Modified Continuous-Moment Sum Rule and the Pomeranchon*

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A modified form of the continuous-moment sum rule is employed to investigate whether or not the Pomeranchon intercept α_P, deviates from its maximal value of unity in the forward direction. This sum rule contains a continuously varying power of the amplitude, in addition to the usual continuously-varying moment. Two particular cases, corresponding to the first and the second powers of the amplitude, are analyzed in terms of unconstrained three-pole models. The two resulting solutions agree within the errors. They have essentially the same value of α_P, viz., 0.988 ± 0.01. Both give excellent fits to high-energy data The value quoted above is favored over unity, although it is consistent with unity.

I. INTRODUCTION

WHETHER or not the Pomeranchuk trajectory $\alpha_P(t)$, where t is the negative square of momentum transfer, attains the maximal intercept of unity in the forward direction ($t=0$) has many important consequences. If $\alpha_P(0)$ (hereafter denoted by α_P) equals unity, then we can understand several experimentally indicated features of high-energy hadron physics, such as the constancy of all total cross sections, the equality of the high-energy limits of all particle-antiparticle total cross sections for a common target, and the purely imaginary nature of all forward elastic scattering amplitudes, among others.[1] However, it has not been possible to verify experimentally some of the predictions following from the assumption of maximal intercept for the Pomeranchon. The dynamical mechanism underlying this assumption is also far from clear. Arguments based on crossing matrices suggest that the Pomeranchon should dominate all other trajectories and the Froissart bound[2] prevents any trajectory above $J=1$ in the forward direction, but there is no theoretical reason why the Pomeranchon intercept cannot be less than unity.[3] As a matter of fact, Cabibbo *et al.* have argued that the present experimental evidence does not rule out the possibility of the intercept being considerably less than unity.[4] These authors and Meshkov and Yodh[5] have given two-pole models with $\alpha_P=0.93$. However, Olsson and Yodh[6] have shown that all existing two-pole models are inconsistent with their continuous-moment sum rule (hereafter abbreviated as CMSR). On the other hand, the only two three-pole models which survive their test are constrained to have $\alpha_P=1$.[6]

Interest in this question was revived recently by Chew and Frazer who have proposed the value 0.99 for α_P.[7] In view of the importance of this question, a thorough investigation seems to be in order. In this paper, we consider this question within the framework of pure Regge-pole theory, since no reliable way of evaluating the contribution of cuts exists.

To this end, we write down a modified form of CMSR which contains a continuously varying power of the crossing-even forward amplitude, in addition to a continuously varying moment. This puts an additional degree of freedom at our disposal. Thus, we can test the stability of our solution by comparing two or more independent equations, corresponding to different powers.

The derivation of the sum rule, together with definitions and normalization, is presented in Sec. II. In Sec. III, we describe the evaluation of the real part of the scattering amplitude and its error from an ordinary dispersion relation. The calculation of the finite-energy integrals and their errors are given in Sec. IV. In Sec. V, we analyze the data in terms of one-, two-, and three-pole models. We also discuss constrained and nonconstrained three-pole models and the stability of the solutions. In the Appendix, we apply the phase representation to determine the number of zeros of a given amplitude.[8,9]

II. MODIFIED CONTINUOUS-MOMENT SUM RULE

A. Definitions and Normalizations

Throughout this paper, $F(\omega)$ denotes the forward crossing-even pion-nucleon scattering amplitude, while ω and q denote the laboratory energy and momentum of the pion. The amplitude $F(\omega)$ is given by

$$F(\omega)=\tfrac{1}{2}[F_{\pi^-p}(\omega)+F_{\pi^+p}(\omega)].$$

It is normalized so that the optical theorem is of the

* Work supported by the U. S. Atomic Energy Commission.

[1] For a complete discussion of this and some of the following points, see G. F. Chew, Comments Nucl. Particle Phys. **1**, 121 (1967).

[2] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[3] In physics "anything which is not prohibited is compulsory," according to Gell-Mann's principle. See O. M. Bilaniuk and E. C. G. Sudarshan, Phys. Today **22**, 43 (1969).

[4] N. Cabibbo, L. Horowitz, J. J. J. Kokkedee, and Y. Ne'eman, Nuovo Cimento **45A**, 275 (1965).

[5] S. Meshkov and G. B. Yodh, Phys. Rev. Letters **19**, 603 (1967).

[6] M. G. Olsson and G. B. Yodh, Phys. Rev. Letters **21**, 1022 (1968). This paper contains a table of references to two-pole and three-pole models.

[7] G. F. Chew and W. R. Frazer, Phys. Rev. **181**, 1914 (1969).

[8] Y. S. Jin and S. W. MacDowell, Phys. Rev. **138**, B1279 (1965).

[9] M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962); Phys. Rev. **130**, 2127 (1963).

