Other methods, such as the point-splitting technique do not seem to have any practical significance.

It is unfortunate that the point-splitting technique cannot be used to calculate the various anomalies that have been discovered. It is seen that this method simplifies the calculation considerably in that it presents a formal expression for the anomaly which is moderately easy to evaluate; see (3.4). Moreover from the structure of the formulas for the anomalies, in the point-splitting context, one may hope to be able to prove results independent of perturbation theory. Perhaps a useful point of view about the point-splitting device is that it provides a clue to the existence of anomalies. The precise value then must be computed by the method relevant to the application—typically by the BJL method.

(d) Although we have found q-number ST in some of our models, their significance to the usual applications of current algebra seems to be minimal. The only important role that ST have had, to our knowledge, has been in connection with Weinberg's first sum rule.[22] It is true that in derivations of that result, frequently the assumption is employed that the ST are c numbers, which is not valid in our models. However that assumption is in fact too strong—the sum rule requires merely the equality of the vacuum expectation values of the ETC between vector currents and axial-vector currents. This equality is maintained in the present investigation, as is seen from (2.7). Thus, the first Weinberg sum rule holds, even though the ST are q numbers.

[22] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

On the other hand, our considerations indicate that the use of canonical commutation relations to draw conclusions concerning the asymptotic behavior of electroproduction amplitudes or the convergence of radiative corrections in general and mass shifts in particular is highly suspect. Not only can the interactions change the values and tensor structure of the commutators,[10] they also can introduce entirely new forms. The usual "proofs" contain strong implicit assumptions concerning the dynamics.

Electromagnetic mass shifts may be of particular interest; if there is a neutral scalar meson, then the Cottingham formula applied to any shift should yield a quadratic divergence even though there are no charged boson fields. To be sure, this divergence would, in a complete theory, be associated with the electromagnetic mass renormalization of the neutral scalar particle,[6] but the Cottingham formula is indifferent to the source of the divergence.

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Three-Dimensional Covariant Integral Equations for Low-Energy Systems*

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Integral equations suitable for the dynamical treatment of strongly interacting particles are derived. The equations can be described as Bethe-Salpeter equations with one particle restricted to the mass shell, resulting in a three-dimensional covariant equation which can be easily interpreted physically. To restore the dynamical terms omitted in the process of restricting one particle to the mass shell, additional kernels are added to the irreducible kernels from the original Bethe-Salpeter equation. The addition of these extra terms leads to a resulting simplification in the kernels themselves, since the new kernels have the same structure as the original ones, with some partial cancellations. Estimates as to the convergence of the procedure and the sizes of the various potentials are given. The special case of the hydrogen atom is discussed briefly, and comments are made on the application of these equations to the nuclear-force problem. Connections between scattering equations and bound-state equations are discussed, and the relativistic normalization condition for bound-state wave functions is derived.

I. INTRODUCTION AND DISCUSSION

Everyone knows that the hydrogen atom can be quite well described by nonrelativistic quantum mechanics and that only the finer details require the application of the ideas of relativistic field theory (in the form of the Bethe-Salpeter equation). On the other hand, even though much progress has been made in the last 20 years toward an understanding of the nuclear force, no one has yet been able to construct a simple reasonably accurate theoretical description of the deuteron. The striking simplicity of the hydrogen atom is sharply contrasted with the complexity of the current models of the nuclear force, which usually have about 10 adjustable parameters and can be said to be as complicated as is theoretically possible.[1] Is it really

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[1] See Rev. Mod. Phys. 39, 495–718 (1967) for a recent review of the status of the nucleon-nucleon interaction.

true that one cannot construct a simple theoretical description of the essential features of the deuteron and, if so, what is the property of the dynamics of the nuclear force which prevents such a description when the hydrogen atom can be handled so well?

Motivated originally by the above considerations, we concern ourselves in this paper to the general problem of the construction of covariant integral equations which satisfactorily describe the behavior of strongly interacting particles. In a subsequent paper these equations are used to analyze the low-energy nucleon-nucleon system.[2]

It is desirable that the equations have a simple physical interpretation and be fairly easy to work with. These conditions are achieved by starting with equations of the Bethe-Salpeter type, but restricting one of the particles to its positive-energy mass shell. The result is a covariant three-dimensional equation. The dynamics must also be correct, of course, and the contributions left out of the Bethe-Salpeter equation by restricting one particle to the mass shell can be reintroduced by enlarging the number of terms one includes in the kernel. One might first think that this would complicate the problem, but as it turns out the opposite is true. The new terms added to the old kernel tend to cancel some of the old terms, resulting in some simplifications and a better physical understanding of the remaining terms, which now (because of the three-dimensional nature of the equation) can be viewed as a sum of relativistic potentials. The nth potential has a range characteristic of the exchange of n quanta, and hence by restricting ourselves to the first n terms we describe the force to a distance of λ/n, where λ is the Compton wavelength of the particle which generates the force. And the equations have the desirable feature that they reduce to Schrödinger or Dirac equations in the nonrelativistic limit. Hence the new equations are especially suitable for the description of low-energy systems, and by way of illustration we show in Sec. III C how an effective Dirac equation for the hydrogen atom can be derived. (See Sec. II B for a verbal summary of how to construct the equations and corresponding kernels and for an explicit example for scalar particles.)

Another advantage of the equations described above is that the wave function one obtains corresponds to one particle off the mass shell, and such solutions are precisely those needed to describe the interaction of loosely bound systems in scattering theory or to use the two-body interaction in the treatment of three-body forces. In such situations it is usually a sufficient approximation to regard one of the particles as a spectator, and hence to restrict it to its mass shell.[3]

In developing the equations in the following sections, we focus attention on the "nonrelativistic" domain; we assume that the kinetic energies of the interacting particles are small compared to their rest mass and that the mass of the exchanged particle which accounts for the interaction is small compared to the mass of the interacting particles. If these conditions are met, then our arguments can be regarded as a derivation of the equations within the dynamical assumptions we make. If these conditions do not obtain, the equations still provide a dynamical model for the strong interactions in the same way that, for example, the Bethe-Salpeter equation in the ladder approximation is a dynamical model.

Our approach differs from that taken recently by Blankenbecler and Sugar, and by Namyslowski.[4] Both of these approaches focus attention on the unitarity condition on the right-hand side—the equations are constructed from the unitarity cuts contributed by the lowest-lying intermediate states, both elastic and inelastic. In a problem involving the form factors of loosely bound systems,[5] we found some time ago that selected pieces from many inelastic contributions to the unitarity sum were necessary to properly describe the dynamics, and we see in Sec. II that this also tends to be true for the low-energy scattering or bound-state problem. Hence our approach focuses attention on the range and strength of the different forces which contribute to the interaction, and not on the presence or absence of right-hand cuts. As it turns out, the contributions we include contain elastic and inelastic cuts. In other aspects, our approach is similar to that of Ref. 4; in particular, a difficulty common to all approaches of this type is that crossing symmetry is not handled very well.

In Sec. II A we outline the arguments used to construct the integral equations, and in Sec. II B we obtain the equations explicitly for some simple cases involving spin-zero "nucleons" and the exchange of spin-zero bosons. In Sec. III we turn to the complications which arise when one extends the ideas of Sec. II to interactions of two spin-$\frac{1}{2}$ nucleons through the exchange of massive scalar and pseudoscalar mesons. We find that the equations developed can be applied to the case of scalar-meson exchange, but that we encounter obstacles in the pseudoscalar case, which explain why it is difficult to treat the nucleon interaction. Among our observations is that the difficulty with the low-energy nucleon-nucleon system results not so much from the intrinsic nature of the strong interactions, but rather from the unfortunate fact that the pion is a pseudoscalar particle [see the discussion following Eq. (3.17)]. We will show in a subsequent paper[2] how this difficulty can be circumvented by considering the two-pion exchange as a single relativistic interaction, which turns out to be equivalent primarily to the exchange of a scalar meson of distributed mass, and hence amenable to the techniques developed here.

In Sec. IV we discuss how to obtain the corresponding

[2] F. Gross (to be published).
[3] F. Gross, Phys. Rev. 140, B410 (1965).
[4] R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966); J. M. Namyslowski, *ibid.* 160, 1522 (1967). See also additional references in both papers.
[5] F. Gross, Phys. Rev. 134, B405 (1964); 136, B140 (1964).

bound-state equation from any integral equation for the scattering amplitude. We also give a simple derivation of the normalization condition for bound-state wave functions.

II. SCALAR THEORY

In this section we develop our general approach for the simplest case possible—two neutral scalar "nucleons" of mass M interacting by the exchange of a neutral scalar meson of mass $\sigma \ll M$.

We begin by writing out the Feynman perturbation series for the interaction of the nucleons by the successive exchange of σ mesons. We consider all σ-meson exchanges, but exclude nucleon self-energy terms and meson-nucleon vertex corrections. These terms could be incorporated later by using phenomenological meson-nucleon form factors or by using information about the meson-nucleon scattering amplitude. The diagrams we consider are shown explicitly in fourth order in Fig. 1 and in sixth order in Fig. 2.

We next examine each Feynman diagram, and decompose it into a finite number of covariant pieces. The method of doing this will be described later. These pieces turn out to be of different orders in the small parameter (σ/M). When all the pieces from all the Feynman diagrams are regarded collectively, we have a new perturbation series. If the dimensionless coupling constant g is not too large, so that

$$\xi = (g^2/32\pi)(\sigma/M) < 1\,, \tag{2.1}$$

then some of the terms in this new series diverge while the remaining terms converge. In the case (2.1), it is easy to construct covariant integral equations which, when iterated, correspond to the diverging parts of the series. Hence, the entire series can be handled by first solving the integral equations and then using perturbation theory. The number of kernels or potentials one needs in the integral equation is finite. When $\xi > 1$, we may still employ this method, since the potentials we must use have shorter and shorter range. However, we must now use an infinite number of potentials with larger and larger couplings if we wish to solve the problem completely.

