Regularization and Ward-Identity Anomalies*

R. W. Brown, Chia-Chang Shih, and Bing-Lin Young
Brookhaven National Laboratory, Upton, New York 11973
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In the context of the free-quark model, we discuss the validity of the naïve Ward identities (WI's) for arbitrary regularized n-point functions of scalar, pseudoscalar, vector, and axial-vector currents. In a simple version of the regularization procedure described by Pauli and Villars, we find that the naïve vector WI's are all automatically satisfied, and that there is a compact necessary condition for the existence of an axial-vector anomaly. Subsequently, this version leads to a large number of anomalous axial-vector WI's (corresponding to the cases $n=2$, 3, 4, and 5). It is shown that this number cannot be reduced, for example, to Bardeen's "minimal" solution without additional counterterms beyond those possible in the general regularization framework—in spite of the framework's well-known ambiguities. We discuss other minimal sets, as well as a symmetry-breaking model in which no further anomalies are found. The explicit forms of the WI anomalies for the general minimal solution is given along with the necessary counterterms.

I. INTRODUCTION

Recently it has been ascertained[1,2] that in the σ model and in spinor electrodynamics the axial-vector divergence equation implied by a naïve manipulation of the field equations is not satisfied in perturbation theory due to the presence of triangle diagrams. Accordingly, Adler[2] has given a modification of the naïve divergence equation; the aforementioned manipulation involves ill-defined field-operator products and hence must be done carefully, a fact pointed out years ago by Schwinger.[3] Careful field-theoretic calculations have since been given which are consistent with Adler's result.[4] Further work with respect to arbitrary orders of perturbation theory[5] and with respect to reduction-formula modifications[6] has also been done.

Specifically it has been seen that the Ward identity (WI) relating the axial-vector–vector–vector (AVV) and the pseudoscalar-vector-vector (PVV) three-point functions (3-pf's) in electrodynamics must be modified since the AVV triangle graph is superficially linearly divergent. No modification in the way of acceptable counterterms could remove the "anomaly."[2] This thus carries over to the hard-pion calculations[7] involving the $\langle AVV\rangle$ and $\langle PVV\rangle$ vertices. Wilson[8] has studied another set of WI's relating the 3-pf's $\langle AAV\rangle$, $\langle PAV\rangle$, and $\langle PPV\rangle$ and found that these identities *can* be satisfied by redefining the ingredient vertices with appropriate counterterms. Examining the general set of 3-pf's for scalar, pseudoscalar, vector, and axial-vector currents in a free-quark model, Gerstein and Jackiw[9] (GJ) found only $\langle AAA\rangle$ and $\langle AVV\rangle$ to have anomalous WI's. In dealing with divergent momentum-integral representations of the 3-pf's and 2-pf's, they prescribed a certain recipe which was used to determine whether or not two infinite quantities were equal. On the strength of this procedure they then argued that *no* n-pf's for $n\geq 4$ were anomalous. In another spinor-field calculation by Bardeen,[10] in which an ϵ separation of the interaction Lagrangian was used, the S matrix was carefully defined by counterterms and some WI's involving 4-pf's and 5-pf's were also found to be anomalous. Since GJ utilized a regularization argument in a portion of their work, their disagreement with Bardeen raises an interesting question concerning regularization and its relation to the general counterterm procedure.

This last remark brings us to the purpose of our paper. We consider a generalization of the framework described in GJ. In it we define general n-pf's by the regularization described by Pauli and Villars[11] and employed in closed-loop perturbation calculations by Steinberger.[12] In this way, all of the pertinent symmetries of a given n-pf are carried over unambiguously to its momentum representation, which is now well defined. Furthermore, no infinite quantities arise in the computation; there is no need to equate one infinity with another. Finally, this approach yields a simple prescription for locating and calculating WI anomalies.

The outline of this work is as follows: In Sec. II, we define our model and notation and discuss the corresponding naïve Ward identities (NWI's). A universal regularization in which all of the n-pf's are regularized in the same way is introduced in Sec. III. There follows an enumeration of those universally regularized n-pf's which have anomalous WI's. In Sec. IV we enlarge the freedom in the definitions of the n-pf's to include general local mass and momentum polynomials and describe which additional NWI's are consequentially satisfied. Section V shows how some of those local polynomials can be identified with ambiguities in the regularization

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scheme. In an effort to make a more realistic calculation of possible anomalies, a symmetry-breaking model in which the strangeness-changing vector currents are not conserved is employed in Sec. VI. Conclusions and discussions comprise Sec. VII.

II. NAIVE WARD IDENTITIES IN A SPINOR MODEL

We begin by defining in the interaction picture the $SU(3)\otimes SU(3)$ currents[13]

$$j_i{}^a(x)\equiv\bar{\psi}(x)\frac{\lambda^a}{2}\Gamma_i\psi(x). \tag{1}$$

The free spinor field ψ is also a column matrix in the internal space which, for our present purposes, represents states with a common (degenerate) mass m. Later on, in Sec. VI, a symmetry-breaking model is considered which admits nondegenerate masses. In (1) we have Gell-Mann's $SU(3)$ matrices λ^a $(\equiv\lambda_a)$, where $a=0, 1, \cdots, 8$. The Dirac matrices $\Gamma_i\equiv 1, i\gamma_5, \gamma_\mu$, and $\gamma_\mu\gamma_5$ imply the scalar, pseudoscalar, vector, and axial-vector currents $j_i{}^a(x)\equiv S^a(x)$, $P^a(x)$, $V_\mu{}^a(x)$, and $A_\mu{}^a(x)$, respectively.

It follows from the free-spinor equation of motion that the divergence equations for the vector and axial-vector currents are

$$\partial^\mu V_\mu{}^a(x)=0,\quad \partial^\mu A_\mu{}^a(x)=2mP^a(x). \tag{2}$$

With the neglect of Schwinger terms, we find for the equal-time commutators that

$$[V_0{}^a(x),j_i{}^b(y)]_{x_0=y_0}=if^{abc}j_i{}^c(x)\delta(\mathbf{x}-\mathbf{y}),$$
$$\left[A_0{}^a(x),\begin{pmatrix}V_\mu{}^b(y)\\ A_\mu{}^b(y)\end{pmatrix}\right]_{x_0=y_0}=if^{abc}\begin{pmatrix}A_\mu{}^c(x)\\ V_\mu{}^c(x)\end{pmatrix}\delta(\mathbf{x}-\mathbf{y}), \tag{3}$$
$$\left[A_0{}^a(x),\begin{pmatrix}S^b(y)\\ P^b(y)\end{pmatrix}\right]_{x_0=y_0}=id^{abc}\begin{pmatrix}P^c(x)\\ -S^c(x)\end{pmatrix}\delta(\mathbf{x}-\mathbf{y}).$$

The n-pf for our currents or, strictly speaking, its Fourier transform is

$$\langle j_1{}^a(k_1)j_2{}^b(k_2)\cdots j_n{}^e(k_n)\rangle$$
$$\equiv\int d^4z_2d^4z_3\cdots d^4z_ne^{-ik_2z_2}\cdots e^{-ik_nz_n}$$
$$\times\langle 0|T(j_1{}^a(0)j_2{}^b(z_2)\cdots j_n{}^e(z_n))|0\rangle, \tag{4}$$

in which we have factored out the four-momentum-conserving δ function. Thus

$$\sum_{i=1}^n k_i=0$$

is left understood. The general first-order (single-momentum contraction) vector WI is, by naïve use of (2) and (3),

$$k_1{}^\mu\langle V_\mu{}^a(k_1)j_2\cdots j_n\rangle=n-1 \text{ of the } (n-1)\text{-pf's}. \tag{5}$$

Also we find, for the axial-vector case, the NWI

$$k_1{}^\mu\langle A_\mu{}^a(k_1)j_2\cdots j_n\rangle=-2mi\langle P^a(k_1)j_2\cdots j_n\rangle$$
$$+n-1 \text{ of the } (n-1)\text{-pf's}. \tag{6}$$

The coefficients of the $(n-1)$-pf's in (5) and (6) are determined by the commutation relations in (3). Higher-order WI's can be found if other currents in $j_2, \cdots, j_n$ are vector or axial-vector ones.

Unfortunately, life is not so simple and (5) plus (6) is hardly the whole story. When one substitutes an explicit momentum-integral representation[14] for (4), the ultraviolet-divergence difficulties prevent an immediate verification of these identities. This is due to the lack of the freedom needed for translating integration variables when there are linear or quadratic divergences present. For the same reason, Bose symmetry or even the more general invariance of (4) under the exchange

$$j_i{}^d(k_i)\leftrightarrow j_j{}^e(k_j) \tag{7}$$

does not always hold in such representations. Thus, displacement invariance in the four-momentum integration variable is indeed expected to be crucial.

In order to proceed, we wish to redefine our n-pf's by using regularization. This will remove not only the infinities present in (4) but also the displacement surface terms.

III. UNIVERSAL REGULARIZATION AND WARD-IDENTITY ANOMALIES

The regularized n-pf is defined as

$$\langle j_1{}^a(k_1)\cdots j_n{}^e(k_n)\rangle_{\text{regularized}}$$
$$\equiv\sum_{i=0}^N C_i\langle j_1{}^a(k_1)\cdots j_n{}^e(k_n)\rangle_{m_i}. \tag{8}$$

Here the C_i's are functions of the set of masses m_j and the subscript m_i on an n-pf refers to the mass of that n-pf's loops. Also,

$$C_0=1,\quad m_0=m.$$

Such an n-pf [Eq. (8)] is free of ultraviolet divergences and integration-displacement ambiguities provided that

$$\sum_{i=0}^N C_im_i{}^\alpha=0,\quad \alpha=0, 1, 2, 3. \tag{9}$$

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[13] Our basic notation and conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964); *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965). In particular, $\epsilon_{0123}=-\epsilon^{0123}\equiv+1$ and $\not{p}\equiv\gamma^\mu p_\mu$. Other more specific notation often follows that of GJ.

[14] We are talking here about closed-loop Feynman diagrams. The usual omission of the disconnected graphs is made.

This can be seen by considering the integral and surface-term calculations in Appendix A; the efficacy of (8) and (9) will also be more transparent as we proceed in the main text. Note that the N in (8) and (9) may be as large as one wishes and that in the end the limit

$$m_i \to \infty, \quad i \neq 0$$

is left understood [albeit constrained to satisfy things like (9)]. In view of such a limit, we demand that the sums

$$\sum_{i=0}^{N} C_i m_i^\alpha \ln m_i^2 \equiv K_\alpha \tag{10}$$

are either finite or consistently redefined as part of renormalization constants. Lastly we ask that

$$\sum_{i=1}^{N} \frac{|C_i|}{m_i} \to 0, \tag{11}$$

which is a stronger condition than is needed in electrodynamics of vector currents.[15] The condition (11) implies, for example, that for $n \geq 5$, the regularized n-pf (8) reduces to our original amplitude (4).

