for example, Predazzi[4] and Roskies.[4]) If we scale out the leading s dependence in Eq. (9), we find

$$G(s,t)\to f(t)\left(\frac{s}{s_0}\right)^{\alpha(t)}=f(t)\left(\frac{s}{0.51}\right)^{\alpha(t)},$$

where s is measured in GeV^2. Thus we see that in this model, the Regge-scale factor is determined by the width of the low-energy resonances. The value of $s_0=0.51$ GeV^2 compares very well with the values found in phenomenological fits.[10]

One can now compute $M^2=4+s_1$, and we find $M=1.24$ GeV, which would suggest that the trajectory arises primarily from the binding of heavy particles, such as $\rho\rho$, ρf, and nucleon-antinucleon.

[10] In the bulk of the phenomenological Regge fits, this parameter is arbitrarily set equal to 1 GeV^2. However, in a few recent fits it was taken to be an adjustable parameter. R. C. Arnold and M. L. Blackmon [Phys. Rev. **176**, 2082 (1968)] found $s_0=0.3$ GeV^2; J. Botke and J. R. Fulco [*ibid.* **182**, 1837 (1969)] obtained limits of $0.1<s_0<0.5$ GeV^2; and D. D. Reeder and K. V. L. Sarma [*ibid.* **172**, 1566 (1968)] obtained several values depending on the particular process ranging from 0.3 to 4.0 GeV^2.