Although our development makes use of a perturbation series which does not generally converge, we feel that the integral equations we derive should be a good approximation, even if the corresponding series does diverge. The principal restriction on the usefulness of our method is the requirement (2.1) (in order that only a finite number of potentials be needed) and the requirement that $\sigma/M \ll 1$, in order that the series converge rapidly.

A. Derivation of the Equation

We will examine the exchange diagrams in second, fourth, and sixth order in some detail. Afterwards, the trends will become clear and we can make the discussion more general.

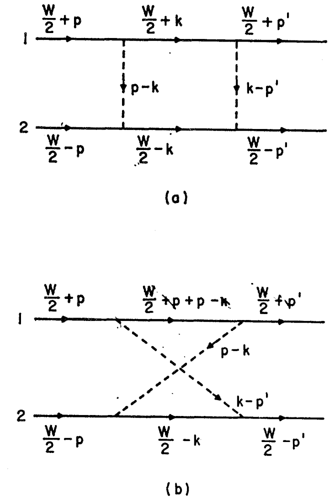

FIG. 1. Feynman diagrams in fourth order included in the dynamical model discussed in the text. Solid lines are "nucleons" and dashed lines mesons.

The interaction Lagrangian is

$$\mathcal{L}_I = g(M\sigma)^{1/2}(\psi^*\psi\phi + \psi^*\psi\phi^*)\,, \tag{2.2}$$

where the factor $(M\sigma)^{1/2}$ has been added to make g dimensionless. Hence the scattering amplitude in second order is attractive:

$$\mathfrak{M}^{(2)} = -g^2 M\sigma/[\sigma^2 - (p-p')^2]. \tag{2.3}$$

If we restrict both particles to the mass shell, we have

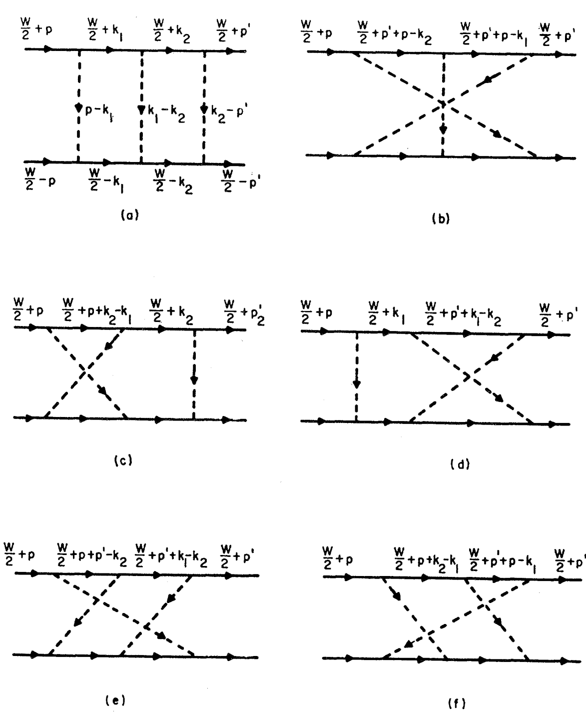

FIG. 2. Feynman diagrams in sixth order included in the dynamical model.

the usual nonrelativistic first Born approximation:

$$\mathfrak{M}^{(2)}=-g^2M\sigma/[\sigma^2+(\mathbf{p}-\mathbf{p}')^2]. \tag{2.4}$$

The first interesting questions arise in fourth order. There are two exchange diagrams, shown in Fig. 1. We think of these diagrams as the relativistic second Born approximation and investigate how they differ from the iteration of (2.4). If the dynamics can be described by the potential (2.4) then the fourth-order terms should not differ significantly from the iteration of (2.4).

First, we consider the box diagram, Fig. 1(a), and again restrict ourselves to the case when both external particles are on the mass shell. The computations are much simpler if we restrict ourselves to forward scattering $\mathbf{p}=\mathbf{p}'$, which should not be too restrictive since at low energies the scattering is mostly S-wave anyway. Hence, in the c.m. system,

$$p_0=p_0'=0, \tag{2.5}$$

and the diagram gives

$$\mathfrak{M}^{(4a)}=\frac{ig^4(M\sigma)^2}{(2\pi)^4}\int\frac{d^4k}{[(E_k-E_p)^2-k_0^2-i\epsilon][(E_k+E_p)^2-k_0^2-i\epsilon][\omega_{p-k}^2-k_0^2-i\epsilon]^2}, \tag{2.6}$$

where

$$\omega_k=(\sigma^2+\mathbf{k}^2)^{1/2},\quad E_k=(M^2+\mathbf{k}^2)^{1/2}. \tag{2.7}$$

We decompose this diagram into three covariant pieces by performing the k_0 integration, using the residue theorem. Closing the contour in the lower half k_0 plane gives

$$\mathfrak{M}^{(4a)}=\mathfrak{M}_+^{(4a)}+\mathfrak{M}_0^{(4a)}+\mathfrak{M}_-^{(4a)}, \tag{2.8}$$

where $\mathfrak{M}_\pm$ are the residues from the positive (negative) energy nucleon poles and $\mathfrak{M}_0$ is the contribution from the (double) meson pole. Hence, if $\omega\equiv\omega_{p-k}$,

$$\mathfrak{M}_+^{(4a)}=-\frac{g^4M^2\sigma^2}{(2\pi)^3}\times\int\frac{d^3k}{8E_kE_p(E_k-E_p)[\omega^2-(E_k-E_p)^2]^2}, \tag{2.9a}$$

$$\mathfrak{M}_0^{(4a)}=\frac{g^4M^2\sigma^2}{(2\pi)^3}\times\int\frac{d^3kN}{(2\omega)^3[(E_k-E_p)^2-\omega^2]^2[(E_k+E_p)^2-\omega^2]^2}, \tag{2.9b}$$

where

$$N=12\omega^2(E_k^2+E_p^2)-10\omega^4-2(\mathbf{k}^2-\mathbf{p}^2)^2$$

and

$$\mathfrak{M}_-^{(4a)}=\frac{g^4M^2\sigma^2}{(2\pi)^3}\times\int\frac{d^3k}{8E_kE_p(E_k+E_p)[\omega^2-(E_k+E_p)^2]^2}. \tag{2.9c}$$

Although these separate pieces of the fourth-order diagram do not appear to be separately covariant, one can easily see that they are by noting that (2.9a) and (2.9c) can be written in a manifestly covariant form:

$$\mathfrak{M}_+^{(4a)}=\frac{g^4M^2\sigma^2}{(2\pi)^3}\times\int\frac{d^4k\delta[(\tfrac{1}{2}W+k)^2-M^2]\theta(\tfrac{1}{2}W+k_0)}{[(\tfrac{1}{2}W-k)^2-M^2][\sigma^2-(p-k)^2]^2}, \tag{2.9d}$$

$$\mathfrak{M}_-^{(4a)}=\frac{g^4M^2\sigma^2}{(2\pi)^3}\times\int\frac{d^4k\delta[(\tfrac{1}{2}W-k)^2-M^2]\theta(k_0-\tfrac{1}{2}W)}{[(\tfrac{1}{2}W+k)^2-M^2][\sigma^2-(p-k)^2]^2}. \tag{2.9e}$$

In a nonrelativistic situation, when the total energy $W\equiv2M+\epsilon$ is close to $2M$ (so that $\epsilon\ll M$), then

$$\mathbf{p}^2=(\tfrac{1}{2}W)^2-M^2\cong M\epsilon\equiv\delta^2. \tag{2.10}$$

In this situation one expects that the intermediate nucleons remain close to their mass shell. Hence we intuitively expect $\mathfrak{M}_+$ to be the dominant term, and $\mathfrak{M}_-$, which picks up the explicit negative-energy pole (contributions from far off the mass shell), to be quite small. Finally, $\mathfrak{M}_0$ which comes from the energy singularities of the pion propagators, should be smaller than $\mathfrak{M}_+$ if retardation effects in the potential are small, as one expects.

It is easy to evaluate the terms (2.9) in the limit when $\mathbf{p}^2\sim\sigma^2\sim\delta^2\ll M^2$, and this is sufficient to bear out these intuitive observations. We observe that the integrand of $\mathfrak{M}_+$ and $\mathfrak{M}_0$ is uniformly convergent as k or $M\to\infty$, and hence the $M\to\infty$ limit can be taken under the integral. Similarly, $\mathfrak{M}_-$ is uniformly convergnet as σ, δ, $\mathbf{p}\to0$ and $k\to\infty$, so this limit may be taken. We obtain:

$$\mathfrak{M}_+^{(4a)}\xrightarrow[M\to\infty]{}-\frac{g^4M\sigma^2}{4(2\pi)^3}\int\frac{d^3k}{(\mathbf{k}^2-\delta^2-i\epsilon)[\sigma^2+(\mathbf{p}-\mathbf{k})^2]^2}=(-g^4/32\pi)M/(\sigma-2i\delta), \tag{2.11a}$$

$$\mathfrak{M}_0^{(4a)}\xrightarrow[M\to\infty]{}\frac{g^4}{(2\pi)^3}\frac{3\sigma^2}{16}\int\frac{d^3k}{[\sigma^2+(p-k)^2]^{5/2}}=\frac{g^4}{32\pi^2}, \tag{2.11b}$$

$$\mathfrak{M}_-^{(4a)}\xrightarrow[M\to\infty]{}\frac{g^4\sigma^2}{64\pi^2M}\int_0^\infty\frac{k^2dk}{E_k(E_k+M)^3}=\frac{g^4\sigma^2}{192\pi^2M^2}. \tag{2.11c}$$

Note that in this limit $\mathfrak{M}_0$ and $\mathfrak{M}_-$ are energy-independent. We have

$$\mathfrak{M}_+^{(4a)}/\mathfrak{M}_0^{(4a)} = -\pi M/(\sigma-2i\delta) \sim M/\sigma \gg 1\,, \quad (2.12)$$

and we see that when $\sigma \ll M$ the total interaction is well approximated by the positive-energy nucleon pole, and that this approximation holds for energies of the order of σ, but certainly not for energies of the order of M. If $\sigma \approx M$, then the interaction is of very short range and is dominated by large-nucleon kinetic energies, which means that the nucleons are far off the mass shell and the other off-mass-shell contributions are important as well.