We shall consider here a simple version of regularization in which the same set of C_i and m_i are used for all of the n-pf's and are such that

$$K_\alpha = 0, \quad \alpha = 0, 1, 2, 3. \tag{12}$$

We call this "universal regularization" (UR). It will be seen that the employment of a common set of C_i leads to a simple determination of WI anomalies. The ambiguities present in the general regularization scheme (e.g., nonzero K_α) are fully considered later in Sec. V.

Before addressing ourselves directly to the vector WI's of the UR n-pf's, let us recall the difficulties in verifying NWI's with unregularized loop amplitudes. One is that upon contraction of a momentum with its corresponding vector or axial-vector current, the $(n-1)$-pf's that result are usually not in "standard form." Their integration variables have to be translated in order to attain the agreed-upon form for a given $(n-1)$-pf. Another difficulty is the unsatisfactory situation of dealing with infinite quantities (i.e., n-pf's for $n \leq 4$). Thus our UR procedure is to the point and the use of a common set of C_i assures us that the $(n-1)$-pf's arising in a WI calculation will be consistently defined.

We may immediately say that the first-order vector NWI's (5) are satisfied by the UR n-pf's, since the integral representations of $(n-1)$-pf's which arise on the right-hand side can now undergo any desired change of integration variable. This, as with other assertions presented here, will be clearer when specific examples are given later.

The axial-vector case is much different, however. With UR we find an anomalous term on the right-hand side of (6)[16]:

$$\Delta(A^a(k_1) j_2 j_3 \cdots j_n) \equiv -2i \sum_{i=0}^{N} C_i (m_i - m) \langle P^a(k_1) j_2 \cdots j_n \rangle_{m_i}. \tag{13}$$

Even though the constants in (10) are set equal to zero in UR, m_i^{-1} terms in $\langle P j_2 \cdots j_n \rangle_{m_i}$ imply a nonvanishing Δ. On the other hand, (13) does restrict the anomalous (first-order) axial-vector WI's to those involving only $n \leq 5$. This follows from the integral expressions for the Feynman loops in Appendix A, viz.,

$$\langle j_1 j_2 \cdots j_n \rangle_M = O(M^{4-n}), \quad n \geq 5.$$

We thus write down the loop-momentum representations for $n \leq 5$ only, and thereupon list the corresponding anomalous axial-vector identities according to (13). Charge-conjugation invariance is assumed in our model, so that our n-pf's remain invariant under $\lambda_a \to c_i \lambda_a^T$ for each current j_i^a, where c_i is $+1$ for the S, P, and A currents and -1 for V.

For the 1-pf,

$$\langle j_1^a(0) \rangle_{\mathrm{UR}} = -i\tfrac{1}{4} \operatorname{Tr}(\lambda_a + c_1 \lambda_a^T) \sum_0^N C_i B_1^i(0) = -i(\tfrac{3}{2})^{1/2} \frac{1+c_1}{2} \delta^{a0} \sum_0^N C_i B_1^i(0), \tag{14}$$

where[17]

$$B_1^i(a) \equiv \int_l \operatorname{Tr}[\Gamma_1 S_i(l+a)], \tag{15}$$

$$S_i(k) \equiv (\not{k} - m_i)^{-1}. \tag{16}$$

Since only the current with the vacuum properties (i.e., $j_1^a = S^a \delta^{a0}$) gives a nonzero 1-pf, there are obviously no WI's (5) and (6) with $n=1$. For the same reason the charge-conjugation information in (14) yields no new restrictions. The momentum integral of (15) is calculated in Appendix A utilizing a finite integration region; it can thus be seen that the UR version of the scalar one-point amplitude vanishes.

We continue with the two-point amplitudes

$$\langle j_1^a(p) j_2^b(-p) \rangle_{\mathrm{UR}} = \tfrac{1}{8} \operatorname{Tr}(\lambda_a \lambda_b + c_1 c_2 \lambda_a^T \lambda_b^T) \times \sum_0^N C_i D_{12}^i(0,p) = \tfrac{1}{2} \delta^{ab} \frac{1+c_1 c_2}{2} \sum_0^N C_i D_{12}^i(0,p), \tag{17}$$

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[15] There the condition need only be Eq. (11) with m_i replaced by m_i^2 (see Ref. 11). The mass parity is even if there are only V currents (cf. Ref. 20).

[16] We have performed—but shall not give here—a general 3-pf calculation in which the limit $m_i \to \infty$, $i \neq 0$ is taken before the k^μ contraction. The result for the anomaly is the same but this procedure requires extensive use of the mass expansion and displacement formulas in Appendix A and is much more tedious than that outlined in the text.

[17] We use the notation $\int_k \equiv \int d^4k/(2\pi)^4$ after Brandt (Ref. 4).

in which

$$D_{12}{}^i(a,b)\equiv\int_l \mathrm{Tr}[\Gamma_1 S_i(l+a)\Gamma_2 S_i(l+b)]. \quad (18)$$

The $n=2$ WI's for $\langle AP\rangle_{\mathrm{UR}}$ and $\langle VV\rangle_{\mathrm{UR}}$ are naïve. That is, they agree with the formal manipulations on (4) and are termed nonanomalous. For instance,

$$p^\mu\langle V_\mu{}^a(p)V_\nu{}^b(-p)\rangle_{\mathrm{UR}}=-\tfrac{1}{2}\delta^{ab}\sum_0^N C_i B_{V_\nu}{}^i(p), \quad (19)$$

since

$$S_i(l+p)\not{p}S_i(l)=S_i(l)-S_i(l+p). \quad (20)$$

The right-hand side of (19) is zero because regularization allows us to shift integration variables, and so this identity is naïve—as expected since it is a vector WI.

In the case of $\langle AP\rangle_{\mathrm{UR}}$, the freedom in shifting integration variables can be used to obtain

$$p^\mu\langle A_\mu{}^a(p)P^b(-p)\rangle_{\mathrm{UR}}=-2mi\langle P^a(p)P^b(-p)\rangle_{\mathrm{UR}}+\delta^{ab}\langle S(0)\rangle_{\mathrm{UR}}+\Delta(AP), \quad (21)$$

using

$$S_i(l+p)\not{p}\gamma_5 S_i(l)=2m_i S_i(l+p)\gamma_5 S_i(l)+\gamma_5 S_i(l)+S_i(l+p)\gamma_5. \quad (22)$$

Here we have employed the reduced 1-pf

$$\langle S(0)\rangle_{\mathrm{UR}}\equiv\sum_0^N C_i B_S{}^i(0). \quad (23)$$

The anomaly is, from (13),

$$\Delta(AP)=-2i\sum_0^N C_i(m_i-m)\langle P^a(p)P^b(-p)\rangle_{m_i}. \quad (24)$$

Since $\langle PP\rangle_m$ has no m^{-1} term in its mass expansion [see (A16)], we have $\Delta(AP)=0$. Therefore, within the framework of UR we say that the $\langle AP\rangle$ WI has no anomaly.

Doing the same sort of things, we go on to find that

$$p^\mu\langle A_\mu{}^a(p)A_\nu{}^b(-p)\rangle_{\mathrm{UR}}=-2mi\langle P^a(p)A_\nu{}^b(-p)\rangle_{\mathrm{UR}}+\Delta(AA), \quad (25)$$

with the nonvanishing anomaly

$$\Delta(AA)=-(1/24\pi^2 i)p^2 p_\nu. \quad (26)$$

One must go beyond UR in order to satisfy the $\langle AA\rangle$ NWI. Obviously, (25) implies that the second-order WI for $p^\mu p^\nu\langle A_\mu A_\nu\rangle$ is anomalous also. Before continuing with the three-vertex loops, we remark that $\langle AV\rangle$, $\langle AS\rangle$, $\langle VP\rangle$, and $\langle SP\rangle$ are zero by the usual pseudotensor argument[18] and that $\langle VS\rangle$ vanishes due to charge-conjugation invariance.

[18] That is, the lack of available independent momenta prohibits the construction of the necessary tensor forms.

The three-vertex loops are

$$\langle j_1{}^a(k)j_2{}^b(p)j_3{}^c(q)\rangle_{\mathrm{UR}}=\frac{i}{16}\sum_0^N C_i[\mathrm{Tr}(\lambda_a\lambda_b\lambda_c+c_1c_2c_3\lambda_a{}^T\lambda_b{}^T\lambda_c{}^T)F_{123}{}^i(p,0,-q)+\mathrm{Tr}(\lambda_a\lambda_c\lambda_b+c_1c_2c_3\lambda_a{}^T\lambda_c{}^T\lambda_b{}^T)F_{132}{}^i(q,0,-p)], \quad (27)$$

with $k+p+q=0$ and

$$F_{123}{}^i(a,b,c)\equiv\int_l \mathrm{Tr}[\Gamma_1 S_i(l+a)\Gamma_2 S_i(l+b)\Gamma_3 S_i(l+c)]. \quad (28)$$

We refer to GJ for a list of all the (first-order) 3-pf NWI's. Of these, the vector ones are all satisfied automatically by our UR amplitudes.

For example,

$$k^\mu\langle V_\mu{}^a(k)A_\nu{}^b(p)A_\lambda{}^c(q)\rangle_{\mathrm{UR}}=f^{abc}[\langle A_\nu(-q)A_\lambda(q)\rangle_{\mathrm{UR}}-\langle A_\nu(p)A_\lambda(-p)\rangle_{\mathrm{UR}}], \quad (29)$$

where we have introduced the reduced 2-pf

$$\langle j_1(p)j_2(-p)\rangle_{\mathrm{UR}}\equiv\tfrac{1}{2}\sum_0^N C_i D_{12}{}^i(0,p). \quad (30)$$

But the counterpart axial-vector WI by (13) and (A18) is

$$p^\nu\langle V_\mu{}^a(k)A_\nu{}^b(p)A_\lambda{}^c(q)\rangle_{\mathrm{UR}}=-2mi\langle V_\mu{}^a(k)P^b(p)A_\lambda{}^c(q)\rangle_{\mathrm{UR}}+f^{abc}[\langle V_\mu(k)V_\lambda(-k)\rangle_{\mathrm{UR}}-\langle A_\mu(-q)A_\lambda(q)\rangle_{\mathrm{UR}}]+\Delta(VAA), \quad (31)$$

with the anomaly

$$\Delta(VAA)=(1/24\pi^2 i)f^{abc}[2p_\mu p_\lambda+3p_\mu q_\lambda+q_\mu p_\lambda-g_{\mu\lambda}(3p^2+q^2+3p\cdot q)]. \quad (32)$$