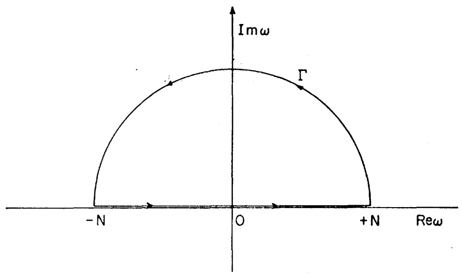

FIG. 1. Finite contour in the upper-half ω plane.

form

$$\mathrm{Im}F(\omega)=\tfrac{1}{2}[\sigma_{\pi^-p}(\omega)+\sigma_{\pi^+p}(\omega)]q=\sigma^{(+)}(\omega)q, \quad (1)$$

where $\sigma_{\pi^\pm p}$ is the $\pi^\pm p$ total cross section. Our definition of the pion-nucleon coupling constant is

$$\frac{f^2}{4\pi}=\frac{g^2}{4\pi}\left(\frac{1}{2M}\right)^2=0.081\pm0.002.$$

The nucleon pole term is given by

$$F_B(\omega)=-2f^2\omega_B/(\omega^2-\omega_B^2),$$

where $\omega_B=1/(2M)$; M is the nucleon mass. Natural units are used everywhere, i.e., $\hbar=c=\mu=1$.

B. Modified Analytic Function and its Analytic Properties

Consider the function

$$g(\omega)=e^{i\pi(\beta-\lambda)}\omega[F(\omega)-F(\mu)-F_B(\omega)+F_B(\mu)]q^{-2\beta}, \quad \beta<1+\tfrac{1}{2}\lambda \quad (2)$$

where β and λ are real continuous variables. The analytic structure of $g(\omega)$ is as follows: It has no poles. Furthermore, the expression in the square brackets in Eq. (2) has exactly two zeros at $\omega=\pm\mu$, as shown in the Appendix. Since λ is a real, continuous variable, this generates branch points in $g(\omega)$ at $\omega=\pm\mu$. Of course, $q^{-2\beta}$ also gives cuts starting at $\omega=\pm\mu$. Thus, the phase factor in the definition of $g(\omega)$ guarantees that $g(\omega)$ is real analytic in the cut ω plane. The restriction $\beta<1+\frac{1}{2}\lambda$ comes from the threshold behavior $q^{\lambda-2\beta}$ of $g(\omega)$, since we want to avoid kinematic poles there.

C. Derivation of the Sum Rule

If we integrate $g(\omega)$ around the closed, finite contour shown in Fig. 1, we get

$$\int_{-N}^{+N} g(\omega)d\omega+\int_\Gamma g(\omega)d\omega=0. \quad (3)$$

In order to evaluate the second term in Eq. (3), we must know $g(\omega)$ for complex ω. For real $\omega\geq N$, we may assume the Regge asymptotic form $g_R(\omega)$ for $g(\omega)$, if N is sufficiently large. It is usually assumed without proof that the same expression $g_R(\omega)$ holds for ω real as well as complex for $|\omega|\geq N$. Here, we shall prove that this equality holds for complex ω also, provided it is understood in the sense described below.[10] We assume the following:

(1) $g(\omega)=g_R(\omega)$, where $g_R(\omega)$ is the Regge asymptotic limit of $g(\omega)$ for *real* $\omega\geq N$.

(2) There are no poles in $g(\omega)$ in the region $|\omega|\geq N$.

If $G(\omega)=g(\omega)-g_R(\omega)$, then integration around the contour in Fig. 2 gives

$$\int_\Gamma G(\omega)d\omega+\int_{-N}^{-R} G(\omega)d\omega +\int_{\Gamma'} G(\omega)d\omega+\int_R^N G(\omega)d\omega=0. \quad (4)$$

Now, by assumption (1), the integrals on the real axis vanish, and if we let Γ' recede to infinity, the Sugawara-Kanazawa theorem enables us to set the Γ' integral equal to zero, too.[11] So we are left with

$$\int_\Gamma g(\omega)d\omega=\int_\Gamma g_R(\omega)d\omega. \quad (5)$$

That is, although $g(\omega)$ and $g_R(\omega)$ may not be equal for complex ω for $|\omega|\geq N$, their integrals are. In other words, as long as the equality is understood in the sense of Eq. (5), $g(\omega)$ and $g_R(\omega)$ are the same for all real and complex ω, provided $|\omega|\geq N$. The rigorous derivation of Eq. (5) does away with the necessity of deriving finite-energy sum rules and CMSR's by assuming superconvergence.[12,13] The latter requires infinite contours and often leads to spurious restrictions on the continuously-varying moment.

Now we can evaluate the second term in Eq. (3), assuming the following asymptotic form for $F(\omega)$:

$$F(\omega)=-\sum_j \frac{\gamma_j e^{-i\pi\alpha_j/2}}{\omega_0 \sin\frac{1}{2}\pi\alpha_j}\left(\frac{\omega}{\omega_0}\right)^{\alpha_j} \quad \text{for } \omega\geq N. \quad (6)$$

$\omega_0=\frac{1}{2}M$ and j run through all the allowed Regge trajectories.