The remaining fourth-order contribution comes from the crossed-box diagram shown in Fig. 1(b). Near threshold we intuitively expect this diagram to be smaller than the uncrossed box, because it requires two mesons to be present at the same time, and hence the nucleons must be farther off their mass shell. A second line of reasoning derives from noting that if we neglect retardation, so that the interaction is instantaneous, then the crossed box is greatly suppressed.[6] Mathematically this comes about because there is no positive-energy nucleon pole in the lower-half complex k_0 plane, so that the dominant term $\mathfrak{M}_+$ is now missing.[7]

The diagram gives

$$\mathfrak{M}^{(4b)} = \frac{ig^4M^2\sigma^2}{(2\pi)^4} \int \frac{d^4k}{[E_{2p-k}{}^2-(E_p-k_0)^2-i\epsilon][E_k{}^2-(E_p-k_0)^2-i\epsilon](\omega^2-k_0{}^2-i\epsilon)^2}. \quad (2.13)$$

Closing the k_0 contour in the lower-half plane, we see that there is no positive-energy nucleon pole, so that

$$\mathfrak{M}^{(4b)} = \mathfrak{M}_0^{(4b)} + \mathfrak{M}_-^{(4b)}. \quad (2.14)$$

Taking the $M \to \infty$ limit as in the uncrossed case gives

$$\mathfrak{M}_0^{(4a)} \xrightarrow[M\to\infty]{} -\frac{3g^4\sigma^2}{32\pi^2} \int_0^\infty \frac{k^2dk}{\omega^5} = \frac{-g^4}{32\pi^2}, \quad (2.15a)$$

$$\mathfrak{M}_-^{(4b)} \xrightarrow[M\to\infty]{} -\frac{g^4\sigma^2}{64\pi^2 M} \int_0^\infty \frac{k^2dk[6M(E_k+M)+4k^2]}{E_k{}^3(E_k+M)^3} = \frac{-g^4\sigma^2}{32\pi^2M^2}. \quad (2.15b)$$

Hence we see that $\mathfrak{M}^{(4b)}$ is smaller than $\mathfrak{M}^{(4a)}$ by a factor of σ/M, and, furthermore, that the dominant terms of $\mathfrak{M}_0^{(4)} = \mathfrak{M}_0^{(4a)} + \mathfrak{M}_0^{(4b)}$ cancel. This means that $\mathfrak{M}_+^{(4a)}$ agrees with the exact fourth-order contribution $\mathfrak{M}^{(4)}$ to order $(\sigma/M)^2$. In calculating $\mathfrak{M}^{(4)}$ we obtain a better answer from the (simple) contribution $\mathfrak{M}_+^{(4a)}$ than we do from the (more complicated) ladder approximation $\mathfrak{M}^{(4a)}$. These cancellations referred to in Sec. I are an important advantage of our procedure over the conventional Bethe-Salpeter approach.

The fact that the crossed graph tends to cancel the box graph has been known for a long time. This cancellation has shown up in a number of different ways. Bethe and Salpeter showed in their classic paper that for neutral scalar particles the instantaneous potential gives correct results to order $(\sigma/M)^2$.[6] In an interesting paper, Charap and Fubini showed that the crossed graph tends to cancel an energy-dependent part of the box graph—making the two together more suitable for a static potential than the box diagram alone.[8] What we see here is that the simple approximation $\mathfrak{M}_+^{(4a)}$ similarly benefits from this cancellation—because of the cancellation it is more accurate than one would expect. For charged theories, the two fourth-order diagrams have different isospin coefficients, so the cancellation is not complete.

We now consider the sixth-order diagrams, all of which are shown in Fig. 2, First, we examine the contributions from the various singularities for the ladder diagram, Fig. 2(a), at threshold (i.e., for simplicity we assume $p=0$ and suppress the $-i\epsilon$):

$$\mathfrak{M}^{(6a)} = \frac{-g^6\sigma^3M^3}{(2\pi)^8} \int \frac{d^4k_1d^4k_2}{D(\omega_{k_1}{}^2-k_{10}{}^2)(\omega_{k_2}{}^2-k_{20}{}^2)[\omega_{k_1-k_2}{}^2-(k_{10}-k_{20})^2]}, \quad (2.16a)$$

where

$$D = [(E_{k_1}-M)^2-k_{10}{}^2][(E_{k_2}-M)^2-k_{20}{}^2][(E_{k_1}+M)^2-k_{10}{}^2][(E_{k_2}+M)^2-k_{20}{}^2]. \quad (2.16b)$$

If we now perform the k_{10} and k_{20} integrations in the complex plane, there will be terms arising from positive-energy nucleon poles (+), meson poles (0), negative-energy nucleon poles (−), and combinations of these. We saw in fourth order that the negative-energy poles gave a very small contribution. We will therefore neglect these residues entirely. Hence, we have

$$\mathfrak{M}^{(6a)} \simeq \mathfrak{M}_{++}^{(6a)} + \mathfrak{M}_{+0}^{(6a)} + \mathfrak{M}_{00}^{(6a)}, \quad (2.17)$$

[6] Even if the interaction is instantaneous, there is still a con- tribution from the crossed box. See the discussion in Sec. III C and the original paper by E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

[7] If we had closed the contour in the upper half-plane we would have encountered two nucleon poles which are still suppressed because they nearly cancel.

[8] J. M. Charap and S. P. Fubini, Nuovo Cimento 14, 540 (1959).

where

$$\mathfrak{M}_{++}{}^{(6a)}=\frac{g^6\sigma^3 M}{64(2\pi)^6}\int\frac{d^3k_1d^3k_2}{(E_{k_1}-M)(E_{k_2}-M)E_{k_1}E_{k_2}[\omega_{k_1}{}^2-(E_{k_2}-M)^2][\omega_{k_1}{}^2-(E_{k_2}-M)^2][\omega_{k_2-k_1}{}^2-(E_{k_1}-E_{k_2})^2]}. \quad (2.18)$$

Again, in the limit $\sigma/M\rightarrow 0$, the above integral can be exactly evaluated. Taking the limit $M\rightarrow\infty$ in (2.18) and performing the angular integrations gives

$$\mathfrak{M}_{++}{}^{(4a)}\xrightarrow[M\to\infty]{}\left(\frac{g^2}{32\pi}\right)^2 g^2\frac{4MC}{\pi^2\sigma}, \quad (2.19)$$

where

$$C=\int_0^\infty\int_0^\infty\frac{dx_1dx_2}{x_1x_2(1+x_1{}^2)(1+x_2{}^2)}\times\ln\left\{\frac{1+(x_1+x_2)^2}{1+(x_1-x_2)^2}\right\}\approx 3. \quad (2.20)$$

In a similar way, we show that the other terms are of the following orders:

$$\mathfrak{M}_{+0}{}^{(6a)}\simeq\mathfrak{M}_{++}{}^{(6a)}(\sigma/M),\quad \mathfrak{M}_{00}{}^{(6a)}\simeq\mathfrak{M}_{++}{}^{(6a)}(\sigma/M)^2. \quad (2.21)$$

The reason for this is that the internal momenta are always of order σ, and the internal energy will be of order σ if we take the residue at a meson pole, or of order σ^2/M if we take the residue at a positive-energy nucleon pole. Hence, for box diagrams in which a particular energy occurs in two meson propagators and two nucleon propagators, the contribution from a meson pole will be proportional to

$$\text{meson pole}\approx(1/\sigma)^5,$$

while if we take a nucleon pole instead, it will be of order

$$\text{nucleon pole}\approx(1/\sigma)^4M/\sigma^2=(1/\sigma)^5M/\sigma.$$

The same argument holds for each box in any ladder diagram. Hence, if we have a ladder diagram involving the exchange of n mesons, the contribution resulting from the n-meson residues will be down by order $(\sigma/M)^n$ compared to the dominant contribution where all the positive-energy nucleon poles are retained.

In a similar manner one can develop a general rule for estimating the size of the other sixth-order diagrams shown in Fig. 2. One notes that in the parts of the diagrams where mesons are crossed (i.e., where the diagrams cannot be separated into two disjoint pieces by cutting nucleon lines only), one of two situations arises:

(i) The positive-energy nucleon poles occur in pairs like

$$(E_k-M+k_0-i\epsilon)^{-1}(E_k-M+k_0-r_0-i\epsilon)^{-1},$$

in which case they are both in the upper half-plane. Consequently, when we close the k_0 contour in the lower half-plane we will avoid them and get no contribution from the positive-energy nucleon poles. [Figure 2(b) with $r_0=0$ and Figs. 2(c) and (d) are of this type.]

(ii) The positive-energy poles occur in clusters like

$$(E_k-M+k_0-i\epsilon)^{-2}(E_{k-r}-M+k_0-r_0-i\epsilon)^{-1}\times(E_r-M+r_0-i\epsilon)^{-1}.$$

The poles in r_0 are now in both halves of the complex plane, but the poles in k_0 are all in the upper half-plane so that when we close the k_0 contour in the lower half-plane we will miss the nucleon pole entirely and obtain $k_0\simeq\sigma$. Hence, the r_0 poles are now separated by a distance of order σ, so that they contribute terms no larger than those already contributed by the meson poles. [Figures 2(e) and 2(f) are of this type.]

In higher-order diagrams the same arguments apply, as the reader can easily confirm for himself.

Our conclusion, then, is that in all irreducible parts of meson-exchange diagrams, the positive-energy nucleon poles will give contributions no larger than meson poles, while in diagrams which are reducible, each positive-energy nucleon pole gives an enhancement of order M/σ over the corresponding meson pole. Note that a cut through a diagram which reduces it is also a two-body unitarity cut, so that we may also say that the diagrams are dominated by the two-body elastic unitary contributions.