As a consistency check, the second-order identity for $k^\mu p^\nu\langle V_\mu A_\nu A_\lambda\rangle_{\mathrm{UR}}$ can be obtained equivalently from (29) or (31) after taking note of (25) and its anomaly. The remaining anomalous $n=3$ first-order axial-vector WI's are

$$p^\nu\langle S^a(k)A_\nu{}^b(p)P^c(q)\rangle_{\mathrm{UR}}=-2mi\langle S^a(k)P^b(p)P^c(q)\rangle_{\mathrm{UR}}+d^{abc}[\langle P(-q)P(q)\rangle_{\mathrm{UR}}-\langle S(k)S(-k)\rangle_{\mathrm{UR}}]+\Delta(SAP), \quad (33)$$

where

$$\Delta(SAP)=(1/24\pi^2 i)d^{abc}(p^2+q^2-p\cdot q), \quad (34)$$

$$k^\mu\langle A_\mu{}^a(k)V_\nu{}^b(p)V_\lambda{}^c(q)\rangle_{\mathrm{UR}}=-2mi\langle P^a(k)V_\nu{}^b(p)V_\lambda{}^c(q)\rangle_{\mathrm{UR}}+\Delta(AVV), \quad (35)$$

where

$$\Delta(AVV)=-(1/8\pi^2 i)d^{abc}\epsilon_{\nu\lambda\alpha\beta}p^\alpha q^\beta \quad (36)$$

and

$$k^\mu\langle A_\mu{}^a(k)A_\nu{}^b(p)A_\lambda{}^c(q)\rangle_{\mathrm{UR}}=-2mi\langle P^a(k)A_\nu{}^b(p)A_\lambda{}^c(q)\rangle_{\mathrm{UR}}+\Delta(AAA), \quad (37)$$

where

$$\Delta(AAA) = -(1/24\pi^2 i) d^{abc} \epsilon_{\nu\lambda\alpha\beta} p^\alpha q^\beta . \tag{38}$$

We thus obtain anomalous second- and third-order WI's for $\langle VAA \rangle_{\rm UR}$ and also an anomalous second-order identity for $\langle SAA \rangle_{\rm UR}$, since it involves (33).

The momentum-integral representation for the general 4-pf is

$$\langle j_1{}^a(k) j_2{}^b(p) j_3{}^c(q) j_4{}^d(t) \rangle_{\rm UR} = -\tfrac{1}{32} \sum_0^N C_i \{ W^{abcd} [H_{1234}{}^i(p,0,-q,-q-t) + c(4) H_{1432}{}^i(t,0,-q,-q-p)]$$
$$+ W^{abdc} [H_{1243}{}^i(p,0,-t,-t-q) + c(4) H_{1342}{}^i(q,0,-t,-t-p)]$$
$$+ W^{acbd} [H_{1324}{}^i(q,0,-p,-p-t) + c(4) H_{1423}{}^i(t,0,-p,-p-q)] \} , \tag{39}$$

in which $k+p+q+t=0$ and

$$H_{1234}{}^i(a,b,c,d) \equiv \int_l \mathrm{Tr}[\Gamma_1 S_i(l+a) \Gamma_2 S_i(l+b) \Gamma_3 S_i(l+c) \Gamma_4 S_i(l+d)] , \tag{40}$$

$$W^{abcd} \equiv \mathrm{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d + c(4) \lambda_a{}^T \lambda_b{}^T \lambda_c{}^T \lambda_d{}^T] , \tag{41}$$

$$c(4) \equiv c_1 c_2 c_3 c_4 . \tag{42}$$

By an explicit calculation of (13) utilizing (A20), we find seven anomalous first-order axial-vector WI's here.[19] Since it is cumbersome to write the anomalous equations here, we merely list the corresponding 4-pf's [which occur on the left-hand side of (6)] in Table I. For completeness we also list the UR 2- and 3-pf's that are associated with anomalies. The anomalous higher-order 4-pf WI's are many, and we shall not enumerate them here.

We come now to the last loops which can give us trouble, those corresponding to the 5-pf

$$\langle j_1{}^a(k) j_2{}^b(p) j_3{}^c(q) j_4{}^d(t) j_5{}^e(u) \rangle_{\rm UR}$$
$$= -\frac{i}{64} \sum_0^N C_i \{ Z^{abcde} [J_{12345}{}^i(p,0,-q,-q-t,-q-t-u) + c(5) J_{15432}{}^i(u,0,-t,-t-q,-t-q-p)]$$
$$+ Z^{abced} [J_{12354}{}^i(p,0,-q,-q-u,-q-u-t) + c(5) J_{14532}{}^i(t,0,-u,-u-q,-u-q-p)]$$
$$+ 10 \text{ further permutations of 2345 synchronized with } bcde \} , \tag{43}$$

with $k+p+q+t+u=0$ and

$$J_{12345}{}^i(a,b,c,d,e) \equiv \int_l \mathrm{Tr}[\Gamma_1 S_i(l+a) \Gamma_2 S_i(l+b) \times \Gamma_3 S_i(l+c) \Gamma_4 S_i(l+d) \Gamma_5 S_i(l+e)] , \tag{44}$$

$$Z^{abcde} \equiv \mathrm{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d \lambda_e + c(5) \lambda_a{}^T \lambda_b{}^T \lambda_c{}^T \lambda_d{}^T \lambda_e{}^T] , \tag{45}$$

$$c(5) \equiv c_1 c_2 c_3 c_4 c_5 . \tag{46}$$

Although the 5-pf's remain unchanged under it, the regularization in (43) which leads to (13) for $n=5$ provides a simple way of calculating the anomalies. One should remember that any anomaly in a 5-pf (first-order) WI is simply a consequence of the UR of the 4-pf [Eq. (39)].

Again it is cumbersome and unenlightening to do more than list those UR 5-pf's which lead to anomalous WI's. There are 11 that do so [inspection of the trace in (A22) leads immediately to this result] and they are given in Table I also. We shall not list the large number of higher-order anomalous WI's that arise for the 5-pf's.

An examination of Table I shows how the tensor structure of our n-pf's gives a clear picture of what is happening. For $n \leq 5$, a necessary condition for the existence of an m^{-1} term in the Appendix-A mass expansions is that the number of P and S in Eq. (13)

TABLE I. Universally regularized n-point functions which have anomalous first-order axial-vector WI's. The boldfaced ones are of abnormal parity and represent Bardeen's minimal set.

One-point	None
Two-point	$\langle AA \rangle$
Three-point	**⟨AAA⟩**, $\langle AAV \rangle$, **⟨AVV⟩**, $\langle APS \rangle$
Four-point	$\langle AAAA \rangle$, **⟨AAAV⟩**, $\langle AAVV \rangle$, **⟨AVVV⟩**, $\langle AVPS \rangle$, $\langle AAPP \rangle$, $\langle AASS \rangle$
Five-point	**⟨AAAAA⟩**, $\langle AAAAV \rangle$, **⟨AAAVV⟩**, $\langle AAVVV \rangle$, **⟨AVVVV⟩**, $\langle AAAPS \rangle$, $\langle AAVPP \rangle$, $\langle AAVSS \rangle$, $\langle AVVPS \rangle$, $\langle APPPS \rangle$, $\langle APSSS \rangle$
Six-point and higher	None

[19] It is a simple matter to find which 4-pf's have anomalous first-order WI's: We need only determine whether or not the trace in (A20) vanishes.

be odd.[20] Since one P is the result of the axial-vector contraction, the n-pf's in Table I all have an *even* number of P and S. This condition is also sufficient for the normal-parity n-pf's—those that have an even number of A and P.

Other than to say that n-pf's for $n\geq 6$ can have anomalous higher-order WI's as soon as they involve the amplitudes shown in Table I, we have come to the end of the UR picture. To remove any of the existing anomalies, one must consider either a more general version of regularization, or the addition of *ad hoc* counterterms. Since it turns out that the regularization ambiguities cannot be of any help in removing any of the anomalies yet present in UR, we find it simpler to present the results of the *ad hoc* modifications first.

IV. GENERAL COUNTERTERMS

In Sec. III we found all the UR n-pf's which do not satisfy first-order NWI's and listed them in Table I. The existence of their corresponding anomalies cannot be taken seriously, however, until all of the freedom in defining the n-pf's is exhausted. This freedom includes adding local polynomials (in m and the momenta). Therefore, we now consider as a more general definition of our n-pf, the UR one plus possible counterterms, the object, of course, being the removal of as many anomalies as possible. The form of these counterterms is quite restricted from the start.

Since each polynomial must have the same dimension and tensor structure as the associated n-pf has, it attains a definite mass parity; the even and odd powers of m cannot be mixed.[20] One notes that the dimension of an n-pf is $(\text{mass})^{4-n}$. Therefore n-pf's for $n\geq 5$ cannot be modified without introducing kinematic or mass singularities (i.e., cannot be redefined by polynomials). Also, with $\epsilon_{\mu\nu\sigma\rho}$, $g_{\mu\nu}$, and constants as the only dimensionless counterterms, the 4-pf's which are rank-1 or rank-3 Lorentz tensors may not be changed by local quantities. An additional remark is that the normal and abnormal (space) parity n-pf's (those which have even and odd numbers, respectively, of A and P) are not mixed by the WI's and hence give rise to two separate "chains" of related n-pf's. Finally, we note that the anomalies Δ of (13) are independent of m.

Taking the previous comments into account and remembering crossing symmetry, we concoct the most general counterterm for each n-pf, $n\leq 4$. Let us define this in terms of the "new" n-pf

$$``\langle j_1j_2\cdots j_n\rangle" \equiv \langle j_1j_2\cdots j_n\rangle_{\rm UR}+\delta(j_1j_2\cdots j_n). \quad (47)$$

For convenience, we define [see (41)]

$$W_1\equiv W^{abcd},\quad W_2\equiv W^{abdc},\quad W_3\equiv W^{acbd} \quad (48)$$

[evaluated at $c(4)=+1$], and the $\bar{W}_i$ as counterparts to (48), but with $c(4)=-1$

$$\bar{W}_1\equiv W^{abcd}[c(4)=-1],\quad \text{etc.} \quad (49)$$

[20] From the momentum-integral representations in Appendix A, it can be seen that the mass parity of a given amplitude is $(-1)^{n_s+n_p}$, where n_s (n_p) is the number of S (P) present.