In the following, we shall assume a three-pole model, unless stated otherwise. The Regge trajectories that can be exchanged for the crossing-even amplitude $F(\omega)$ are then the P, P', and P''. In order to evaluate the second term in Eq. (3), we substitute Eqs. (5) and (6) into Eq. (3), and note that the experimental data are con-

[10] This proof was completed with the help of Professor M. Sugawara.
[11] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961).
[12] M. G. Olsson, Phys. Rev. **171**, 1681 (1968).
[13] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

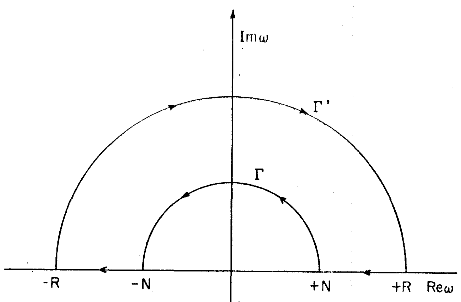

FIG. 2. Contour used to derive Eq. (5).

sistent with $F(\mu)=0$. If we define

$$A_j=\frac{\gamma_j e^{-i\pi\alpha_j/2}}{\omega_0\sin\frac{1}{2}\pi\alpha_j}(\omega/\omega_0)^{\alpha_j}\,,\quad j=1,\,2,\,3$$

$$B(\omega)=F_B(\omega)-F_B(\mu)\,, \tag{7}$$

then for all reasonable values of the P, P', P'' parameters, $1\gg|A_2/A_1|\gg|A_3/A_1|\gg|B/A_1|$. Therefore, we can expand $g(\omega)$ in a convergent binomial series. This enables us to perform the integration around the finite contour Γ. We are led to the following sum rule:

$$I(\beta,\lambda)=\frac{1}{N}\int_0^N dq\left(\frac{q}{N}\right)^{1-2\beta}\mathrm{Im}\{e^{i\pi(\beta-\lambda)}[F(\omega)-B(\omega)]^\lambda\}$$

$$=\sum_{l,m,n}\frac{\Gamma(\lambda+1)\Lambda_1{}^{\lambda-l}\Lambda_2{}^{l-m}\Lambda_3{}^{m-n}b^n}{\Gamma(\lambda-l+1)\Gamma(l-m+1)\Gamma(m-n+1)\Gamma(n+1)}$$

$$\times\left(\frac{N}{\omega_0}\right)^x\frac{\sin\frac{1}{2}\pi(2\beta-x)}{2-2\beta+x}\,,\quad \beta<1+\tfrac{1}{2}\lambda. \tag{8}$$

l, m, and n are integers and

$$x=\alpha_1(\lambda-l)+\alpha_2(l-m)+\alpha_3(m-n)\,,$$

$$B(\omega)=F_B(\omega)-F_B(\mu)=bq^2/(\omega^2-\omega_B{}^2)\,,$$

$$b=2f^2\omega_B/(1-\omega_B{}^2)\,,$$

$$\Lambda_j=\gamma_j/\omega_0\sin(\tfrac{1}{2}\pi\alpha_j)\,,\quad j=1,\,2,\,3\,.$$

If λ is fractional, the binomial series will be infinite. However, since it is convergent, it can be terminated after a finite number of terms to give any desired accuracy. Our sum rule Eq. (8) reduces to many previous ones such as that of Igi,[14] Dolen, Horn, and Schmid,[13] and Olsson[15] as particular cases. Although our sum rule is valid for all real λ, the $\lambda=1$ and $\lambda=2$ cases are sufficient for our purposes. So we give them explicitly. The $\lambda=1$ case, which is the ordinary CMSR, gives

$$I_1(\beta)=\frac{1}{N}\int_0^N dq\left(\frac{q}{N}\right)^{1-2\beta}\mathrm{Im}\left[e^{i\pi\beta}\left(\frac{F(\omega)-B(\omega)}{N}\right)\right]$$

$$=\sum_j\frac{\Lambda_j}{N}\left(\frac{N}{\omega_0}\right)^{\alpha_j}\frac{\sin\frac{1}{2}\pi(2\beta-\alpha_j)}{2-2\beta+\alpha_j}+\frac{b\sin\pi\beta}{2N(1-\beta)}\,,$$

$$\beta<\tfrac{3}{2}. \tag{9}$$

The $\lambda=2$ case gives

$$I_2(\beta)=\frac{1}{N}\int_0^N dq\left(\frac{q}{N}\right)^{1-2\beta}\mathrm{Im}\left[e^{i\pi\beta}\left(\frac{F(\omega)-B(\omega)}{N}\right)^2\right]$$

$$=\sum_j\left(\frac{\Lambda_j}{N}\right)^2\left(\frac{N}{\omega_0}\right)^{2\alpha_j}\frac{\sin\pi(\beta-\alpha_j)}{2(1-\beta+\alpha_j)}+\frac{b^2\sin\pi\beta}{2N^2(1-\beta)}$$

$$+\sum_{i\neq j}\frac{2\Lambda_j\Lambda_i}{N^2}\left(\frac{N}{\omega_0}\right)^{\alpha_i+\alpha_j}\frac{\sin\frac{1}{2}\pi(2\beta-\alpha_i-\alpha_j)}{2-2\beta+\alpha_i+\alpha_j}$$

$$+\sum_j\frac{2b\Lambda_j}{N^2}\left(\frac{N}{\omega_0}\right)^{\alpha_j}\frac{\sin\frac{1}{2}\pi(2\beta-\alpha_j)}{2-2\beta+\alpha_j}\,,\quad\beta<2 \tag{10}$$

where $N=5.03$ GeV/c.