From the above considerations we obtain the following rough estimate for the size of a $2n$th-order irreducible diagram:

$$\mathfrak{M}_0{}^{(2n)}\approx\frac{(g^2M\sigma)^n}{\sigma^2}\left(\frac{K}{(2\pi)^3M^2}\right)^{n-1}=\left(g^2\frac{M}{\sigma}\right)\left(\frac{g^2K}{(2\pi)^3}\frac{\sigma}{M}\right)^{n-1}, \quad (2.22)$$

where K is some effective constant depending on the details of the three-dimensional momentum integrations which can be obtained only by comparing the individual diagrams. In what follows, we will use the letter n to represent the number of meson exchanges in any diagram. If a $2n$th-order diagram can be reduced r ways, then the leading term comes from the r positive-energy nucleon poles so that the diagram is enhanced:

$$\mathfrak{M}_r{}^{(2n)}\approx\left(g^2\frac{M}{\sigma}\right)\left(\frac{g^2K}{(2\pi)^3}\frac{\sigma}{M}\right)^{n-1}\left(\frac{M}{\sigma}\right)^r. \quad (2.23)$$

It is now clear how to construct the covariant integral equations referred to at the beginning. In any order of perturbation theory, the largest pieces of the Feynman diagrams come from the positive-energy nucleon poles in the ladder diagram. If we introduce the effective expansion parameter

$$\eta = g^2 K/(2\pi)^3, \tag{2.24}$$

we see that at threshold the dominant pieces of the ladder sum give a series like

$$\mathfrak{M}_1 \approx (g^2 M/\sigma)(1-\eta+\eta^2-\cdots+\eta^m-\cdots). \tag{2.25}$$

In fact, by examining (2.4), (2.11a), and (2.19), we conjecture that $K \simeq \frac{1}{4}\pi^2$ is a representative value, so that the effective-expansion parameter is something like

$$\eta_1 = g^2/32\pi. \tag{2.26}$$

If $\eta_1 > 1$, we think of the series (2.25) as a symbolic representation of the integral equation which can be written to correspond to this series. This integral equation is a simple three-dimensional covariant equation, and we take it as the first approximation to the scattering problem. The kernel for the integral equation is a form of the one-meson-exchange potential. We will write this equation explicitly below.

The next largest series of terms we would have to consider comes from pieces of Feynman diagrams where there are at least $n-2$ two-body unitarity cuts (or $n-2$ ways to reduce the diagram). In these diagrams we pick up the $n-2$ positive-energy nucleon poles, and one remaining meson pole. The only diagrams of this type are ladder diagrams, or ladderlike diagrams which contain at most one crossed-meson box, as in Fig. 1(b) or Figs. 2(c) and 2(d). To sum such terms as this, requires the addition of a new potential or kernel to the integral equation corresponding to a part of the two-meson-exchange contribution. This kernel is the sum of the meson-pole contributions from the crossed and uncrossed box, Figs. 1(a) and 1(b). Adding this potential gives us more than the terms referred to above; in particular it also gives us all iterations of the new potential itself. The terms which involve only the iteration of the new potential include only $n =$ even, and go like

$$\begin{aligned}\mathfrak{M}_{\frac{1}{2}n-1}{}^{(2n)} &\approx [g^2(M/\sigma)][\eta_1(\sigma/M)]^{n-1}(M/\sigma)^{\frac{1}{2}n-1} \\ &= (g^4/32\pi)[\eta_1{}^2(\sigma/M)]^{n-1}.\end{aligned} \tag{2.27}$$

Hence the effective two-meson-exchange expansion parameter is

$$\eta_2 \simeq \eta_1{}^2(\sigma/M), \tag{2.28}$$

and the iteration of the two-meson potential at threshold looks like

$$\mathfrak{M}_2 \approx (g^4/32\pi)(1+\eta_2+\eta_2{}^2+\cdots+\eta_2{}^m+\cdots). \tag{2.29}$$

The pattern is now clear. If $\eta_2 > 1$, then we again think of this series as a symbolic representation of some of the terms which would be generated by iterating the integral equation with one- and two-meson-exchange potentials. If $\eta_2 < 1$, then the series converges, and it is not necessary to sum the two-meson-exchange potential to all orders. We assume if $\eta_2 < 1$ that the one-meson-exchange potential would be sufficient, and that two-meson effects could be treated as a perturbation. Of course, if η_2 is close to unity, we would probably want to include the two-meson potential anyway.

In a similar fashion we could add, if necessary, three-meson potentials. These would be computed from the sixth-order graphs shown in Fig. 2; we would take all those pieces derived from two-meson poles or one-meson pole and a nucleon pole buried in irreducible parts of the diagram. The effective expansion parameter for the three-pion-exchange contributions would be

$$\eta_3 \simeq \eta_1[\eta_1(\sigma/M)]^2. \tag{2.30}$$

For the p-meson-exchange terms we expect an expansion parameter like

$$\eta_p \simeq \eta_1[\eta_1(\sigma/M)]^p. \tag{2.31}$$

Hence, as long as $\xi = \eta_1(\sigma/M) < 1$ [Eq. (2.1)], there will always be a p such that $\eta_p < 1$, no matter how large η_1 is. For these theories we can sum up all of the divergent terms with a three-dimensional covariant integral equation with a finite number of potentials or interaction kernels (p of them). The method is clumsy if p is greater than 3.

At this point we observe that because of the cancellation [Eqs. (2.10b) and (2.14a)] in the two-meson potential for neutral theories, the one-meson-exchange term includes all the major effects up to three-meson exchange. In the case of charged exchange, there will be no such cancellation and the two-meson potentials can be expected to be important. This turns out to be extremely important in the case of the deuteron.

B. Summary and Example

We now summarize the prescription for writing down the wave equation and the corresponding potentials. One begins with the usual Bethe-Salpeter equation but restricts one of the particles to the mass shell. The precise form of the equation can be obtained by formally integrating over the internal energy k_0, picking up the positive-energy pole, and ignoring the k_0 singularities in the kernels and wave function (or scattering amplitude). These singularities are restored (as described in Sec. I) by adding terms to the original sum of irreducible diagrams which defines the potential. To get the new potential involving the exchange of n quanta, we take *all* diagrams with n quanta exchanged, whether reducible or not, and do the internal energy integrations picking up *all* poles except positive-energy nucleon poles corresponding to two-body unitarity cuts in reducible diagrams. These contributions have already been included in the iteration of lower-order potentials. Note that the nth-order potential contains pieces of inelastic cuts involving n mesons as referred to in Sec. I. Finally, we note that the arguments in Sec. II A serve merely to show that in the nonrelativistic domain one has reason to believe that this is a convergent procedure.

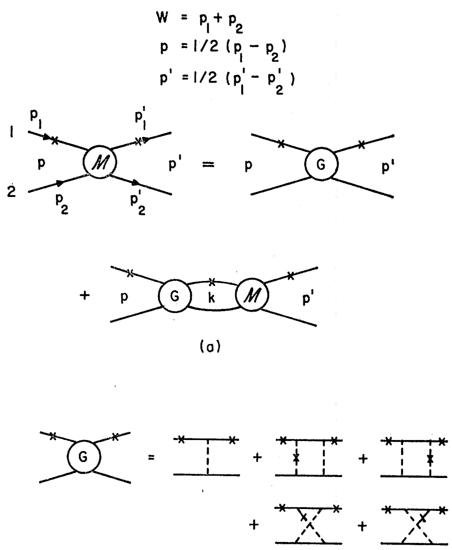

FIG. 3. Symbolic representation of the integral equation and potentials (up to two-boson-exchange terms). A cross on any particle line means that the particle is restricted to its mass shell.

As an example of the preceding, we give the wave equation if one- and two-meson potentials are retained. This relativistic wave equation is a Bethe-Salpeter equation with one of the nucleons (particle 1) on the mass shell (the variables are defined in Fig. 3):

$$\mathfrak{M}(p,p'W,)=G(p,p',W)-\int\frac{d^3k\,G(p,kW)\mathfrak{M}(k,p',W)}{(2\pi)^3(2E_k-W)2WE_k}. \quad (2.32)$$

If particle 1 is on the mass shell, then

$$(\tfrac{1}{2}W+p)^2=M^2,\quad (\tfrac{1}{2}W+p')^2=M^2, \quad (2.33)$$

which means that

$$p_0=E_p-\tfrac{1}{2}W,\quad p_0'=E_{p'}-\tfrac{1}{2}W, \quad (2.34)$$

so that $\mathfrak{M}$ and G no longer depend on the full four-momentum but only on the three-momentum. However, the restriction to three-momentum is done in a covariant fashion, so that $\mathfrak{M}$ and G are still covariant objects. They are simply scattering amplitudes with particle 2 off the mass shell. In the same way, the three-dimensional k integration is actually covariant.

The kernels are

$$G(p,k,W)=G^{(1)}(p,k,W)+G^{(2)}(p,k,W), \quad (2.35)$$

where

$$G^{(1)}(p,k,W)=\frac{-g^2M\sigma}{\sigma^2+(\mathbf{p}-\mathbf{k})^2-(E_p-E_k)^2}, \quad (2.36a)$$

$$G^{(2)}(p,k,W)=G^{(4a)}(p,k,W)+G^{(4b)}(p,k,W), \quad (2.36b)$$

and

$$G^{(4a)}(p,k,W)=\frac{-g^4M^2\sigma^2}{2(2\pi)^3}\int\frac{d^3q[\omega_{p-q}{}^2-(\omega_{k-q}+E_k-E_p)^2]^{-1}}{\omega_{k-q}[E_q{}^2-(\omega_{k-q}+E_k)^2][E_q{}^2-(W-E_k-\omega_{k-q})^2]}+k\leftrightarrow p, \quad (2.36c)$$

$$G^{(4b)}(p,k,W)=\frac{-g^4M^2\sigma^2}{2(2\pi)^3}\int\frac{d^3q[\omega_{p-q}{}^2-(\omega_{k-q}+E_k-E_p)^2]^{-1}}{\omega_{k-q}[E_{p+k-q}{}^2-(\omega_{k-q}-E_p)^2][E_q{}^2-(W-E_k-\omega_{k-q})^2]}+k\leftrightarrow p. \quad (2.36d)$$

The term $G^{(4a)}$ comes from the meson-pole contributions to the uncrossed box while $G^{(4b)}$ is the meson-pole contributions to the crossed box. For completeness we should include the negative-energy nucleon poles in these potentials also, but since they are much smaller we have ignored them. These kernels are covariant.