Further, let

$$y\equiv 1/24\pi^2 i. \quad (50)$$

By straightforward (but tedious) algebra, we find that *all* of the normal-parity NWI's can now be satisfied if there are certain relations between the counterterms in the normal-parity chain. The normal-parity set is[21]

(1) $n=1$:

$$\delta(S)=-6^{1/2}ia_1\delta^{a0}m^3; \quad (51)$$

(2) $n=2$:

$$\begin{aligned}
&\delta(VV)=a_2\delta^{ab}(p^2g_{\mu\nu}-p_\mu p_\nu),\\
&\delta(AA)=\delta(VV)+\delta^{ab}(yp^2g_{\mu\nu}+4a_3m^2g_{\mu\nu}),\\
&\delta(AP)=-2ia_3\delta^{ab}mp_\mu,\\
&\delta(PP)=\delta^{ab}(a_3p^2+a_1m^2),\\
&\delta(SS)=\delta^{ab}[(a_3-y)p^2+a_4m^2];
\end{aligned} \quad (52)$$

(3) $n=3$:

$$\begin{aligned}
&\delta(VVV)=a_2f^{abc}[g_{\mu\nu}(k-p)_\lambda+g_{\mu\lambda}(q-k)_\nu+g_{\nu\lambda}(p-q)_\mu],\\
&\delta(VAA)=\delta(VVV)+yf^{abc}[3g_{\mu\nu}k_\lambda-3g_{\mu\lambda}k_\nu+g_{\nu\lambda}(p-q)_\mu],\\
&\delta(VAP)=-2ia_3f^{abc}mg_{\mu\nu},\\
&\delta(AAS)=i(4a_3-6y)d^{abc}mg_{\mu\nu},\\
&\delta(VPP)=a_3f^{abc}(p-q)_\mu,\\
&\delta(VSS)=(a_3-y)f^{abc}(p-q)_\mu,\\
&\delta(APS)=d^{abc}[a_3(q-p)_\mu+3yp_\mu],\\
&\delta(PPS)=\tfrac{1}{2}i(a_4-a_1)d^{abc}m,\\
&\delta(SSS)=\tfrac{3}{2}i(a_4-a_1-4y)d^{abc}m;
\end{aligned} \quad (53)$$

(4) $n=4$:

$$\begin{aligned}
\delta(VVVV)=&\tfrac{1}{8}a_2[g_{\mu\nu}g_{\lambda\sigma}(-W_1-W_2+2W_3)\\
&+g_{\mu\lambda}g_{\nu\sigma}(2W_1-W_2-W_3)\\
&+g_{\mu\sigma}g_{\nu\lambda}(-W_1+2W_2-W_3)],\\
\delta(AAAA)=&\delta(VVVV)+\tfrac{1}{2}y[g_{\mu\nu}g_{\lambda\sigma}(-W_1-W_2+W_3)\\
&+g_{\mu\lambda}g_{\nu\sigma}(W_1-W_2-W_3)\\
&+g_{\mu\sigma}g_{\nu\lambda}(-W_1+W_2-W_3)],\\
\delta(AAVV)=&\delta(VVVV)+\tfrac{1}{8}y[g_{\mu\nu}g_{\lambda\sigma}(-W_1-W_2+2W_3)\\
&+g_{\mu\lambda}g_{\nu\sigma}(3W_1-3W_2)\\
&+g_{\mu\sigma}g_{\nu\lambda}(-3W_1+3W_2)],\\
\delta(VVPP)=&-\tfrac{1}{8}a_3g_{\mu\nu}(W_1+W_2-2W_3),\\
\delta(VVSS)=&-\tfrac{1}{8}(a_3-y)g_{\mu\nu}(W_1+W_2-2W_3),\\
\delta(AAPP)=&-\tfrac{1}{8}a_3g_{\mu\nu}(W_1+W_2+2W_3)+\tfrac{3}{8}yg_{\mu\nu}(W_1+W_2),\\
\delta(AASS)=&-\tfrac{1}{8}a_3g_{\mu\nu}(W_1+W_2+2W_3)\\
&+\tfrac{3}{4}yg_{\mu\nu}(W_1+W_2+W_3),\\
\delta(AVSP)=&-\tfrac{1}{8}ig_{\mu\nu}[a_3(\bar{W}_1-\bar{W}_2-2\bar{W}_3)+3y(\bar{W}_2+\bar{W}_3)],\\
\delta(PPPP)=&\tfrac{1}{16}(a_1-a_4)(W_1+W_2+W_3),\\
\delta(SSPP)=&\tfrac{1}{16}(a_4-a_1)(W_1+W_2-W_3)-\tfrac{3}{8}y(W_1+W_2),\\
\delta(SSSS)=&\tfrac{1}{2}(-16y+a_1-a_4)(W_1+W_2+W_3).
\end{aligned} \quad (54)$$

[21] The notation is the following: The momenta and $SU(3)$ indices are consistent with the n-pf definitions of Sec. III, and from left to right in a given n-pf the Lorentz indices are μ, ν, λ, σ, and ρ when needed.

All of those δ's which are not listed are necessarily zero.

It is interesting that there are still four arbitrary constants a_i in Eqs. (51)-(54). Thus the normal-parity n-pf's still have some ambiguity in their definitions even after demanding that they satisfy the vector and axial-vector NWI's. However, we notice that a number of UR n-pf's had to be changed even though they originally led to no anomalies (see Table I). This shows the interlocking relationship arising from the imposition of NWI's.

The abnormal-parity series is a different story. For example, the $\langle AAA\rangle$ amplitude cannot be modified in a crossing-symmetric way at all, and any modification of $\langle AVV\rangle$ will ruin its vector NWI's. The latter result turns out to be an example of a general problem: Fixing up one abnormal-parity WI often spoils another.

Explicitly, the only nonzero δ's in the abnormal series are[21]

$$\begin{aligned}\delta(AVV)&=b_1d^{abc}\epsilon_{\mu\nu\lambda\alpha}(p-q)^\alpha\,,\\ \delta(AAAV)&=b_2(\bar{W}_1+\bar{W}_2-\bar{W}_3)\epsilon_{\mu\nu\lambda\sigma}\,,\\ \delta(AVVV)&=b_3(\bar{W}_1-\bar{W}_2-\bar{W}_3)\epsilon_{\mu\nu\lambda\sigma}\,.\end{aligned}\tag{55}$$

Hence the abnormal-parity anomalies will depend on the constants b_i. Taking the redefinitions (55) into account, we find that the abnormal axial-vector anomalies now are[21]

$$\begin{aligned}\Delta(AAA)&=-yd^{abc}\epsilon_{\nu\lambda\alpha\beta}p^\alpha q^\beta\,,\\ \Delta(AVV)&=(2b_1-3y)d^{abc}\epsilon_{\nu\lambda\alpha\beta}p^\alpha q^\beta\,,\\ \Delta(AAAV)&=\tfrac{1}{8}i\epsilon_{\nu\lambda\sigma\alpha}\{\bar{W}_1[b_1(q+2t)^\alpha+8ib_2k^\alpha+y(t-p)^\alpha]\\ &\quad+\bar{W}_2[b_1(3t-k)^\alpha+8ib_2k^\alpha+y(3t-k)^\alpha]\\ &\quad+\bar{W}_3[-b_1(p+2t)^\alpha-8ib_2k^\alpha+y(q-t)^\alpha]\}\,,\\ \Delta(AVVV)&=\tfrac{1}{8}i\epsilon_{\nu\lambda\sigma\alpha}\\ &\quad\times\{\bar{W}_1[b_1(3q+k)^\alpha-8ib_3k^\alpha-3y(p+t)^\alpha]\\ &\quad+\bar{W}_2[-b_1(3t+k)^\alpha+8ib_3k^\alpha+3y(p+q)^\alpha]\\ &\quad+\bar{W}_3[-b_1(3p+k)^\alpha+8ib_3k^\alpha+3y(t+q)^\alpha]\}\,,\\ \Delta(AAAAA)&=\tfrac{1}{32}(y-16ib_2)\epsilon_{\nu\lambda\sigma\rho}\\ &\quad\times\sum_{\text{all perm. of }bcde}\epsilon(bcde)Z^{abcde}\;[c(5)=1]\,,\\ \Delta(AVVVV)&=-\tfrac{1}{32}(3y-16ib_3)\epsilon_{\nu\lambda\sigma\rho}\\ &\quad\times\sum_{\text{all perm. of }bcde}\epsilon(bcde)Z^{abcde}\;[c(5)=1]\,,\\ \Delta(AAAVV)&=\tfrac{1}{16}\epsilon_{\nu\lambda\sigma\rho}[(y+8ib_2-8ib_3)\\ &\quad\times(Z^{abcde}+Z^{abdce}+Z^{acbed}+Z^{acebd})\\ &\quad+(3y-16ib_3)Z^{abdec}+(y-16ib_2)Z^{adcbe}]_{c(5)=1}\\ &\quad-(d\leftrightarrow e)\,,\end{aligned}\tag{56}$$

where $\epsilon(bcde)=1$ (-1) for even (odd) permutations of $bcde$. We emphasize that the above anomalies refer to WI's obtained by contracting on the first (left-most) axial-vector current.

If the b_i's introduced in (55) are nonvanishing, there results abnormal-parity vector WI anomalies as well. We define Δ_V as the vector analog of Δ. So, *boldfacing that vector current which is contracted*, the possible vector anomalies are[21]

$$\begin{aligned}\Delta_V(A\mathbf{V}V)&=b_1d^{abc}\epsilon_{\mu\lambda\alpha\beta}p^\alpha q^\beta\,,\\ \Delta_V(AAA\mathbf{V})&=b_2\epsilon_{\mu\nu\lambda\alpha}t^\alpha(\bar{W}_1+\bar{W}_2-\bar{W}_3)\,,\\ \Delta_V(A\mathbf{V}VV)&=\tfrac{1}{8}i\epsilon_{\mu\lambda\sigma\alpha}p^\alpha[\bar{W}_1(b_1-8ib_3)\\ &\quad+\bar{W}_2(-b_1+8ib_3)+\bar{W}_3(-2b_1+8ib_3)]\,.\end{aligned}\tag{57}$$

All abnormal-parity WI's not referred to in (56) and (57) have no anomalies. In particular, the set of modifications (55) still leave us with no 5-pf vector WI anomalies.

It follows that we cannot avoid anomalies in the abnormal-parity series. Depending on the choices for the b_i, we may have a number of minimal possibilities. One can choose to eliminate all of the vector anomalies by setting

$$b_1=b_2=b_3=0\,.\tag{58}$$

This and the redefinitions (51)-(54) comprise the "minimal" solution found by Bardeen.[10] The only anomalous first-order WI's are axial-vector ones and pertain to the seven boldfaced abnormal-parity n-pf's in Table I; their anomalies are obtained by combining (56) and (58).

The choices

$$2b_1=48ib_2=16ib_3=3y\tag{59}$$

limit the number of axial-vector anomalies to the WI's involving $\langle AAA\rangle$, $\langle AAAV\rangle$, and $\langle AVVV\rangle$. Concomitantly there are vector anomalies in the WI's for the latter two amplitudes and for $\langle AVV\rangle$. Here we have only six separate anomalous WI's (not counting the variants obtained by crossing) embracing four amplitudes.

Some notes are in order here. It is impossible to define away $\Delta(AAA)$ and $\Delta(AAAV)$. Also, the fact that we can find a set of b_i which eliminates *all* 5-pf anomalies is not surprising since there are no integration-displacement troubles there. If we wish to minimize the number of n-pf's with anomalous WI's, (59) is as good as we can do; one can let b_1 be arbitrary, however, since there are still only four amplitudes involved. This class of solutions then includes $b_i=0$, which revives the vector NWI for $\langle AVV\rangle$.