On the left-hand side of sum rules (9) and (10), the same amplitude occurs in two different powers and, in each case, the varying moment β weights the data differently. Thus, we are extracting information from the given data more efficiently than it was obtained from previous CMSR analyses.

We note that the sum rules obtained by taking discrete values of β, for a fixed λ, are complementary but not independent, since all moments are not equally effective. Therefore, it is essential to take into account the entire range of β. For a given intercept α_j, we can find a value of λ and a value of β which makes one of the denominators very small. This emphasizes that particular trajectory. As β and λ vary, different trajectories are emphasized. A solution obtained in this way will be a stable one.

In order to evaluate the integrals in our sum rules, we must know both real and imaginary parts of the amplitude. The imaginary parts can be computed from the optical theorem. However, since the data on the real part are scanty, we calculate it from an ordinary dispersion relation, as described in the following section.

III. EVALUATION OF REAL PART AND ITS ERRORS FROM AN ORDINARY DISPERSION RELATION

The real part can be calculated from the dispersion relation

$$\mathrm{Re}F(q^2)=\frac{2f^2\omega_Bq^2}{(\omega^2-\omega_B{}^2)(1-\omega_B{}^2)}+\frac{2q^2}{\pi}P\int_0^\infty\frac{dq'\sigma^{(+)}(q')}{q'^2-q^2}. \tag{11}$$

[14] K. Igi, Phys. Rev. Letters **19**, 76 (1962); Phys. Rev. **130**, 820 (1963).
[15] M. G. Olsson, Nuovo Cimento **57A**, 420 (1968).

For purposes of numerical evaluation, we may rewrite (11) as

$$\mathrm{Re}F(q^2)=\frac{2f^2\omega_B q^2}{(\omega^2-\omega_B{}^2)(1-\omega_B{}^2)}+\frac{2q^2}{\pi}\int_0^{Q_0}\frac{dq'[\sigma^{(+)}(q')-\sigma^{(+)}(q)]}{q'^2-q^2}-\frac{q}{\pi}[\tfrac{1}{2}A-\sigma^{(+)}(q)]\ln\left(\frac{Q_0-q}{Q_0+q}\right)+\frac{Bq^2}{\pi}\int_{Q_0}^{\bar{Q}}\frac{dq'}{q'^c(q'^2-q^2)}+\frac{Bq^2}{\pi\bar{Q}^{1+c}}\sum_{n=0}^{\infty}\frac{(Q_R)^{2n}}{2n+1+c}, \quad (12)$$

where

$$Q_0=8\ \mathrm{GeV}/c,\quad \bar{Q}=5Q_0,\quad Q_R=q/\bar{Q}.$$

The parameters A, B, and c extrapolate the high-energy cross sections as follows[16]:

$$\sigma^{(+)}=\tfrac{1}{2}A+B/2q^c \quad \text{for } q\geq Q_0. \quad (13)$$

The last term in Eq. (12) comes from the series expansion of the integrand of the $\bar{Q}-\infty$ integration. Since $Q_R\leq\frac{1}{5}$, the series converges rapidly and only a few terms are needed. The first integral is evaluated from experimentally measured quantities by means of the piece-wise Simpson rule.

The errors are calculated as follows.[17] First of all, we note that all data are subject to systematic errors and that accurate estimates of these errors do not exist in every case. Therefore, we include only the statistical errors and treat all total cross sections as uncorrelated. However, the parameters A, B, and c are correlated. Now, if $X=X(Z_i,S_j)$ is a function of n independent variables Z_i and m correlated variables S_j, the error in X is given by[17]

$$\delta X=\left[\sum_{i=1}^{n}\left(\frac{\partial X}{\partial Z_i}\delta Z_i\right)^2+\sum_{j,k}\frac{\partial X}{\partial S_j}\frac{\partial X}{\partial S_k}(H^{-1})_{jk}\right]^{1/2}, \quad j,k=1,2,\cdots,m. \quad (14)$$

Here X may be the real part of the amplitude (12), or it may be any complicated function of the total cross sections and the parameters A, B, and c. The H_{jk} is the error matrix which we calculated from the high-energy data and the parameters A, B, and c.[16] Since our calculations incorporate the latest data on πN scattering, they will be very useful for evaluation of sum rules. For this reason, we give $\mathrm{Re}F(\omega)$, $\mathrm{Im}F(\omega)$, and their errors in Table I.