At this point one could adopt the philosophy that the kernels (2.36) should be treated exactly and Eq. (2.32) solved numerically using these kernels. One would argue that even though the dynamics has been accurately described only to order $(\sigma/M)^2$ by these kernels, an exact treatment guarantees Lorentz invariance and may introduce relativistic terms which are meaningful, even though comparable terms have already been neglected. We think there is much to be said for this point of view, but we will now adopt a different one—we will work in the c.m. system, and neglect all terms of order $(\sigma/M)^2$ or higher. This means that in the two-meson-exchange kernels, we neglect all terms of order σ/M.

We restrict the external momenta to order σ. Examination of the integrals show that they converge and that internal momenta can also be regarded as of order σ. Hence, neglecting all terms of order $(\sigma/M)^2$, we have the approximate equations

$$\mathfrak{M}(p,p',W)=G(p,p',W)-\int\frac{d^3k\,G(p,k,W)\mathfrak{M}(k,p',W)}{(2\pi)^3 4M(\mathbf{k}^2-\delta^2)} \quad (2.37)$$

and

$$G^{(1)}(p,k,W)=\frac{-g^2M\sigma}{\sigma^2+\boldsymbol{\Delta}^2}, \quad (2.38a)$$

$$G^{(4a)}(p,k,W)=-G^{(4b)}(p,k,W)=\frac{g^4\sigma^2}{8\pi^2}\int_\sigma^\infty\frac{d\Delta'}{(\Delta'^2-\sigma^2)^{1/2}}\frac{1}{4\Delta'^2+\boldsymbol{\Delta}^2}, \quad (2.38b)$$

where

$$\boldsymbol{\Delta}=\mathbf{p}-\mathbf{k},\quad \delta^2=M\epsilon=M(W-2M). \quad (2.39)$$

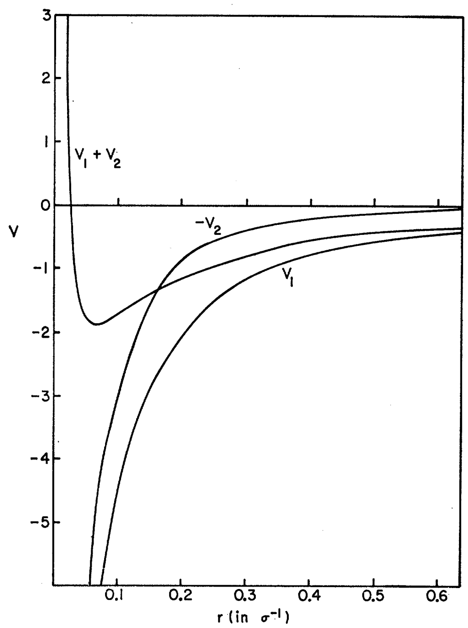

FIG. 4. Graphs of the potentials for the numerical example given in Sec. II B.

Hence, in this approximation the potential is local and the equation, when cast into position space, becomes a Schrödinger equation for the scattering amplitude.

The bound-state equation corresponding to Eq. (2.37) (see Sec. IV) is

$$\Gamma(p)=-\int\frac{d^3kG(p,k,W)\Gamma(k)}{(2\pi)^3 4M(\mathbf{k}^2-\delta^2)}. \tag{2.40}$$

Thus, in position space if we define

$$\psi(r)=\int\frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{(2\pi)^3}\frac{\Gamma(p)}{\mathbf{p}^2-\delta^2}d^3p, \tag{2.41}$$

we obtain

$$(-\nabla^2/M-\epsilon)\psi(r)=-V(r)\psi(r), \tag{2.42}$$

where

$$V(r)=\int\frac{d^3\Delta}{(2\pi)^3}\frac{G(p,k,M_B)e^{-i\Delta\cdot\mathbf{r}}}{4M^2}. \tag{2.43}$$

In this example

$$V(t)=V^{(1)}(r)+V^{(2)}(r), \tag{2.44a}$$

where

$$V^{(1)}(r)=-2\left(\frac{g^2}{32\pi}\frac{\sigma}{M}\right)\frac{e^{-\sigma r}}{r}, \tag{2.44b}$$

$$V^{(2)}(r)=\frac{8\lambda}{\pi}\left(\frac{g^2}{32\pi}\frac{\sigma}{M}\right)^2\int_\sigma^\infty\frac{d\sigma'}{(\sigma'^2-\sigma^2)^{1/2}}\frac{e^{-2\sigma' r}}{r}. \tag{2.44c}$$

The constant λ would be zero if this were a theory of scalar particle exchange because of the cancellation between the crossed and uncrossed two-meson diagrams. If it were a theory of isovector "nucleons" and mesons, then the crossed and uncrossed box give different isospin factors:

$$\begin{aligned}&\delta_{aa'}\delta_{bb'}+\delta_{ab}\delta_{a'b'} \quad \text{(uncrossed box)},\\ &-\delta_{aa'}\delta_{bb'}-\delta_{ab'}\delta_{ba'} \quad \text{(crossed box)},\end{aligned} \tag{2.45}$$

where the initial nucleons have isospin a, b and the final nucleons a', b'.

In this case

$$\lambda=\delta_{ab}\delta_{a'b'}-\delta_{ab'ba'}=\epsilon_{aa'i}\epsilon_{ibb'}, \tag{2.46}$$

which means that the second-order potential has the effect of the exchange of an equivalent distribution of heavy isovector mesons. In the actual case of nucleons interacting by the exchange of pseudoscalar mesons, a similar cancellation occurs which results in the second-order potential being largely equivalent to the exchange of a distribution of scalar mesons.

In Fig. 4 we show a graph of the potentials (2.44) for the hypothetical case when $\lambda=1$, $g^2/4\pi=14$, and $\sigma/M=1/7$ so that $\eta_1(\sigma/M)=\frac{1}{4}$. In this case, the second-order potential provides a repulsive core.

III. SPINOR THEORY

Here we extend the ideas developed in Sec. II to the case of two spin-$\frac{1}{2}$ fermions of equal mass M interacting by the exchange of a scalar meson, a pseudoscalar meson, and a massless vector meson. As we shall see, the presence of spin complicates the situation, particularly in the physically interesting case of pseudoscalar-meson exchange.

A. Scalar-Meson Exchange

The interaction Lagrangian for scalar-meson exchange is

$$\mathcal{L}_I=g\bar{\psi}\psi\phi+g\bar{\psi}\psi\phi^*. \tag{3.1}$$

The second-order scattering amplitude is again attractive. Restricting ourselves to the cases when the external particles are on the mass shell and the scattering is in the forward direction gives

$$\begin{aligned}M_S^{(2)}&=[-g^2/\sigma^2-(p-p')^2][\bar{u}(\mathbf{p})u'(\mathbf{p}')]_1\\&\qquad\times[\bar{u}(-\mathbf{p})u'(-\mathbf{p}')]_2\\&=(-g^2/\sigma^2)(\chi^*\chi')_1(\chi^*\chi')_2.\end{aligned} \tag{3.2}$$

We now look at the fourth-order case, where we see the first signs of the complications which will confront us when we deal with spin. The uncrossed box, Fig. 1(a), gives us (in the forward direction)

$$M_S^{(4a)}=\bar{u}_1(\mathbf{p})\bar{u}_2(-\mathbf{p})\mathfrak{M}_S^{(4a)}u_1'(\mathbf{p})u_2'(-\mathbf{p}), \tag{3.3a}$$

where

$$\mathfrak{M}_S^{(4a)}=\frac{ig^4}{(2\pi)^4}\int\frac{d^4k[M+\gamma_1\cdot(\frac{1}{2}W+k)][M+\gamma_2\cdot(\frac{1}{2}W-k)]}{[(E_k-E_p)^2-k_0^2-i\epsilon][(E_k+E_p)^2-k_0^2-i\epsilon](\omega^2-k_0^2-i\epsilon)^2} \tag{3.3b}$$

and the symbols were defined in the Sec. II. Following the reasoning developed in Sec. II B, we calculate the leading-order contribution by neglecting terms of order p/M. We have

$$M_S^{(4a)}=(\chi^*\cdot\chi')_1(\chi^*\cdot\chi')_2 I_S^{(4a)}, \tag{3.4a}$$

where now

$$I_S^{(4a)}=\frac{ig^4}{(2\pi)^4}\int\frac{d^4k(4M^2-k_0^2)}{[(E_k-E_p)^2-k_0^2-i\epsilon][(E_k+E_p)^2-k_0^2-i\epsilon](\omega^2-k_0^2-i\epsilon)^2}$$
$$=(4/\sigma^2)\mathfrak{M}^{(4a)}+I_P^{(4a)}, \tag{3.4b}$$

where $\mathfrak{M}^{(4a)}$ was evaluated in Sec. II. It is the new quantity $I_P^{(4a)}$ which gives us the complication. This is because when the contributions from the various k_0 poles in I_P are separated out, each separate term diverges. The sum of the terms converges, however, and in fact it converges to a result considerably smaller than the leading term, and can ultimately be neglected. Ultimately, in this case, the complication is a trivial one.