It is mathematically possible to eliminate all of the anomalies if we introduce counterterms with singularities at $m=0$ (i.e., an m^{-1} series). These generally violate the Weinberg asymptotic behavior[22] of our loops and may ruin the soft-pion extrapolation.[23] For completeness, however, we give the prescription for these counterterms here. According to Sec. III and Appendix

[22] S. Weinberg, Phys. Rev. **118**, 838 (1960).
[23] In this regard see the discussion in Ref. 8.

A, one can write

$$\langle j_1 j_2 \cdots j_n \rangle_{\mathrm{UR}} = \sum_{r=r_0}^{\infty} \frac{A_r}{m^r}, \quad r_0 = \max(n-4, 1). \quad (60)$$

The general counterterm $\hat{\delta}$ is then

$$\hat{\delta}(j_1 j_2 \cdots j_n) = \sum_{r=r_0}^{\infty} \frac{B_r}{m^r} \quad (61)$$

if

$$\begin{aligned} B_r &= -A_r \quad \text{for} \quad r \leq r_1 \\ &= \ \ 0 \quad \text{for} \quad r \geq r_1 + 1, \end{aligned} \quad (62)$$

where r_1 is the total number of S and P currents in the given n-pf. We omit derivation details and examples, giving only some essential points which enable one to understand how such a recipe works.

Since r_1 is the same on both sides of a vector WI, the counterterm (61) will not ruin the UR result, which is already nonanomalous. This is because the coefficients of a given power of m^{-1} satisfy the vector WI also. In the axial-vector case, we saw that the UR anomalies are independent of m, meaning that only the zeroth-power equation is anomalous. The $2mi\langle P j_2 \cdots j_n \rangle_{\mathrm{UR}}$ term which has an extra P in it can be redefined to include the anomaly via (61) but this feeds back into other WI's. Taking into account the mass parity mentioned earlier,[20] we can convince ourselves by induction on examples that the series does truncate. In fact, only a maximum of three nonvanishing terms is needed in (61) for any of the UR n-pf's.

Before considering in Sec. V the ambiguities involved in the Pauli-Villars regularization, let us categorize briefly what has been found:

(a) The UR n-pf's satisfy all (first-order) NWI's except for the 23 axial-vector ones implied by Table I. All of these amplitudes vanish as $m \to \infty$.

(b) The UR n-pf's plus the local counterterms (explicitly given in the preceding discussion) represent redefined amplitudes which give rise to only a few unavoidable abnormal-parity anomalies. These amplitudes include polynomials in m; hence there are a number of nonvanishing (and even divergent) amplitudes at $m = \infty$.

(c) An "unphysical" solution with no anomalies whatsoever can be constructed via (61). The corresponding n-pf's would vanish at $m = \infty$ but have bad behavior at $m = 0$.

V. AMBIGUITIES IN PAULI-VILLARS REGULARIZATION

The logarithm sums (10) represent an ambiguity in our UR procedure. This is a special case of the general situation in which different sets of C_i are used for different n-pf's. We ask here whether the corresponding ambiguities in these individually regularized (IR) n-pf's might yet include the polynomial counterterms of Sec. IV. This thus takes into account the full uncertainties in the Pauli-Villars regularization for our problem and determines whether, for example, Bardeen's minimal solution can be obtained in such a framework.

The answer is that some of the polynomials can be swallowed up into the regularization definitions but that none of the anomalies in Table I is so removed. A sketch of the path to this answer follows.

The IR n-pf's are

$$\begin{aligned} \langle j_1 \cdots j_n \rangle_{\mathrm{IR}} &\equiv \sum_i C_i(j_1 \cdots j_n) \langle j_1 \cdots j_n \rangle_{m_i} \\ &= \langle j_1 \cdots j_n \rangle_{\mathrm{UR}} + \sum_i C_i(j_1 \cdots j_n) \\ &\quad \times \sum_\alpha f^\alpha(j_1 \cdots j_n) m_i^\alpha \ln m_i^2. \end{aligned} \quad (63)$$

The quantities f^α are calculated from the integral representations and the resulting mass expansions in Appendix A. We note that these vanish for $n \geq 5$ and also, as detailed calculations show, for all abnormal-parity loops. Let us now consider some of the remaining cases.

We have, as an example,

$$\langle VV \rangle_{\mathrm{IR}} = \langle VV \rangle_{\mathrm{UR}} + y\delta^{ab}(p^2 g_{\mu\nu} - p_\mu p_\nu) \times \sum_i C_i(VV) \ln m_i^2. \quad (64)$$

If $y \sum_i C_i(VV) \ln m_i^2 = a_2$, we see from (52) that indeed the counterterm $\delta(VV)$ can be included in the regularization definition. On the other hand,

$$\begin{aligned} \langle AA \rangle_{\mathrm{IR}} &= \langle AA \rangle_{\mathrm{UR}} + y\delta^{ab}(p^2 g_{\mu\nu} - p_\mu p_\nu) \sum_i C_i(AA) \ln m_i^2 \\ &\quad - 6y\delta^{ab} g_{\mu\nu} \sum_i C_i(AA) m_i^2 \ln m_i^2, \end{aligned} \quad (65)$$

which shows that $\delta(AA)$ *cannot* be incorporated into the logarithm sum ambiguities. The quantity $\sum_i C_i(AA) m_i^2 \times \ln m_i^2$ can only be a multiple of m^2. Another example with this same trouble is

$$\langle APS \rangle_{\mathrm{IR}} = \langle APS \rangle_{\mathrm{UR}} - \tfrac{3}{2} i y d^{abc}(p-q)_\mu \times \sum_i C_i(APS) \ln m_i^2. \quad (66)$$

It can be shown that the remaining cases in which the logarithm terms do not have the same momentum and tensor dependence as the δ's are $\langle VAA \rangle$, $\langle AAAA \rangle$, $\langle AAVV \rangle$, $\langle AASS \rangle$, $\langle AAPP \rangle$, $\langle AVSP \rangle$, and $\langle SSPP \rangle$. Therefore, *the ambiguities in regularization are of no help in removing any of the anomalies referred to in Table I.* We must include counterterms ancillary to the regularization ones in order to obtain the minimal sets discussed earlier.

There are some interesting sidelights that result from the calculation of f^α's. The f^α's all vanish whenever the corresponding δ's are necessarily zero. Furthermore, all of the arbitrary constants a_i of Eqs. (51)–(54) can be absorbed into the logarithm sums. Thus the arbitrariness of the δ's is a partial measure of the ambiguity in the regularization procedure.

In any event, we conclude that universal regularization provides an accurate gauge of the minimal number of anomalous WI's present in the general Pauli-Villars framework.

VI. ANOMALIES IN SYMMETRY-BREAKING MODEL

So far we have discussed WI anomalies for the case of exact $SU(3)$ symmetry where all of the vector currents are conserved. However, in the real world, the strangeness-changing vector currents are not conserved and the exact $SU(3)$ symmetry is necessarily broken. Since, for example, hard-meson and K_{l3} form-factor calculations involve WI's for these vector currents, it is of importance to consider the effect of the broken symmetry on our previous anomaly analysis.

In order to investigate this effect we employ a model proposed by Gell-Mann, Oakes, and Renner.[24] This model can be introduced into our framework by assuming that the quark masses split according to their strangeness quantum number. The p and n nonstrange quarks have the same mass m_p, which is different than that for the strange Λ quark, m_Λ. Our spinor field

$$\psi(x)\equiv\begin{bmatrix}\psi_p(x)\\ \psi_n(x)\\ \psi_\Lambda(x)\end{bmatrix}$$

now satisfies

$$(i\gamma\cdot\partial-\bar{M})\psi=0\,, \tag{67}$$

where

$$\bar{M}\equiv\begin{bmatrix}m_p & 0 & 0\\ 0 & m_p & 0\\ 0 & 0 & m_\Lambda\end{bmatrix}=\bar{m}I-\frac{1}{\sqrt{3}}\delta m\lambda_8\,, \tag{68}$$

$$\bar{m}\equiv\tfrac{1}{3}(2m_p+m_\Lambda)\,,$$

$$\delta m\equiv m_\Lambda-m_p\,.$$

The divergence equations (2) are now changed to

$$\begin{aligned}\partial^\mu V_\mu{}^a(x)&=(2/\sqrt{3})\delta m f^{8ab}S^b(x)\,,\\ \partial^\mu A_\mu{}^a(x)&=2[\bar{m}\delta^{ab}-(1/\sqrt{3})\delta m d^{8ab}]P^b(x)\\ &\equiv 2M^{ab}P^b(x)\,.\end{aligned} \tag{69}$$

Hence the vector NWI (5) is changed to

$$k_1{}^\mu\langle V_\mu{}^a(k_1)j_2\cdots j_n\rangle=-(2/\sqrt{3})\delta m i f^{8ab}\langle S^b(k_1)j_2\cdots j_n\rangle +n-1\text{ of }(n-1)\text{-pf's} \tag{70}$$

and the naïve axial-vector identity (6) becomes

$$k_1{}^\mu\langle A_\mu{}^a(k_1)j_2\cdots j_n\rangle=-2M^{ab}i\langle P^b(k_1)j_2\cdots j_n\rangle +n-1\text{ of }(n-1)\text{-pf's}. \tag{71}$$

Of course, when $\delta m=0$, everything reverts to the exact symmetry of Sec. II.

The unregularized momentum-integral representation for an n-pf is now

$$\langle j_1{}^{a_1}(k_1)\cdots j_n{}^{a_n}(k_n)\rangle=-\tfrac{1}{2}(\tfrac{1}{2}i)^n[L^{a_1,a_2,\cdots,a_n}(k_2,k_3,\cdots,k_n) +\text{synchronized permutations of } a_j,\, k_j,\text{ and }\Gamma_j \text{ for } j=2,3,\cdots,n\text{ with }(n-1)!\text{ terms in all}]\,, \tag{72}$$

where we have used

$$\begin{aligned}&L^{a_1,a_2,\cdots,a_n}(k_2,k_3,\cdots,k_n)\\ &\equiv\int_l \mathrm{Tr}[\lambda_{a_1}\Gamma_1\bar{S}(l+k_2)\lambda_{a_2}\Gamma_2\bar{S}(l)\lambda_{a_3}\Gamma_3\bar{S}(l-k_3)\\ &\cdots\lambda_{a_n}\Gamma_n\bar{S}(l-k_3-\cdots-k_n)+c_1c_2\cdots c_n\lambda_{a_1}{}^T\Gamma_1\bar{S}(l+k_2)\\ &\times\lambda_{a_2}{}^T\Gamma_2\bar{S}(l)\lambda_{a_3}{}^T\Gamma_3\bar{S}(l-k_3)\\ &\cdots\lambda_{a_n}{}^T\Gamma_n\bar{S}(l-k_3-\cdots-k_n)]\,.\end{aligned} \tag{73}$$

The propagator $\bar{S}$ is no longer a multiple of the internal space unit matrix I,

$$\bar{S}(k)\equiv\begin{bmatrix}S_p(k) & 0 & 0\\ 0 & S_p(k) & 0\\ 0 & 0 & S_\Lambda(k)\end{bmatrix}=\Lambda_pS_p(k)+\Lambda_\Lambda S_\Lambda(k)\,, \tag{74}$$

with

$$\begin{aligned}S_p(k)&\equiv(\boldsymbol{k}-m_p)^{-1}\,,\\ S_\Lambda(k)&\equiv(\boldsymbol{k}-m_\Lambda)^{-1}\end{aligned} \tag{75}$$

and the projection operators

$$\begin{aligned}\Lambda_p&\equiv\tfrac{2}{3}I+(1/\sqrt{3})\lambda_8\,,\\ \Lambda_\Lambda&\equiv\tfrac{1}{3}I-(1/\sqrt{3})\lambda_8\,;\end{aligned} \tag{76}$$

consequently there is no simple factorization of the internal matrix trace in (73).