IV. EVALUATION OF SUM RULES AND THEIR ERRORS

In Sec. III, we described the calculation of $\mathrm{Re}F(\omega)$, $\mathrm{Im}F(\omega)$, and their errors. Now we can evaluate $I(\beta,\lambda)$ and its error for given β and λ. For $\lambda=1$, β must be smaller than $\frac{3}{2}$. However, the value $\beta=\frac{3}{2}$ would require a subtraction at the threshold and because of the large errors there, we take β in the range $-2\leq\beta\leq 1$. Similarly, for $\lambda=2$, we take $-2\leq\beta\leq\frac{3}{2}$ instead of $-2\leq\beta\leq 2$. In the ranges mentioned, the integrands are finite everywhere. Then, we use the piece-wise Simpson rule to evaluate $I(\beta,\lambda)$. For calculation of the errors, note that the Simpson rule enables us to write

$$I(\beta,\lambda)=\sum_i (c_i x_i+d_i y_i),$$

where x_i, y_i are the real and imaginary parts of the amplitude at the momentum q_i and c_i, d_i are functions of momenta, N, β, and λ. Now, if the real and imaginary parts were completely independent, the error could be calculated from Eq. (14) with only the first sum retained. It turns out that the errors calculated in this way are unrealistically small. However, since the real part is calculated from the imaginary parts, there is a certain degree of correlation between the two and also between the real parts at different energies. In view of the fact that we have included only the statistical errors which are quite small, we calculate the errors by coherent addition, i.e.,

$$\delta I(\beta,\lambda)=\sum_i (|c_i|\delta x_i+|d_i|\delta y_i).$$

A different evaluation may lead to a somewhat different estimate of errors. However, this will change primarily the χ^2, and will not appreciably effect the solution, since we have taken into account the whole range of β. As we shall see, our solutions derived from the $\lambda=1$ and $\lambda=2$ cases are indeed mutually consistent, indicating strong stability. In Figs. 3 and 4, we have plotted $I_1(\beta)$ and $I_2(\beta)$ with their errors. The continuous curves are calculated from the three-pole model, solutions $C1$ and $C2$ (see Table II). The agreement is excellent.

V. ANALYSIS OF DATA IN TERMS OF REGGE PARAMETERS

A. $\lambda=1$ Case

As described in Sec. IV, we have evaluated $I_1(\beta)$ and its error for 61 values of β in the range $-2\leq\beta\leq 1$. This information can be analyzed in terms of Regge-pole

[16] We use the following experimental data. The scattering lengths: $a_1=0.192\pm0.004$, $a_3=-0.096\pm0.002$ in natural units, from G. Hohler, G. Ebel, and J. Giesere, Z. Physik **180**, 430 (1964). The low-energy data are from V. S. Barashenkov and V. M. Maltsev, Forschr. Physik **9**, 549 (1961). The intermediate-energy data are from A. A. Carter *et al.*, Phys. Rev. **168**, 1457 (1968); A. Diddens *et al.*, Phys. Rev. Letters **10**, 262 (1963); A. Citron *et al.*, Phys. Rev. **144**, 1101 (1966). The high-energy data are from K. J. Foley *et al.*, Phys. Rev. Letters **19**, 330 (1967).

[17] J. Orear, University of California, Lawrence Radiation Laboratory Report No. UCRL-8417, 1958 (unpublished).

TABLE I. Real and imaginary parts of F are tabulated as functions of the pion lab momentum, q. The units of q are GeV/c and the other quantities are in natural units. The real part and its error are calculated from Eq. (12).