Since this I_P term will be important in the following, we show how this comes about by a detailed examination:

$$I_P^{(4a)}=\frac{-ig^4}{(2\pi)^4}\int\frac{d^4k\,k_0^2}{[(E_k-E_p)^2-k_0^2-i\epsilon][(E_k+E_p)^2-k_0^2-i\epsilon](\omega^2-k_0^2-i\epsilon)^2}. \tag{3.5}$$

Integration over k_0 and closing the contour in the lower half-plane gives the three types of terms referred to in Sec. II:

$$I_{P+}^{(4a)}=\frac{-g^4}{(2\pi)^3}\int\frac{d^3k(E_k-E_p)}{8ME_k[\omega^2-(E_k-E_p)^2]^2}, \tag{3.6a}$$

$$I_{P0}^{(4a)}=\frac{g^4}{(2\pi)^3}\int\frac{d^3k[-8(\mathbf{p}\cdot\mathbf{k})\mathbf{k}^2-4M^2\mathbf{k}^2+12(\mathbf{p}\cdot\mathbf{k})^2+8M^2(\mathbf{p}\cdot\mathbf{k})-4M^2(\mathbf{p}^2+\sigma^2)]}{4\omega[\omega^2-(E_k-E_p)^2]^2[\omega^2-(E_k+E_p)^2]^2}, \tag{3.6b}$$

$$I_{P-}^{(4a)}=\frac{g^4}{(2\pi)^3}\int\frac{d^3k(E_k+E_p)}{8ME_k[\omega^2-(E_k+E_p)^2]^2}. \tag{3.6c}$$

Note that the $+$ and $-$ integrals diverge linearly while (3.6b) diverges logarithmically. For this reason we can no longer assume that $\mathbf{k}^2\approx\sigma^2$, and hence all the $\mathbf{k}^2$ terms have been retained in the above expressions. Adding the three terms together, retaining the leading terms only, we obtain

$$I_P^{(4a)}\cong\frac{g^4}{4(2\pi)^3}\int\frac{d^3k}{[\omega^2-(E_k-E_p)^2]^2[\omega^2-(E_k+E_p)^2]^2}$$
$$\times\{(1/\omega)[-8(\mathbf{p}\cdot\mathbf{k})\mathbf{k}^2+4\mathbf{k}^2(\sigma^2-\mathbf{p}^2)-4M^2(\mathbf{p}-\mathbf{k})^2+12(\mathbf{p}\cdot\mathbf{k})^2-4\sigma^2M^2]$$
$$-(1/E_k)[-8(\mathbf{p}\cdot\mathbf{k})\mathbf{k}^2+4\mathbf{k}^2(\sigma^2-\mathbf{p}^2-M^2)+4(\mathbf{p}\cdot\mathbf{k})^2+4M^2\mathbf{p}^2]\}, \tag{3.7}$$

which is now convergent. We next expand the denominators, assuming $\mathbf{p}$, and $\sigma\ll M$ only (k unrestricted) change variables of integration from $\mathbf{k}$ to $\mathbf{k}'=\mathbf{k}-\mathbf{p}$ and do the angular integrations. When the leading terms are retained, we have

$$I_P^{(4a)}\cong\frac{-g^4}{32\pi^2M^2}\int_0^\infty\frac{k'^2dk'}{k'^2+\sigma^2}\left(\frac{1}{\omega_{k'}}-\frac{1}{E_{k'}}\right)$$
$$=(-g^4/32\pi^2M^2)[\ln(M/\sigma)-1]. \tag{3.8}$$

By comparison with (2.11), we see that the troublesome term I_P is smaller than the first term in (3.4b) by about a factor of $(\sigma/M)^3\ln(M/\sigma)$ which can be quite small.

Since the three divergent terms give a small contribution, they can be neglected and it is still possible to construct a simple convergent low-energy relativistic theory for spin-$\frac{1}{2}$ particles with scalar exchange in a manner similar to that discussed in Sec. II. To this end, it is useful to introduce the identity

$$\frac{M+\gamma\cdot k_1}{2M}=\frac{E_k+k_{10}}{2E_k}u(\mathbf{k})\bar{u}(\mathbf{k})-\frac{E_k-k_{10}}{2E_k}v(-\mathbf{k})\bar{v}(-\mathbf{k}), \tag{3.9}$$

which expresses the fact that the positive-energy off-shell nucleon of momentum $\mathbf{k}$ can be regarded as a superposition of a positive-energy mass-shell nucleon of momentum $\mathbf{k}$ and a negative-energy mass-shell nucleon of momentum $-\mathbf{k}$ (antinucleon traveling in the opposite direction so that the quantum numbers will come out right). Now in the high-$\mathbf{k}$ limit, the divergence comes from the $u(\mathbf{k})\bar{u}(\mathbf{k})$ and $v(-\mathbf{k})\bar{v}(-\mathbf{k})$ terms which go like $|\mathbf{k}|$ as $|\mathbf{k}|\to\infty$. Hence these terms can be made convergent by dividing by a power of $|\mathbf{k}|$ in some covariant fashion. An example is given by the choice

$$\mathfrak{M}_S(p,p'W)=G(p,p'W)-\frac{M^4}{(2\pi)^3}\int\frac{d^4k\,G(p,k,W)[M+\gamma_1\cdot k_1][M+\gamma_2\cdot k_2]\mathfrak{M}_S(k,p',W)\delta(M^2-k_1^2)\theta(k_{10})}{(M^2-k_2^2)(M^2-k^2)^2}. \tag{3.10}$$

Equation (3.10) is explicitly covariant. After the k_0 integration has been performed, the identity (3.9) gives us

$$\begin{aligned}\frac{M^2}{M^2-k^2}(M+\gamma_1\cdot k_1)&\to\frac{2M^3}{E_p(2E_k-E_p)}[u(\mathbf{k})\bar{u}(\mathbf{k})]_1,\\ \frac{M^2}{M^2-k^2}(M+\gamma_2\cdot k_2)&\to\frac{2M^3}{E_p(2E_k-E_p)}\left(\frac{E_p}{E_k}u(-\mathbf{k})\bar{u}(-\mathbf{k})-\frac{(E_k-E_p)}{E_k}v(\mathbf{k})\bar{v}(\mathbf{k})\right)_2,\end{aligned} \tag{3.11}$$

and both of these terms are now convergent in $|\mathbf{k}|$. They differ from the exact expressions by the factor $M^2/(M^2-k^2)$ so that we make an error of

$$1-\frac{M^2}{E_p(2E_k-E_p)}=\frac{2E_pE_k-2M^2-p^2}{E_p(2E_k-E_p)}\approx\frac{\mathbf{k}^2}{M^2}, \tag{3.12}$$

and hence (3.10) is accurate to order M^{-2}.

It would now be a straightforward matter to exapnd (3.10) to order M^{-1} and obtain the effective lowest-order potentials as we did at the end of Sec. II for the scalar case. In such a treatment one sees that the $[v(\mathbf{k})\bar{v}(\mathbf{k})]_2$ term is down by M^{-3} and consequently must be neglected. To order M^{-2}, we must include the positive-energy spinor states only.

B. Pseudoscalar-Meson Exchange

For the pseudoscalar interaction Lagrangian, we take

$$\mathfrak{L}_I=ig\bar{\psi}\gamma^5\psi\phi+ig\bar{\psi}\gamma^5\psi\phi^*. \tag{3.13}$$

As in Sec. III A, if we restrict ourselves to the case when the scattering is in the forward direction,

$$\begin{aligned}M_P{}^{(2)}&=\frac{g^2}{\sigma^2-(p-p')^2}[\bar{u}(p)\gamma^5u'(p')]_1[\bar{u}(-\mathbf{p})\gamma^5u'(-\mathbf{p}')]_2\\&=0,\end{aligned} \tag{3.14}$$

because of the off-diagonal nature of the γ^5 matrix.

For the fourth-order box diagram in the forward direction, we have

$$M_P{}^{(4a)}=\bar{u}_1(\mathbf{p})\bar{u}_2(-\mathbf{p})\,\mathfrak{M}_P{}^{(4a)}u_1'(\mathbf{p})u_2'(-\mathbf{p}), \tag{3.15}$$

where

$$\mathfrak{M}_P{}^{(4a)}=\frac{ig^4}{(2\pi)^4}\int\frac{d^4k[M-\gamma_1\cdot(\frac{1}{2}W+k)][M-\gamma_2\cdot(\frac{1}{2}W-k)]}{[(E_k-E_p)^2-k_0^2-i\epsilon][(E_k+E_p)^2-k_0^2-i\epsilon](\omega^2-k_0^2-i\epsilon)^2}. \tag{3.16}$$

Again, neglecting terms of order p/M, we have

$$M_P{}^{(4a)}=(\chi^*\cdot\chi')_1(\chi^*\cdot\chi')_2I_P{}^{(4a)}, \tag{3.17}$$

where $I_P{}^{(4a)}$ is precisely the term introduced in Eq. (3.5). Hence, the result (3.8) applies to the pseudoscalar case.

We now can see quite clearly why the pseudoscalar case gives so much trouble. The pseudoscalar interaction is intrinsically relativistic. By this we mean that in fourth order the retardation of the potential cannot be ignored—it gives the major contribution, and the full interaction must be treated in order to obtain a finite result. The source of the difficulty is not so much that the γ^5 interaction strongly favors particle-antiparticle coupling (or nucleon pair production), but rather that the γ^5 interaction suppresses the interaction of positive-energy states [cf. Eq. (3.14)] to the point where in fourth order the dominant contribution comes from the meson singularities, and the nucleon singularities serve only the secondary role of providing a built-in cutoff.

We are now in a position to answer some of the questions posed in Sec. I. Since there is no nonrelativistic theory of the one-pion-exchange (OPE) force, one should not be surprised that simple nonrelativistic models built around OPE potentials have never worked. Furthermore, the iteration of the OPE force is not the most important force in the intermediate region; there is a large contribution from two-pion-exchange (TPE) processes which is not merely a relativistic generalization of the second OPE Born approximation. Since the dominant contribution to the TPE force comes from the retardation of the pions, it is an intrinsically new force. Although it has been known for a long time that a large attractive force in the intermediate region is needed to bind the deuteron, a common speculation is that this force must be due to some significant low-energy S-wave pion-pion interaction. It has not always been appreciated that this attraction comes partly from a straightforward analysis of the crossed and uncrossed box diagrams.[2]

C. Massless Vector-Meson Exchange

As a practical application of the ideas developed so far, we consider the case of the hydrogen atom. We show how we can derive an effective Dirac equation for the interaction of the electron in the field of the proton. This effective equation has already been discussed by Grotch and Yennie.[9]

Three new features enter the hydrogen problem: (i) The fermion masses are unequal, (ii) the exchanged boson has zero mass, and (iii) the exchanged boson is a vector particle. We first discuss the unequal-mass problem in the spin-zero case.