In dealing with the anomalies in WI's for (72), we can make good use of our earlier degenerate-mass calculations by means of the expansion[25]

$$S_\Lambda(k)=S_p(k)+S_p(k)\delta mS_p(k) +S_p(k)\delta mS_p(k)\delta mS_p(k)+\cdots\,. \tag{77}$$

That is, m_p can play the part of m, and δm can be considered as due to a unitary-singlet scalar interaction with zero momentum transfer. This procedure also avoids regularization complications which would arise if a closed loop with different propagator masses were present. Thus, we may still use our old regularization here. Altogether, then, the broken-symmetry n-pf's (72) can be rewritten via (74) and (77) as an infinite series of degenerate-mass m_p loops.

[24] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[25] We consider this as a proper expansion in the perturbative sense. The quantity δm is essentially a mass counterterm.

Since we have reinterpreted the symmetry breaking as a mass splitting,, the results of Bardeen[10] imply that we should find *no change in the mass-degenerate anomalies* listed in Sec. IV—if general counterterms are allowed. This turns out to be the case. A description of the calculation involving the infinite series in δm is relegated to Appendix B; we merely list the results here.

If the degenerate-mass m_p loops are universally regularized, we find that:

(i) The vector-current NWI's (70) are all satisfied.

(ii) *All* of the normal-parity 2-, 3-, and 4-pf axial-vector WI's [see Eq. (71)] are anomalous as well as the normal-parity 5-pf ones listed in Table I.

(iii) No *new* abnormal-parity axial-vector identities beyond those already listed in Table I are anomalous. These conclusions are reasonable since we have seen that UR anomalies correspond to the existence of m^{-1} terms in n-pf's, a fact that carries over to the nondegenerate case.

Now if the degenerate m_p loops arising from the use of (77) are redefined with auxiliary counterterms as in Sec. IV, all of the normal-parity anomalies [see case (ii) above] are removed. This means that the nondegenerate normal-parity n-pf's can be modified by polynomials in δm and m_p such that they satisfy Eqs. (70) and (71). However, as before, the abnormal-parity WI's [see case (iii) above] cannot be redefined so as to remove *all* of their anomalies. But the anomalies are still the same (independent of δm) as the degenerate-case ones in Eqs. (56) and (57).

We remark that, in principle, it is also possible to remove all anomalies even in this case if the pathological m^{-1} series mentioned earlier could be used. Because of the expansion (77), this requires an infinite number of counterterms for each n-pf.

VII. CONCLUSION

The purpose of this work has been twofold. First, we desired to put the notion of a WI anomaly into the language of the Pauli-Villars regularization. Second, we wanted to find the explicit forms of the general "minimal" set of anomalies (that set corresponding to the WI's relating the abnormal-parity n-pf's for $n\leq 5$) in a quark model which includes symmetry-breaking effects. This would give some guide to the validity of certain hard-meson calculations.

We have found that the usual regularization is not adequate[26] for redefining n-pf's in the free-quark model, since acceptable *ad hoc* counterterms can be introduced which further reduce the number of anomalous WI's (however, it required a pathological m^{-1} series to eliminate all of them). The philosophy here is that the NWI's are dynamical criteria. A simple form of regularization did lead automatically to the naïve forms for the conserved-vector-current WI's and a compact formula [Eq. (13)] for the spurious axial-vector terms. The former development is to be expected since regularization was intended for gauge-invariant calculations. The latter is independent of the quark mass m and therefore the limit $m\rightarrow 0$ (corresponding to $\partial^\mu A_\mu{}^a=0$).

Depending upon whether one wants to preserve the naïve forms of all of the vector-current WI's, to remove as many as possible of the axial-vector anomalies, or whatever else, the general minimal set of abnormal-parity Δ's has been given explicitly in Sec. IV. The question concerning which set should be chosen is a dynamical one and has partially been answered in that $\langle AVV\rangle$ seems to require an anomaly in its axial-vector WI—without any in its vector WI.[2] This provides a resolution of the π^0-decay puzzle[1] and requires $b_1=0$ in Eq. (55). Additional experimental information is needed in order to determine the specific values for the other b_i; in electrodynamics, however, gauge invariance would require the set (58) given by Bardeen. It should be stated that the higher-order WI's and their anomalies can be found by contracting on the first-order set enumerated in Sec. IV. Generalizing the free-quark model to include nonconserved strangeness-changing vector currents, we discovered no change in the anomalies found in the exact-symmetry case. This generalization presented in Sec. VI yields no freedom with which one could modify our Δ's.

Since only the abnormal-parity n-pf's for $n\leq 5$ enter into anomalous (first-order) WI's, the kinematic structure of an n-pf seems to be at the heart of the matter. This has been noted by Wilson in a calculation comparing the WI's for $\langle AVV\rangle$ and $\langle AAV\rangle$.[8]

For a concluding remark, we mention that the coefficients of the anomalies we found are, of course, model-dependent. For example, the quark-model result predicts a π^0 lifetime about nine times smaller than the accepted experimental value; a successful prediction, however, can be obtained in the Han-Nambu model.[27]

Note added in proof. C. R. Hagen has pointed out recently [Phys. Rev. (to be published)] that the ϵ separation method has certain inconsistencies; consequently, he has advocated that a particular form of the regularization technique is the appropriate one to employ. We have also discovered work by W.-K. Tung [Phys. Rev. (to be published)] and by D. Amati, C. Bouchiat, and J.-L. Gervais [Orsay Report (unpublished)] in which Ward-identity anomalies have been investigated using regularization.

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[26] After the completion of this work, it has come to our attention that C. W. Kim, W. W. Repko, and A. Sato [Johns Hopkins Report (unpublished)], as well as J.-L. Gervais and B. W. Lee [Orsay Report (unpublished)], have discussed the inadequacies of the usual regularization in the σ model.

[27] M. Y. Han and Y. Nambu, Phys. Rev. **139B**, 1006 (1965). In this connection see S. Okubo, *ibid.* **179**, 1629 (1969), and Appendix B of Ref. 2.

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APPENDIX A

This appendix is divided into three sections. The momentum integrals which arise in our n-pf representations are evaluated in (i). The displacement "surface terms" for the $n=1$, 2, and 3 cases are given next in (ii); from which it follows that the regularization described in the text does indeed remove these ambiguities. Finally, in (iii) we exhibit the mass expansions of the loop integrals singling out the $\ln m^2$ and m^{-1} terms so vital to our work.

(i) Momentum Integrals

In momentum space, the integration region is considered to be large but finite if there are ultraviolet divergences present. In particular, we follow the method of Akhiezer and Berestetsky[28] in which the finite invariant symmetric region is denoted by N. The reader is referred to Appendix III of their book for the details but is reminded here that *a change of variables also changes* N. Note that we can obtain all of the momentum integrals needed below by taking the appropriate derivatives of the results in Ref. 28 and that the usual Feynman parametrization is put to good use.

The one-vertex loop [see Eq. (15)] with fermion mass m is found to be[29]

$$B_1(a)\equiv B_1{}^0(a)\equiv\int_{l,(N)}\mathrm{Tr}[\Gamma_1 S(l+a)]=(16\pi^2 i)^{-1}\times\mathrm{Tr}\{\Gamma_1[m^3(\ln m^2+A_0)-m^3-mA_2+\tfrac{1}{2}ma^2]-\tfrac{1}{2}\Gamma_1 \boldsymbol{a}[A_2+m^2-\tfrac{1}{3}a^2]\}, \quad (A1)$$

where

$$S(k)\equiv S_0(k)\equiv(\boldsymbol{k}-m)^{-1}.$$

Here $A_0=A_0(N)$ and $A_2=A_2(N)$ are "constants" which diverge logarithmically and quadratically, respectively, as N is expanded to infinity.

The two-vertex spinor loop, following the notation of Eqs. (18) and (A1), can be expressed as

$$D_{12}(a,b)\equiv\int_{l,(N)}\mathrm{Tr}[\Gamma_1 S(l+a)\Gamma_2 S(l+b)]=(16\pi^2 i)^{-1}\int_0^1 dx\,\mathrm{Tr}\{\Gamma_1 P_x(a)\Gamma_2 P_x(b)(\ln D_x+A_0)+\tfrac{1}{2}\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha[D_x(\ln D_x-\tfrac{1}{2}+A_0)-\tfrac{1}{2}A_2+\tfrac{1}{3}p_x{}^2]+\tfrac{1}{2}\Gamma_1 P_x(a)\Gamma_2 \boldsymbol{p}_x+\tfrac{1}{2}\Gamma_1\boldsymbol{p}_x\Gamma_2 P_x(b)-\tfrac{1}{6}\Gamma_1\boldsymbol{p}_x\Gamma_2\boldsymbol{p}_x\}. \quad (A2)$$

We have introduced some convenient notation which will also be of value later:

$$P_t(k)\equiv\boldsymbol{p}_t+\boldsymbol{k}+m,\quad D_t\equiv\Delta_t+p_t{}^2, \quad (A3)$$

where in the case of (A2)

$$p_x\equiv-ax-b(1-x),\quad \Delta_x\equiv m^2-a^2x-b^2(1-x). \quad (A4)$$

For the three-vertex loop we write the $i=0$ term of Eq. (28) as

$$F_{123}(a,b,c)\equiv\int_{l,(N)}\mathrm{Tr}[\Gamma_1 S(l+a)\Gamma_2 S(l+b)\Gamma_3 S(l+c)]=(16\pi^2 i)^{-1}\int_0^1 ydydx\,\mathrm{Tr}\{\Gamma_1 P_y(a)\Gamma_2 P_y(b)\Gamma_3 P_y(c)D_y{}^{-1}+\tfrac{1}{2}[\Gamma_1 P_y(a)\Gamma_2\gamma_\alpha\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2 P_y(b)\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3 P_y(c)](\ln D_y+\tfrac{1}{2}+A_0)+\tfrac{1}{6}[\Gamma_1\boldsymbol{p}_y\Gamma_2\gamma_\alpha\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2\boldsymbol{p}_y\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3\boldsymbol{p}_y]\}, \quad (A5)$$

where in terms of (A4),

$$p_y\equiv p_x y-c(1-y),\quad \Delta_y\equiv\Delta_x y+(m^2-c^2)(1-y). \quad (A6)$$