q	ReF	ImF
0	0±0.030	0
0.108	0.615±0.114	0.394±0.034
0.115	0.873±0.107	0.368±0.025
0.140	1.661±0.264	0.963±0.068
0.150	2.321±0.184	0.962±0.069
0.168	3.431±0.448	2.061±0.386
0.206	4.440±0.464	4.367±0.537
0.219	5.353±0.596	5.106±0.487
0.237	5.030±0.414	7.770±0.198
0.242	5.065±1.162	7.721±0.367
0.254	5.697±0.381	10.015±0.205
0.271	3.031±0.695	12.516±0.248
0.277	2.153±0.292	12.903±0.156
0.292	0.309±0.331	13.706±0.339
0.303	−1.420±0.297	14.261±0.346
0.320	−4.466±0.397	13.486±0.393
0.337	−5.978±0.324	11.636±1.091
0.353	−6.571±0.757	10.755±0.189
0.385	−7.344±0.296	8.147±0.225
0.424	−6.619±0.397	7.327±0.189
0.454	−6.567±0.427	6.492±0.488
0.565	−4.561±0.347	5.452±0.062
0.573	−4.331±0.252	5.435±0.249
0.595	−3.732±0.687	5.602±0.059
0.711	−2.300±0.478	7.851±0.041
0.812	−1.245±0.237	7.619±0.040
0.931	0.346±0.224	12.020±0.044
1.050	−3.375±0.233	14.920±0.047
1.089	−3.994±0.236	14.214±0.051
1.201	−2.433±0.253	14.481±0.058
1.433	−3.332±0.316	20.079±0.062
1.476	−4.272±0.338	20.440±0.064
1.604	−5.681±0.380	20.400+0.073
1.644	−5.699±0.402	20.366±0.175
1.719	−5.519±0.454	20.659±0.094
1.785	−5.648±0.515	21.221±0.107
1.825	−5.236±0.462	21.310±0.202
1.851	−5.022±0.474	21.826±0.122
2.035	−5.377±0.664	23.853±0.207
2.102	−5.903±0.625	24.952±0.171
2.135	−6.150±0.625	24.853±0.150
2.245	−5.679±0.675	26.472±0.149
2.346	−6.135±0.830	27.612±0.167
2.456	−6.929±1.046	28.726±0.200
2.470	−6.914±0.859	28.748±0.401
2.520	−6.881±0.862	29.342±0.013
2.620	−7.215±0.914	30.259±0.013
2.656	−7.388±0.927	30.660±0.200
2.720	−7.740±0.974	31.093±0.014
2.820	−7.889±1.037	31.868±0.014
2.866	−8.058±1.064	32.291±0.212
2.920	−8.229±1.107	32.631±0.015
3.020	−8.394±1.179	33.407±0.015
3.067	−8.427±1.214	33.746±0.247
3.120	−8.457±1.267	34.262±0.012
3.220	−8.836±1.339	35.151±0.012
3.277	−8.794±1.384	35.486±0.194
3.320	−8.777±1.425	36.009±0.013
3.420	−9.042±1.508	36.900±0.013
3.520	−9.199±1.596	37.809±0.013
3.620	−9.419±1.688	38.704±0.014
3.687	−9.636±1.754	39.303±0.180
3.720	−9.718±1.787	39.516±0.014
3.820	−9.820±1.880	40.417±0.010
3.930	−10.089±1.990	41.364±0.010
4.030	−10.601±2.094	42.163±0.010
4.107	−10.237±2.175	42.526±0.251
4.130	−10.209±2.218	42.973±0.010
4.230	−10.779±2.311	43.740±0.011
4.330	−10.764±2.419	44.571±0.011
4.430	−10.911±2.533	45.355±0.011
4.530	−11.024±2.650	46.153±0.012
4.630	−11.127±2.770	46.965±0.012
4.730	−11.247±2.892	47.805±0.012
4.830	−11.370±3.018	48.615±0.012
4.930	−11.494±3.146	49.445±0.013
5.030	−11.613±3.277	50.242±0.013

models. First, we consider the one-pole model. We minimize the difference between the two sides of Eq. (9) with respect to the trajectory and the residue of the Pomeranchon. We find a unique solution with $\alpha_P(0)=0.83$ and $\gamma_P=23.12$. However, this disagrees badly with high-energy total cross sections, as pointed out by Olsson.[15] For two-pole models, we verify Olsson and Yodh's conclusions,[6] viz., those solutions which give a good fit to the high-energy data give very bad χ^2 to our sum rule and vice versa. The following is a typical two-pole solution:

$$\alpha_P=0.989\pm0.01\,,\quad \gamma_P=12.41\pm0.3\,,$$
$$\alpha_{P'}=0.54\ \pm0.02\,,\quad \gamma_{P'}=11.69\pm0.4\,.$$

This gives a $\chi^2=57$ for 61 values of β, and a $\chi^2=217$ for 8 experimental high-energy total cross sections.

This inability of the two-pole models to satisfy the high-energy and the low-energy constraints simultaneously indicates a more complex system of singularities. Within the framework of pure Regge theory this conflict can be avoided by introducing a third pole P''. One may associate the latter with the broad $J=0$, isoscalar $\pi\pi$ resonance ϵ.[15] Or, one may look upon it as a manifestation of a cut in the angular momentum plane. The P' can be associated with f_0, while the Pomeranchon, which is generated by the nonresonating background and has a small slope, may be a flat trajectory without any known particle associated with it.

TABLE II. Comparison of constrained and unconstrained three-pole models. $(\chi^2)_{\text{H.E.}}$ is the χ^2 for 8 experimental high-energy total cross sections. The χ_1^2 and χ_2^2 correspond to Eqs. (9) and (10), respectively.

Solution	α_1	γ_1	α_2	γ_2	α_3	γ_3	χ_1^2	χ_2^2	$(\chi^2)_{\text{H.E.}}$
C1	0.988 ±0.01	12.980±0.2	0.440 ±0.02	11.324±0.3	−0.500 ±0.03	9.856±0.3	7.9	14.5	2.07
C2	0.9879±0.01	12.900±0.2	0.4579±0.02	11.193±0.3	−0.5572±0.04	10.121±0.3	8.3	9.8	2.92
O3	1	12.08	0.49	11.81	−0.5	9.48	23.3	42.7	2.14