For unequal masses, we define the internal momenta as (particle 1 is the proton with mass M; particle 2 is the electron with mass m)

$$k_1=[M/(m+M)]W+k\,,\quad k_2=[m/(M+m)]W-k\,, \tag{3.18}$$

where, when both particles are on the mass shell in the c.m. system,

$$k_0=(mE_k-Me_k)/(M+m)\,,\quad e_k\equiv(m^2+\mathbf{k}^2)^{1/2}. \tag{3.19}$$

The fourth-order uncrossed box for scalar photons is

$$\mathfrak{M}_\gamma{}^{(4a)}=\frac{ie^4}{(2\pi)^4}\int\frac{d^4k}{[(k-p)^2-i\epsilon][(k-p')^2-i\epsilon](E_k{}^2-\{[M/(M+m)]W+k_0\}^2-i\epsilon)(e_k{}^2-\{[m/(M+m)]W-k_0\}^2-i\epsilon)}. \tag{3.20}$$

The k_0 singularities in the complex k_0 plane are shown in Fig. 5 for the particularly simple case when $\mathbf{p}=\mathbf{p}'=0$. Note that the k_0 pole at the positive-energy proton $(k_0=E_k-M)$ remains separated from the potential poles in the lower half-plane until $|\mathbf{k}|\approx M$, while the positive-energy electron pole remains separated from the potential poles in the upper half-plane only for $|\mathbf{k}|$ of the order of m. Hence, closing the contour in the lower half-plane and retaining the proton pole should provide a better approximation than taking the electron pole in the upper half-plane. Specifically, we shall see below that it is easy to approximate the proton-pole term to order M^{-2}, while in the same approximation the electron-pole term would be accurate to order m^{-2} only. This is merely a complicated way to express the fact that it is more accurate to neglect the off-mass-shell behavior of the heavy proton that it is to neglect the off-mass-shell behavior of the lighter electron.

If we specialize (3.20) to $\mathbf{p}=\mathbf{p}'$, and $|\mathbf{p}|=i\delta$, where δ^2 is twice the reduced mass times the binding energy (we are in the bound-state region below threshold), we obtain

$$\mathfrak{M}_{\gamma+}{}^{(4a)}=\frac{-e^4}{2(2\pi)^3}\times\int\frac{d^3k}{E_k[(\mathbf{k}-\mathbf{p})^2-(E_p-E_k)^2]^2[e_k{}^2-(e_p+E_p-E_k)^2]}. \tag{3.21}$$

Retaining all terms to order M^{-1} but neglecting terms of order M^{-2} gives

$$\mathfrak{M}_{\gamma+}{}^{(4a)}\cong\frac{-e^4}{2(2\pi)^3}\int\frac{d^3k}{(\mathbf{k}^2+\delta^2)(\mathbf{k}-\mathbf{p})^4[M+e_p]}$$
$$=(-e^4/64\pi M\delta^3)M/(M+e_p)\,. \tag{3.22}$$

Had we taken the electron pole instead of the proton pole, (3.22) would have been accurate to order m^{-2} only. Had we evaluated (3.21) above threshold, the result would diverge corresponding to the existence of the infinite Coulomb phase factor.[10]

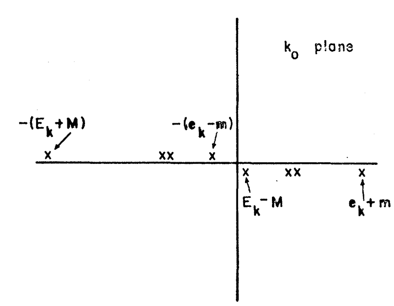

FIG. 5. Location of the singularities in the complex k_0 plane of the integrand of Eq. (3.20) for $\mathbf{p}=\mathbf{p}'=0$.

[9] H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969).
[10] H. Suura and D. R. Yennie, Phys. Rev. Letters **10**, 69 (1963).

In evaluating the contribution from the boson poles of (3.20), we encounter an infrared divergence which is a consequence of the zero mass of the bosons. To treat this difficulty it is necessary only to include the crossed fourth-order diagram, which cancels the divergence and gives a result smaller than (3.21) (the bosons are neutral). To see how this cancellation works, we add the crossed diagram to (3.20), restricting $\mathbf{p}=\mathbf{p}'$, to obtain

$$\mathfrak{M}_{\gamma}{}^{(4)}=\frac{ie^4}{(2\pi)^4}\int\frac{d^4k[E_k{}^2+E_{2p-k}{}^2-2E_p{}^2-2(p_0-k_0)^2]}{[(k-p)^2-i\epsilon]^2[E_k{}^2-(E_p-p_0+k_0)^2-i\epsilon][E_{2p-k}{}^2-(E_p+p_0-k_0)^2-i\epsilon][e_k{}^2-(e_p+p_0-k_0)^2-i\epsilon]}$$
$$=\frac{2ie^4}{(2\pi)^4}\int\frac{d^4k}{[(k-p)^2-i\epsilon][E_k{}^2-(E_p-p_0+k_0)^2-i\epsilon][E_{2p-k}{}^2-(E_p+p_0-k_0)^2-i\epsilon][e_k{}^2-(e_p+p_0-k_0)^2-i\epsilon]}. \quad (3.23)$$

We may now pick up the potential pole contribution from (3.23), which to lowest order in M^{-1} and m^{-1} is

$$\mathfrak{M}_{\gamma 0}{}^{(4)}\cong\frac{e^4}{(2\pi)^3}\int\frac{d^3k}{8M^2m(\mathbf{k}-\mathbf{p})^4}=e^4/64\pi M^2m\delta\,. \quad (3.24)$$

Consequently, we see that the potential poles are suppressed by

$$\mathfrak{M}_{\gamma 0}{}^{(4)}/\mathfrak{M}_{\gamma +}{}^{(4a)}\cong-\delta^2/Mm=-2\epsilon/(M+m)\,, \quad (3.25)$$

where ϵ is the binding energy.

We now are in a position to discuss the realistic case with spin. The idea is to obtain a potential correct in the fourth order up to small terms of the size $(M)^{-2}$. Our previous estimate (3.25) shows that the photon poles can be expected to contribute terms of the order of $(Mm)^{-1}$ to the potential. To avoid having to calculate these terms explicitly, we make use of the old trick and use the gauge invariance of the theory to modify the interaction so that these photon poles are suppressed by an additional factor of M^{-1}, and thus may be ignored. The modification we make is

$$\gamma^0\to\gamma^0-\boldsymbol{\gamma}\cdot\boldsymbol{\Delta}\Delta^0/\boldsymbol{\Delta}^2\,, \quad (3.26)$$

where Δ is the four-momentum carried by the interacting photon. It is easy to show that any modification of the form $\gamma^\mu\to\gamma^\mu-\gamma\cdot\Delta f(\Delta)$, where f is any arbitrary (not necessarily covariant) function of Δ will leave the sum of all photon-exchange diagrams unchanged.[11] The noncovariant choice (3.26) therefore does not change the over-all result, but is convenient because it suppresses the photon-pole contributions to the fourth-order diagrams by an extra factor of M^{-1}, as we will now demonstrate.

We now proceed with the calculation. The decomposition (3.9) for the proton-projection operator shows that the $v\bar{v}$ term is of order M^{-1} when the photon poles are taken (it is zero at the positive-energy proton pole), and the estimate (3.24) shows that the photon poles are already down by M^{-1} with respect to the leading term, so that the negative-energy proton term may be eliminated in all cases. Furthermore, to order M^{-1} the proton spinors reduce to

$$\bar{u}_p(\mathbf{p})\left[\gamma^0-\frac{\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{k})(p-k)^0}{(\mathbf{p}-\mathbf{k})^2}\right]u_p(\mathbf{k})\to -\frac{(p-k)^2}{(\mathbf{p}-\mathbf{k})^2}+\frac{(p-k)^0}{(\mathbf{p}-\mathbf{k})^2}\left(\frac{\mathbf{p}^2-\mathbf{k}^2}{2M}\right), \quad (3.27a)$$

$$\bar{u}_p(\mathbf{p})\gamma^i u_p(\mathbf{k})\to(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma^i+\sigma^i\boldsymbol{\sigma}\cdot\mathbf{k})/2M\,. \quad (3.27b)$$

Since the second term in (3.27)a is of order M^{-2} at the proton pole, and M^{-1} at the photon poles, it may be neglected in all cases. Finally, we have, to order M^{-1}, the contributions from (3.27a) to both the crossed and uncrossed diagrams[12]:

$$M_{V(0,0)}{}^{(4)}=\frac{ie^4}{(2\pi)^4}\times\int d^4k\,\bar{u}_e(-\mathbf{p})\left[\gamma^0-\frac{\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{k})}{(\mathbf{p}-\mathbf{k})^2}(p-k)^0\right](m-\gamma\cdot\{[mW/(M+m)]-k\})^{-1}\left[\gamma^0-\frac{\boldsymbol{\gamma}\cdot(\mathbf{k}-\mathbf{p}')(k-p)^0}{(\mathbf{k}-\mathbf{p}')^2}\right]u_e(-\mathbf{p}')$$
$$\times\frac{1}{(\mathbf{p}-\mathbf{k})^2(\mathbf{k}-\mathbf{p}')^2}\left[\frac{1}{E_k{}^2-(E_p-p_0+k_0)^2-i\epsilon}+\frac{1}{E_{p+p'-k}{}^2-(E_p+p_0-k_0)^2-i\epsilon}\right]. \quad (3.28)$$

[11] R. P. Feynman, Phys. Rev. **76**, 769 (1949). More precisely, this argument applies only to amplitudes in which the external particles are on the mass shell. For off-mass-shell amplitudes and bound-state wave functions the gauge transformation *does* change the amplitude, but this change makes no contribution to any physically observable amplitudes or matrix elements.