The four-vertex amplitude [see Eq. (40)] is

$$H_{1234}(a,b,c,d)\equiv\int_{l,(N)}\mathrm{Tr}[\Gamma_1 S(l+a)\Gamma_2 S(l+b)\Gamma_3 S(l+c)\Gamma_4 S(l+d)]=(16\pi^2 i)^{-1}\int_0^1 z^2dzydydx\times\mathrm{Tr}\{-\Gamma_1 P_z(a)\Gamma_2 P_z(b)\Gamma_3 P_z(c)\Gamma_4 P_z(d)D_z{}^{-2}+\tfrac{1}{2}[\Gamma_1 P_z(a)\Gamma_2 P_z(b)\Gamma_3\gamma_\alpha\Gamma_4\gamma^\alpha+\Gamma_1 P_z(a)\Gamma_2\gamma_\alpha\Gamma_3 P_z(c)\Gamma_4\gamma^\alpha+\text{four more perms.}]D_z{}^{-1}+\tfrac{1}{4}(g^{\alpha\beta}g^{\delta\epsilon}+g^{\alpha\delta}g^{\beta\epsilon}+g^{\alpha\epsilon}g^{\beta\delta})\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\gamma_\epsilon(\ln D_z+\tfrac{5}{6}+A_0)\}, \quad (A7)$$

with

$$p_z\equiv p_y z-d(1-z),\quad \Delta_z\equiv\Delta_y z+(m^2-d^2)(1-z), \quad (A8)$$

[28] A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (State Technico-Theoretical Literature Press, Moscow, 1953) (English translation by Consultants Bureau Enterprises, Inc., New York, 1953).

[29] We do not require four-momentum conservation at the vertices of our loops; this enables us to examine the ambiguities due to shifts of the integration variables.

in terms of (A4) and (A6). Finally, following Eq. (44), the five-vertex loop is (where we may allow $N\to\infty$)

$$J_{12345}(a,b,c,d,e)\equiv\int_l \mathrm{Tr}[\Gamma_1S(l+a)\Gamma_2S(l+b)\Gamma_3S(l+c)\Gamma_4S(l+d)\Gamma_5S(l+e)]=(16\pi^2i)^{-1}\int_0^1 w^3dw\, z^2dzydydx$$
$$\times\mathrm{Tr}\{2\Gamma_1P_w(a)\Gamma_2P_w(b)\Gamma_3P_w(c)\Gamma_4P_w(d)\Gamma_5P_w(e)D_w^{-3}-\tfrac{1}{2}[\Gamma_1P_w(a)\Gamma_2P_w(b)\Gamma_3P_w(c)\Gamma_4\gamma_\alpha\Gamma_5\gamma^\alpha$$
$$+\Gamma_1P_w(a)\Gamma_2P_w(b)\Gamma_3\gamma_\alpha\Gamma_4P_w(d)\Gamma_5\gamma^\alpha+\text{eight more perms.}]D_w^{-2}+\tfrac{1}{4}(g^{\alpha\beta}g^{\delta\epsilon}+g^{\alpha\delta}g^{\beta\epsilon}+g^{\alpha\epsilon}g^{\beta\delta})[\Gamma_1P_w(a)\Gamma_2\gamma_\alpha\Gamma_3\gamma_\beta$$
$$\times\Gamma_4\gamma_\delta\Gamma_5\gamma_\epsilon+\Gamma_1\gamma_\alpha\Gamma_2P_w(b)\Gamma_3\gamma_\beta\Gamma_4\gamma_\delta\Gamma_5\gamma_\epsilon+\text{three more perms.}]D_w^{-1}\}\,,\quad\text{(A9)}$$

where, compounding (A4), (A6), and (A8),

$$p_w\equiv p_zw-e(1-w),\quad \Delta_w\equiv\Delta_zw+(m^2-e^2)(1-w).\quad\text{(A10)}$$

(ii) Loop Integration Displacements

We concern ourselves here with the change in values of our loops corresponding to a change in the integration variable l *but with the same integration region N*. The change $l\to l+s$ inside the integrals of (A1), (A2), (A5), (A7), and (A9) is equivalent to the changes $a\to a+s$, $b\to b+s$, etc., as the case may be. The one-vertex displacement ambiguity is then from (A1)

$$B_1(a+s)-B_1(a)=(16\pi^2i)^{-1}\,\mathrm{Tr}\{\tfrac{1}{2}\Gamma_1ms\cdot(s+2a)+\tfrac{1}{6}\Gamma_1\boldsymbol{a}s\cdot(s+2a)-\tfrac{1}{2}\Gamma_1\boldsymbol{s}[A_2+m^2-\tfrac{1}{3}(s+a)^2]\}\,,\quad\text{(A11)}$$

which, although infinite as $N\to\infty$, will be removed under regularization.

A shift of s in the two-vertex loop (A2) results in the surface term

$$D_{12}(a+s,\,b+s)-D_{12}(a,b)=-\tfrac{1}{6}(16\pi^2i)^{-1}\times\mathrm{Tr}[\Gamma_1\boldsymbol{s}\Gamma_2(\tfrac{1}{2}\boldsymbol{s}-\boldsymbol{a}+2\boldsymbol{b}+3m)+\Gamma_1(\tfrac{1}{2}\boldsymbol{s}+2\boldsymbol{a}-\boldsymbol{b}+3m)\times\Gamma_2\boldsymbol{s}-\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha s\cdot(s+a+b)],\quad\text{(A12)}$$

also removed by regularization. For $a=0$, $b=p$ we agree with Eq. (33) of GJ. There is also a displacement problem for the three-vertex term (A5):

$$F_{123}(a+s,\,b+s,\,c+s)-F_{123}(a,b,c)=-\tfrac{1}{12}(16\pi^2i)^{-1}\times\mathrm{Tr}(\Gamma_1\boldsymbol{s}\Gamma_2\gamma_\alpha\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2\boldsymbol{s}\Gamma_3\gamma^\alpha+\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3\boldsymbol{s}).\quad\text{(A13)}$$

This difference also agrees with GJ [see their Eq. (29)] modulo a factor in their loop definition. Regularized 3-pf's will evidently be shift-invariant.

The n-pf integrals for the cases $n\geq4$ have no displacement ambiguities. For example, (A7) is only logarithmically divergent, so that one expects no shift trouble in the four-vertex loop. Since loops with more vertices are not even divergent, the displacement problem associated with the linear and quadratic divergences is expected not to be present for $n\geq4$. In any case we may see this quantitatively by noting, as an example, that combinations like p_z+a, p_z+b, and $\Delta_z+p_z^2$ in (A7) are invariant under $a\to a+s$, $b\to b+s$, etc.

(iii) Mass Expansions

This section is devoted to displaying the $\ln m^2$ and m^{-1} terms of the previously written loops which are required in the anomaly calculations. We begin by observing that the one-vertex loop has no m^{-1} terms, but that its mass expansion is seen from (A1) to be of the form

$$B_1(a)=(16\pi^2i)^{-1}\,\mathrm{Tr}[b_1mA_2+b_2A_2+b_3m^3(\ln m^2+A_0)+b_4m^3+b_5m^2+b_6m+b_7],\quad\text{(A14)}$$

where the relevant coefficient is

$$b_3=\Gamma_1.\quad\text{(A15)}$$

Equation (A2) expands as

$$D_{12}(a,b)=(16\pi^2i)^{-1}\,\mathrm{Tr}[d_1A_2+d_2m^2(\ln m^2+A_0)+d_3m(\ln m^2+A_0)+d_4(\ln m^2+A_0)+d_5m^2+d_6m+d_7+d_8/m+O(1/m^2)],\quad\text{(A16)}$$

with the coefficients of interest being

$$\begin{aligned}d_2&=\Gamma_1\Gamma_2+\tfrac{1}{2}\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha,\\ d_3&=\tfrac{1}{2}[\Gamma_1(\boldsymbol{a}-\boldsymbol{b})\Gamma_2-\Gamma_1\Gamma_2(\boldsymbol{a}-\boldsymbol{b})],\\ d_4&=-\tfrac{1}{6}[\Gamma_1(\boldsymbol{a}-\boldsymbol{b})\Gamma_2(\boldsymbol{a}-\boldsymbol{b})+\tfrac{1}{2}(a-b)^2\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha],\\ d_8&=-\tfrac{1}{6}(a-b)^2d_3.\end{aligned}\quad\text{(A17)}$$

Note that an expansion like (A16) automatically separates out A_2 and A_0, since their presence is related to $m\to\infty$ divergence of our loops. That is to say, the $O(m^{-2})$ terms are independent of A_0 and A_2.

Similarly, from (A5) we have

$$F_{123}(a,b,c)=(32\pi^2i)^{-1}\,\mathrm{Tr}[f_1m(\ln m^2+A_0)+f_2(\ln m^2+A_0)+f_3m+f_4+f_5/m+O(1/m^2)],\quad\text{(A18)}$$

where we need

$$\begin{aligned}f_1&=\tfrac{1}{2}(\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3+\Gamma_1\gamma_\alpha\Gamma_2\Gamma_3\gamma^\alpha+\Gamma_1\Gamma_2\gamma_\alpha\Gamma_3\gamma^\alpha),\\ f_2&=\tfrac{1}{6}[\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3(2\boldsymbol{c}-\boldsymbol{a}-\boldsymbol{b})+\Gamma_1\gamma_\alpha\Gamma_2(2\boldsymbol{b}-\boldsymbol{a}-\boldsymbol{c})\Gamma_3\gamma^\alpha+\Gamma_1(2\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c})\Gamma_2\gamma_\alpha\Gamma_3\gamma^\alpha],\\ f_5&=\tfrac{1}{6}(a^2+b^2+c^2-a\cdot b-a\cdot c-b\cdot c)(\Gamma_1\Gamma_2\Gamma_3-f_1)+\tfrac{1}{12}(a^\alpha a^\beta+b^\alpha b^\beta+c^\alpha c^\beta+a^\alpha b^\beta+a^\alpha c^\beta+b^\alpha c^\beta)\\&\quad\times[\{\Gamma_1,\gamma_\alpha\}\Gamma_2\gamma_\beta\Gamma_3+\Gamma_1\{\Gamma_2,\gamma_\alpha\}\Gamma_3\gamma_\beta+\Gamma_1\gamma_\beta\Gamma_2\{\Gamma_3,\gamma_\alpha\}]+\Gamma_1\boldsymbol{a}\Gamma_2\boldsymbol{b}\Gamma_3+\Gamma_1\boldsymbol{a}\Gamma_2\Gamma_3\boldsymbol{c}+\Gamma_1\Gamma_2\boldsymbol{b}\Gamma_3\boldsymbol{c}\\&\quad+\Gamma_1\boldsymbol{a}\Gamma_2\{\Gamma_3,\gamma\cdot\Sigma_f\}+\{\Gamma_1,\gamma\cdot\Sigma_f\}\Gamma_2\boldsymbol{b}\Gamma_3+\Gamma_1\{\Gamma_2,\gamma\cdot\Sigma_f\}\Gamma_3\boldsymbol{c}\,,\end{aligned}\quad\text{(A19)}$$