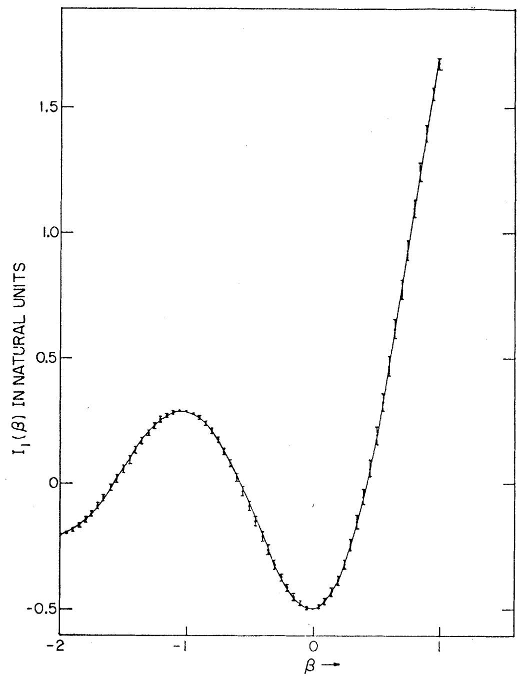

FIG. 3. Plot of $I_1(\beta)$ (in natural units) versus β. $I_1(\beta)$ and its error are calculated from the integral in Eq. (9). The range of β is $-2 \leq \beta \leq 1$ and the interval is 0.05. The continuous curve is calculated from the parameters of our solution $C1$, given in Table II.

Next, we consider the three-pole model. As first pointed out by Della Selva *et al.*,[18] we find that the controlling parameter is α_P. This means that the χ^2 is more sensitive to small variations in α_P than to those in other parameters. Now, if the errors in the sum rule are big (8–10%), then one cannot hope to determine small deviations of α_P from unity. This forced some earlier authors[5,15] to fix α_P at unity. In our case, the errors are at most 3–4%. So we can hope to make an unconstrainted three-pole fit to the sum rule. In Table II, we present the solution $C1$ which gives the lowest χ^2 to an unconstrained three-pole fit. Now an important question is: How stable is this solution for small perturbations in the value of the parameters? As noted earlier, the χ^2 is most sensitive to small perturbations in α_P. It turns out that all the solutions having χ^2 values close to one another do not differ appreciably in their values for α_P. The variations in other parameters are somewhat larger but are still within the errors. Any solution with appreciably different α_P will have considerably larger χ^2.

If we take Olsson's constrained three-pole solution[15] $O3$ (see Table II) and evaluate its χ^2 value from our data, we find a $\chi^2=23$ for the $\lambda=1$ case, as opposed to $\chi^2=7.9$ for our solution $C1$. In order to investigate this point we, too, made a three-pole fit with $\alpha_P=1$ and $\alpha_{P''}=-\frac{1}{2}$. We find that such constrained fits give consistently larger χ^2, the lowest being about 50% higher than that for our solution $C1$.

Another interesting point is that, in the cases of one- and two-pole fits, if we start the run with $\alpha_P=1$ and appropriate values for other parameters, the output (minimized) solutions often had values of α_P somewhat larger than unity. This is one more indication of the inadequacy of the one- and two-pole models. However, this never happened in case of three-pole fits, despite the fact that some of the runs were initiated with $\alpha_P=1$ and 1.01.

B. $\lambda=2$ Case

Here we repeat the minimization process only for the unconstrained three-pole fits. The solution we obtain here is denoted by $C2$. Its parameters and characteristics are given in Table II. Now we can test the stability of the solutions $C1$ and $C2$ by comparing their χ^2 for both $\lambda=1$ and $\lambda=2$ cases. This is done in Table II, where we also compare our solutions with one of the typical constrained solutions, viz., the $O3$ solution of Olsson[15] where $\alpha_P=1$ and $\alpha_{P''}=-\frac{1}{2}$. The important thing that emerges from this analysis is that the Pomeranchon intercept comes out essentially the same in both cases, $C1$ and $C2$.

VI. CONCLUSION

In conclusion, we find that although the value of α_P is consistent with unity within the errors, the value 0.99

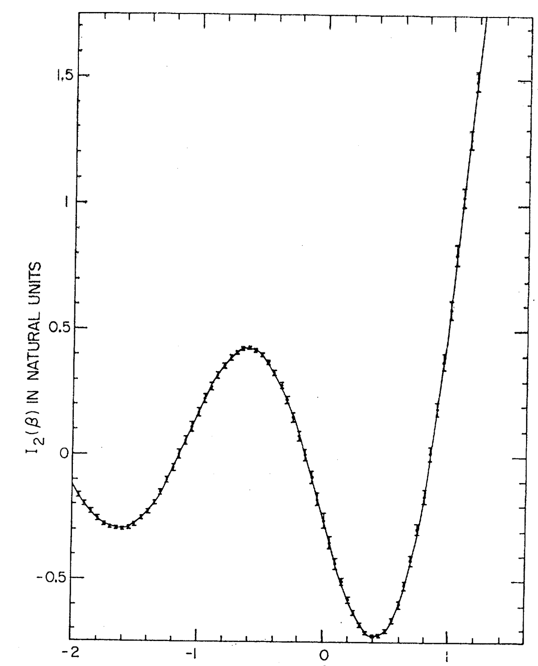

FIG. 4. Plot of $I_2(\beta)$ (in natural units) versus β. $I_2(\beta)$ and its error are calculated from the integral in Eq. (10). The range of β is $-2 \leq \beta \leq \frac{3}{2}$ and the interval is 0.05. The continuous curve is calculated from the parameters of our solution $C2$, given in Table II.

[18] A. Della Selva, L. Masperi, and R. Odorico, Nuovo Cimento **55A**, 602 (1968).