[12] We use the notation $M_{V(0,0)}$ for the $\gamma^0\gamma^0$ part of the vector exchange. In Eq. (3.33) below we write M_V for the *total* vector exchange (including the $\gamma^i\gamma^i$ part).

Note that the transformation (3.26) leads to a cancellation of the photon poles from (3.28) so that to order M^{-1} the (0,0) part of the scattering amplitude no longer has any photon poles. Hence, the k_0 integration in (3.28) gives only the proton pole, and at this pole

$$p_0-k_0=E_p-E_k\cong(\mathbf{p}^2-\mathbf{k}^2)/2M\,, \tag{3.29}$$

so that to order M^{-1}

$$M_{V(0,0)}{}^{(4)}=\frac{-e^4}{(2\pi)^3}\times\int d^3k\,\bar{u}_e(-\mathbf{p})\left[\gamma^0-\frac{\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{k})(\mathbf{k}^2-\mathbf{p}^2)}{2M(\mathbf{p}-\mathbf{k})^2}\right][m-\gamma^0(W-E_k)-\boldsymbol{\gamma}\cdot\mathbf{k}]^{-1}\left[\gamma^0-\frac{\boldsymbol{\gamma}\cdot(\mathbf{k}-\mathbf{p}')(\mathbf{p}'^2-\mathbf{k}^2)}{2M(\mathbf{k}-\mathbf{p}')^2}\right]u_e(-\mathbf{p}')\Big/2M(\mathbf{p}-\mathbf{k})^2(\mathbf{p}'-\mathbf{k})^2. \tag{3.30}$$

Finally, using the fact that

$$\bar{u}_p(\mathbf{p})\gamma^0[(\mathbf{p}^2-\mathbf{k}^2)/2M]u_p(\mathbf{k})\cong\bar{u}_p(\mathbf{p})\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{k})\bar{u}_p(\mathbf{k})=\boldsymbol{\gamma}_p\cdot(\mathbf{p}-\mathbf{k})\,, \tag{3.31}$$

where

$$\gamma_p\equiv\bar{u}_p\gamma u_p\,, \tag{3.32}$$

we can write, including the $\boldsymbol{\gamma}$ terms (3.27b),

$$M_V{}^{(4)}\cong\frac{-e^4}{(2\pi)^3}\int\frac{d^3k\,\bar{u}_e(-p)}{2M}\left[\frac{\gamma^0\gamma_p{}^0}{(\mathbf{p}-\mathbf{k})^2}+\frac{\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{k})\boldsymbol{\gamma}_p\cdot(\mathbf{p}-\mathbf{k})}{(\mathbf{p}-\mathbf{k})^4}-\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}_p}{(\mathbf{p}-\mathbf{k})^2}\right]\times[m-\gamma^0(W-E_k)-\boldsymbol{\gamma}\cdot\mathbf{k}]\left[\frac{\gamma^0\gamma_p{}^0}{(\mathbf{k}-\mathbf{p}')^2}+\frac{\boldsymbol{\gamma}\cdot(\mathbf{k}-\mathbf{p}')\boldsymbol{\gamma}_p\cdot(\mathbf{k}-\mathbf{p}')}{(\mathbf{k}-\mathbf{p}')^4}-\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}_p}{(\mathbf{k}-\mathbf{p}')^2}\right]u_e(-\mathbf{p}')\,. \tag{3.33}$$

Hence, the fourth-order term now appears as the iteration of an effective interaction which can be written as

$$G(p,k)=-[e^2/(\mathbf{p}-\mathbf{k})^2](\gamma^0\gamma_p{}^0-\gamma_\perp{}^i\gamma_{\perp p}{}^i)\,, \tag{3.34}$$

where

$$\gamma_\perp{}^i\equiv\gamma^i-\boldsymbol{\gamma}\cdot\boldsymbol{\Delta}\Delta^i/\boldsymbol{\Delta}^2\,, \tag{3.35}$$

and the correct integral equation which sums this interaction is

$$\Gamma(p)=-\int\frac{d^3k\,G(p,k)[m-\gamma^0(W-E_k)-\boldsymbol{\gamma}\cdot\mathbf{k}]^{-1}\Gamma(k)}{(2\pi)^3 2M}. \tag{3.36}$$

The corresponding position-space bound-state wave equation is precisely the equations used recently by Grotch and Yennie.[9] It is a Dirac equation for the electron with an effective potential given by (3.34). The terms of order M^{-2} and the other terms that we have neglected could be evaluated explicitly to give a small fourth-order potential. These "small" terms give a larger contribution than one might expect because of the fact that the integrals diverge, and are cut off only by the form factor of the nucleon arising from the strong interactions. (For details see Ref. 9).

IV. CONNECTION BETWEEN BOUND-STATE AND SCATTERING EQUATIONS

Here, we show the simple way in which bound-state equations can be obtained from any of the scattering equations discussed in this paper.

The equations we have discussed can be written in the general form

$$\mathfrak{M}(p,p',W)=G(p,p',W)+\frac{i}{(2\pi)^4}\int d^4k\,G(p,k,W)\times\Delta(W,k)\,\mathfrak{M}(k,p',W)=G(p,p',W)+G(p,k,W)\Delta(W,k)\mathfrak{M}(k,p',W)\,, \tag{4.1}$$

where, in the second form, integration over d^4k and multiplication by $i/(2\pi)^4$ is implied (for any repeated variable), G is any interaction kernel, p and p' are the relative four-momenta of the initial and final states defined previously, and W is the total energy in the c.m. system. The function $\Delta(W,k)$ is the appropriate free two-body propagator. Different choices of Δ lead to different types of relativistic wave equations. The original Bethe-Salpeter equation corresponds to both particles being off the mass shell, so that for two bosons of equal mass M, we have

$$\Delta(W,k)=[M^2-(\tfrac{1}{2}W+k)^2-i\epsilon]^{-1}\times[M^2-(\tfrac{1}{2}W-k)^2-i\epsilon]^{-1}\,, \tag{4.2}$$

while for the class of equations suggested in this paper, we have

$$\Delta(W,k)=[M^2-(\tfrac{1}{2}W-k)^2]^{-1}\delta[M^2-(\tfrac{1}{2}W+k)^2]\times\theta(\tfrac{1}{2}W+k_0)\,, \tag{4.3}$$

corresponding to one particle on the mass shell. The Blankenbecler-Sugar[3] choice corresponds to

$$\Delta(W,k)=2\pi\int_{4M^2}^{\infty} ds(s-W^2)^{-1}\delta[M^2-(\tfrac{1}{2}s^{1/2}+k)^2] \times\delta[M^2-(\tfrac{1}{2}s^{1/2}-k)^2], \quad (4.4)$$

which is equivalent to keeping only the two-body cut.

We can write (4.1) in an alternative form. First, we note that

$$\mathfrak{M}(p,p',W)=G(p,p',W)+\mathfrak{M}(p,k,W)\Delta(W,k) \times G(k,p',W). \quad (4.1')$$

Multiplying (4.1) by $\mathfrak{M}(r,p,W)\Delta(W,p)$, integrating over p, and using (4.1′) gives a nonlinear version of (4.1)

$$\mathfrak{M}(r,p',W)=G(r,p',W)+\mathfrak{M}(r,p,W)\Delta(W,p)\mathfrak{M}(p,p',W) -\mathfrak{M}(r,p,W)\Delta(W,p)G(p,k,W)\Delta(W,k)\mathfrak{M}(k,p',W). \quad (4.5)$$

One can easily see that this is equivalent to (4.1) by iterating; the last term on the right-hand side is just sufficient to cancel out the overcounting produced by the second term on the right-hand side.

The equations which describe bound states can be obtained directly from (4.5). One simply demands that (4.5) hold in the vicinity of a bound state. Two equations are obtained—the bound-state wave equation and the normalization condition for the bound-state wave function.

To obtain these equations, we assume that, in the vicinity of the bound state of mass M_B, we have a pole

$$\mathfrak{M}(p,p',W)=\Gamma^\dagger(p)\Gamma(p')/(M_B{}^2-W^2)+R(p,p',W), \quad (4.6)$$

where $\Gamma^\dagger$ is the adjoint of Γ (describing the time-reversed coupling constant) and R is a background function with no singularities at $W^2=M_B{}^2$. Inserting (4.6) into (4.5) and demanding that the coefficient of the double pole be zero on the right-hand side gives the bound-state equation:

$$\Gamma(p)=G(p,k,M_B)\Delta(M_B,k)\Gamma(k), \qquad \Gamma^\dagger(p)=\Gamma^\dagger(k)\Delta(M_B,k)G(k,p,M_B). \quad (4.7)$$

These equations also guarantee that R does not contribute to the single-pole term. Finally, equating the residues of the single-pole term gives a new condition

$$1=-\Gamma^\dagger(p)(\partial/\partial W^2)\Delta(W,p)|_{W^2=M_B{}^2}\Gamma(p)+\Gamma^\dagger(p) \times(\partial/\partial W^2)[\Delta(W,p)G(p,k,W)\Delta(W,k)]\Gamma(k)|_{W^2=M_B{}^2}. \quad (4.8)$$

This is the normalization condition for the wave function. Performing the differentiation of the last term on the right-hand side and using (4.7) gives an alternative form for the normalization condition, which we write out explicitly as

$$1=i\int\int\frac{d^4p\,d^4p'}{(2\pi)^4}\Gamma^\dagger(p)\frac{\partial}{\partial W^2}\left[\Delta(W,p)\delta^4(p-p') +\frac{i}{(2\pi)^4}G(p,p',W)\right]\Bigg|_{W^2=M_B{}^2}\Gamma(p'). \quad (4.9)$$

In the special case (4.2), this normalization condition is identical to the usual normalization condition given for Bethe-Salpeter amplitudes.

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