with

$$\Sigma_f \equiv -\tfrac{1}{3}(a+b+c).$$

Going on to (A7), the expansion is

$$H_{1234}(a,b,c,d)=(96\pi^2 i)^{-1} \times \mathrm{Tr}[h_1(\ln m^2+A_0)+h_2+h_3/m+O(1/m^2)], \quad \text{(A20)}$$

where

$$h_1=\tfrac{1}{4}(g^{\alpha\beta}g^{\delta\epsilon}+g^{\alpha\delta}g^{\beta\epsilon}+g^{\alpha\epsilon}g^{\beta\delta})\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\gamma_\epsilon, \quad \text{(A21)}$$

$$h_2=-\tfrac{1}{2}[h_{1234}(a)+h_{2341}(b)+h_{3412}(c)+h_{4123}(d)],$$

in which

$$h_{ijkn}(p)\equiv\Gamma_i(\not p+\Sigma_h)\Gamma_j[2\Gamma_k\Gamma_n-\gamma_\alpha\Gamma_k\gamma^\alpha\Gamma_n -\gamma_\alpha\Gamma_k\Gamma_n\gamma^\alpha-\Gamma_k\gamma_\alpha\Gamma_n\gamma^\alpha],$$

$$\Sigma_h\equiv-\tfrac{1}{4}(a+b+c+d).$$

The five-vertex loop (A9) yields

$$J_{12345}(a,b,c,d,e)=(384\pi^2 i)^{-1}\,\mathrm{Tr}[j_1/m+O(1/m^2)], \quad \text{(A22)}$$

with

$$j_1=2\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5-\tfrac{1}{2}[\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3\Gamma_4\Gamma_5 +\Gamma_1\gamma_\alpha\Gamma_2\Gamma_3\gamma^\alpha\Gamma_4\Gamma_5+\text{eight more perms.}] +\tfrac{1}{4}(g^{\alpha\beta}g^{\delta\epsilon}+g^{\alpha\delta}g^{\beta\epsilon}+g^{\alpha\epsilon}g^{\beta\delta})[\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\gamma_\epsilon\Gamma_5 +\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\Gamma_5\gamma_\epsilon+\text{three more perms.}]. \quad \text{(A23)}$$

Our final remark is that the loops with six or more vertices are $O(m^{-2})$.

APPENDIX B

In this appendix we shall describe the calculation which has led us to the conclusions in Sec. VI. Equation (77) is crucial here since the nondegenerate-mass n-pf's can be related through its use to a series of degenerate-mass amplitudes, allowing the arguments of Secs. II–V to then be applied term by term.

The starting point is Eq. (72). This expression can be written explicitly in terms of S_p and S_Λ by (74); in turn, the latter propagator is expressed as an infinite series in the former via (77). Thus each n-pf is an infinite series in loop integrals all with the same mass m_p circulating through them. These loops can then be regularized according to the text discussion with, say, UR. That is, a finite sum of regulator loops (each with a large mass m_i) is added to each m_p loop and, furthermore, δm is considered as an external parameter. This can be shown to be the same as regularizing the original nondegenerate n-pf by assigning the regulator mass m_i to m_p and $m_i+\delta m$ to m_Λ.

Let us consider, in more detail, these ingredient *degenerate-mass loops*. In order to develop some notation, the unregularized degenerate-mass n-pf may be expressed in the following fashion:

$$\langle j_1^{a_1}(k_1)j_2^{a_2}(k_2)\cdots j_n^{a_n}(k_n)\rangle_{\text{degenerate mass}} = -\tfrac{1}{2}(\tfrac{1}{2}i)^n\{I^{a_1,a_2,\cdots,a_n}\bar{L}_{1,2,\dots,n}(k_2,k_3,\cdots,k_n) +\text{synchronized perms. of } a_2,\cdots,a_n;\ 2,\cdots,n;\ \times k_2,\cdots,k_n,\text{ such that there are } \times\tfrac{1}{2}[(n-1)!]\text{ terms in all}\}, \quad \text{(B1)}$$

with the internal-symmetry term

$$I^{a_1,a_2,\cdots,a_n}\equiv\mathrm{Tr}[\lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_n} +c_1c_2\cdots c_n\lambda_{a_1}{}^T\lambda_{a_2}{}^T\cdots\lambda_{a_n}{}^T] \quad \text{(B2)}$$

and its loop-momentum-integral coefficient

$$\bar{L}_{1,2,\dots,n}(k_2,k_3,\cdots,k_n) \equiv\int_l \mathrm{Tr}[\Gamma_1 S(l+k_2)\Gamma_2 S(l)\Gamma_3 S(l-k_3) \cdots\Gamma_n S(l-k_3-\cdots-k_n)+c_1c_2\cdots c_n\Gamma_1 S(l+k_n) \times\Gamma_n S(l)\Gamma_{n-1}S(l-k_{n-1}) \cdots\Gamma_2 S(l-k_{n-1}-\cdots-k_2)]. \quad \text{(B3)}$$

Through the help of a lemma, the description of the NWI's for the n-pf (B1) can be made in terms of these quantities I and $\bar{L}$. Before stating this lemma, we make the observation that each of the $(n-1)$-pf's on the right-hand side of an n-pf NWI is always multiplied by an f or d coefficient [see Eq. (3)]. Therefore the commutation relations

$$2if^{abc}\lambda^c=[\lambda^a,\lambda^b],\quad 2d^{abc}\lambda^c=\{\lambda^a,\lambda^b\} \quad \text{(B4)}$$

imply that the internal variables on that side of the equation can each be written as a sum of two I's that also occur on the left-hand side.

Now we can make good use of the following lemma: After a momentum contraction on (B1), the coefficient $\bar{L}$ (B3) of a given internal variable term I (B2) goes over to the coefficient of the same I which arises on the right-hand side of the associated NWI—aside from the possible need for a shift in the integration variable. The proof of this statement is straightforward.

As a result of the preceding discussion, we may shift our attention to the WI's for the individual $\bar{L}$'s. Indeed, in practice the removal of an anomaly in a given WI (see Sec. IV) amounts to the removal of the anomalies in the individual $\bar{L}$ WI's. The counterterms in Eqs. (51)–(55) can be looked upon as sums of I's (e.g., the W's of Sec. III are I's for the case $n=4$), each I having a coefficient which can be interpreted as a modification of $\bar{L}$.

At this point, we return to the nondegenerate-mass n-pf and note that in its δm expansion, the internal variables now differ from the I's. We obtain a degenerate-mass n-pf plus an infinite series in

$$(\delta m)^{l-n}\bar{L}_{1,2,\dots,l}(k_2,k_3,\cdots,k_n), \quad \text{(B5)}$$

with $l \geq n+1$. This expression (B5) is a sum of two loops corresponding to an original degenerate-mass pair with the addition of $l-n$ scalar vertices, each with zero momentum transfer k_i. A detailed examination shows that a lemma analogous to the previous one holds for the nondegenerate-mass case, and thus if each $\bar{L}$ in (B5) satisfies its WI then the corresponding WI for n-pf is also naïve.

In searching for WI anomalies with respect to nondegenerate n-pf's, we note that in their δm expansions, those loops with $l \geq 6$ already satisfy their individual WI's; those loops with $l \leq 5$ can be modified according to the Eqs. (51)–(55). Our previous minimal solution in the degenerate-mass case has been examined term by term in $\bar{L}$. It therefore guarantees our minimal solution again in the present nondegenerate case. It is worthwhile to note that the linear independence of the W's and Z's (introduced in Sec. III) makes the term-by term balancing of individual $\bar{L}$ WI's easily understood.

Let us attempt to clarify what has been said so far. In the *normal*-parity case, each $\bar{L}$ either satisfies its NWI or can be modified to do so; therefore, by our preceding arguments, the corresponding nondegenerate-mass n-pf's can be defined so as to satisfy their WI's. The anomalies present in the *abnormal* nondegenerate-mass WI's are the same as those in the degenerate case. This is true because an abnormal-parity loop with any scalar vertices has no m^{-1} terms, and thus the δm expansion shows that no new anomalies are introduced by the mass breaking.

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Quark Model and the Pomeranchuk Theorem*

YUKIO TOMOZAWA

Randall Laboratory of Physics, The University of Michigan, Ann Arbor, Michigan 48104

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It has been pointed out that the quark model may not be compatible with the Pomeranchuk theorem. We investigate one such model in more detail and discuss the implications of violation of the Pomeranchuk theorem, if any.

I. INTRODUCTION

The equal-time commutation relation (ETCR) of the axial-vector currents has led to the Adler-Weisberger sum rule,[1] upon using the partially conserved axial-vector current (PCAC) assumption[2] and the infinite-momentum technique.[3] Applying a similar operation on the matrix element of the ETCR which involves the divergence of the axial-vector current, and using the subtraction method that has been described in Ref. 4, one can derive a superconvergence sum rule[5] for the zero-mass-pion nucleon scattering amplitude. In particular, it has been pointed out[4] that the ETCR of pion fields which are considered as bound states of the quark-antiquark system may be incompatible with the Pomeranchuk theorem[6] (P theorem). In this paper, we discuss such a possibility further and elaborate on its experimental implications.

In Sec. II, the model is specified, and in Sec. III, we show that the violation of the P theorem is related to the nonvanishing bare masses of quarks. Section IV is devoted to a discussion concerning the validity of the P theorem.

II. QUARK MODEL

We consider a quark model in which the interaction Lagrangian respects the $SU(3)\times SU(3)$ symmetry and the violation of the symmetry is due only to the mass terms. This is the model that has been discussed by several authors.[7–11]

The interaction Lagrangian may be taken as[12,13]

$$\begin{aligned}-\mathfrak{L}_I = {} & g_V\bar{\psi}i\gamma_\mu\psi\bar{\psi}i\gamma_\mu\psi + g_A\bar{\psi}i\gamma_\mu\gamma_5\psi\bar{\psi}i\gamma_\mu\gamma_5\psi \\ & + G(\bar{\psi}i\gamma_\mu\lambda_j\psi\bar{\psi}i\gamma_\mu\lambda_j\psi - \bar{\psi}i\gamma_\mu\gamma_5\lambda_j\psi\bar{\psi}i\gamma_\mu\gamma_5\lambda_j\psi) \\ & + G'(\bar{\psi}\lambda_j\psi\bar{\psi}\lambda_j\psi - \bar{\psi}\gamma_5\lambda_j\psi\bar{\psi}\gamma_5\lambda_j\psi \\ & + \bar{\psi}\lambda_0\psi\bar{\psi}\lambda_0\psi - \bar{\psi}\gamma_5\lambda_0\psi\bar{\psi}\gamma_5\lambda_0\psi)\,, \quad (1)\end{aligned}$$

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