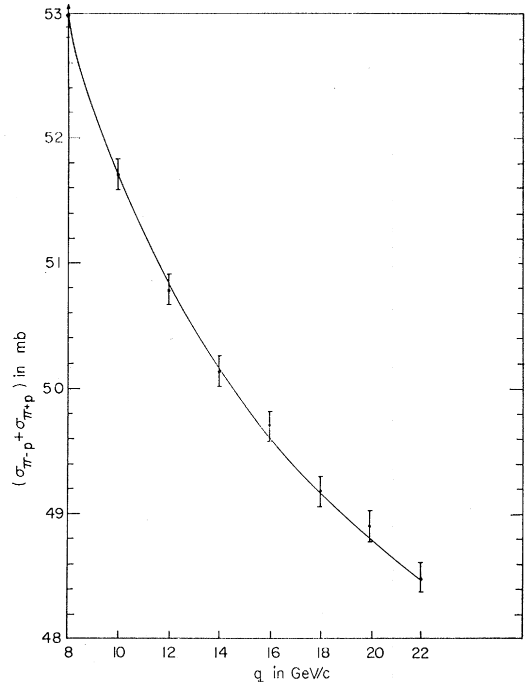

FIG. 5. Total cross-section sum $\sigma_{\pi^-p}+\sigma_{\pi^+p}$ in mb versus q in GeV/c. The data are from Foley *et al.*, Ref. 16. The continuous curve is calculated from our solution *C*1, Table II.

seems to be favored over unity. We list the following points in support of this:

(1) Both the $\lambda=1$ and $\lambda=2$ sum rules give essentially the same value $\alpha_P=0.988$.

(2) Both solutions give excellent fits to the observed high-energy total cross sections, as can be seen from Table II and Fig. 5.

(3) Solutions constrained to $\alpha_P=1$ give consistently higher χ^2 for the sum-rule fits.

(4) In the case of three-pole models, even if we start a run with $\alpha_P=1$ or 1.01, the minimized output solutions always give α_P less than unity.

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APPENDIX

Here we describe briefly the method of determination of the number of zeros of a given amplitude. Although the method is quite general, we consider the pion-nucleon scattering in the forward direction, for definiteness.

Consider a function $A(\omega)$ satisfying the following properties: It is analytic in the ω plane, except for a finite number of poles and cuts on the real axis. $A(\pm\omega)=\pm A(\omega)$. It is bounded polynomially as $|\omega|\to\infty$. It is real-analytic, i.e., $A^*(\omega)=A(\omega^*)$. Now we define a phase $\delta(\omega)$ of $A(\omega)$ as follows:

$$A(\omega)=\pm|A(\omega)|e^{i\delta(\omega)}, \tag{A1}$$

where the upper (lower) sign is to be taken if $\mathrm{Re}A(\mu)$ is positive (negative). For the first case, $0<\delta(\omega)<\pi$ and for the second, $-\pi<\delta(\omega)<0$. If $\mathrm{Re}A(\mu)=0$, continuity arguments determine the sign.

If we assume further that $\delta(\infty)$ is finite, we can show that[9]

$$A(\omega)=\frac{P_z(\omega)}{Q_p(\omega)}\exp\left[\frac{2q^2}{\pi}P\int_\mu^\infty\frac{d\omega'\omega'\delta(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)}\right], \tag{A2}$$

where z and p are, respectively, the number of zeros and poles of $A(\omega)$, while $P_z(\omega)$ and $Q_p(\omega)$ are polynomials of order z and p. The asymptotic behavior of $A(\omega)$ is

$$A(\omega)=\omega^{z-p-(2/\pi)\delta(\infty)}. \tag{A3}$$

Here, z is even (odd) if $A(\omega)$ is crossing-even (odd).

Now we take $A(\omega)=F(\omega)-F(\mu)-F_B(\omega)+F_B(\mu)$, where all the quantities are defined in the text. By construction, this amplitude has two zeros at $\omega=\pm\mu$ and no poles. We must show that these are the only zeros of $A(\omega)$. Since $\mathrm{Re}A(\mu)=0$, one must turn to continuity arguments to determine the sign in Eq. (A1). From Table I, it is easy to see that $\mathrm{Re}A(\omega)>0$ for $\omega\to+\mu$. Thus, the upper sign is implied in Eq. (A1). Therefore, the phase is in the range $0<\delta(\omega)<\pi$ for all ω. Comparing (A3) with the Regge asymptotic behavior, we get

$$z=p+(2/\pi)\delta(\infty)+\alpha_P. \tag{A4}$$

The Regge behavior also determines the phase $\delta(\infty)$ as

$$\delta(\infty)=\pi-\tfrac{1}{2}\pi\alpha_P. \tag{A5}$$

From Eqs. (A4) and (A5), we obtain

$$z=p+2. \tag{A6}$$

Note that this result is independent of whether or not α_P is unity. Now since $A(\omega)$ has no poles, $z=2$. This means that $A(\omega)$ has exactly two zeros which are located at $\omega=\pm\mu$.

The odd amplitude can be treated similarly. The crucial result is Eq. (A3) which is valid for both even and odd amplitudes.