Anomalous Commutators and the Box Diagram

DAVID G. BOULWARE*
University of Washington, Seattle, Washington 98105

AND

R. JACKIW†
Layman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
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In continuation of previous investigations, commutator anomalies associated with the box diagram are exhibited. It is found that q-number Schwinger terms are present in the general case for commutators of currents which are bilinear in fermion fields. Nevertheless, the Weinberg sum rule is not violated. A c-number Schwinger term, which involves three derivatives of a δ function, is also found.

I. INTRODUCTION

EVER since Schwinger[1] publicized the necessary existance of a gradient term in the $[j^0, j^k]$ equal-time commutator (ETC), the nature of this Schwinger term (ST) has remained undetermined. In theories where the ST is calculable by canonical procedures, its character, i.e., whether it is a c or q number, is obviously established by inspection. Examples of such theories are scalar electrodynamics, the σ model, and the algebra of fields. However in quark models, where the current is bilinear in fermion fields, the ST is not given by naive canonical manipulations and its character is more recondite. In the literature there appear conflicting statements, averring both a c- and q-number character to this object.[2] Evidently the ETC is not well defined, at least in perturbation theory, and different results are arrived at, depending on the definition one settles upon.

Of the various prescriptions for calculating the ETC that can be adopted, one has recently been the object of considerable attention: the high-energy method of Bjorken,[3] and of Johnson and Low[4] (BJL). With this approach, Johnson and Low[4] exhibited the existence of q-number ST, in Fermion current ETC's, associated with the singularities of the triangle graph.[5] Specifically the vacuum–single-particle-state matrix element of the ST was found to be nonzero. In the present paper we continue the BJL analysis to exhibit the ST associated with (fermion) box diagrams.[6] We also exhibit a *third* derivative of the δ-function c-number ST, which is rarely mentioned in the literature.

Section II contains the principal analysis and results of this investigation. In Sec. III we compare the present conclusions with those obtained by the point-splitting technique of Schwinger.[1] Concluding remarks about the significance of these results comprise Sec. IV.

II. HIGH-ENERGY DETERMINATION OF COMMUTATORS

The BJL definition of an ETC is obtained as follows: Consider the covariant T^* product of two operators[7]

$$T_{AB}^*(q)=\int d^4x\, e^{iqx}\langle\alpha|T^*A(x)B(0)|\beta\rangle. \tag{2.1}$$

This object is calculable for example by the covariant Feynman-Dyson perturbation theory. Drop all polynomials in q_0, obtaining the T product

$$T_{AB}(q)=\int d^4x\, e^{iqx}\langle\alpha|TA(x)B(0)|\beta\rangle. \tag{2.2}$$

The ETC between A and B is then, *by definition*[8]

$$C_{AB}(\mathbf{q})=\lim_{q_0\to\infty} -iq_0T_{AB}(q)$$
$$\equiv\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle. \tag{2.3}$$

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† Junior Fellow, Society of Fellows. Present address: Massachusetts Institute of Technology, Cambridge, Mass. 02139.

[1] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[2] The full range of opinion is bracketed by D. G. Boulware, Phys. Rev. **151**, 1024 (1966), who asserts the c-number property in spinor electrodynamics; and R. A. Brandt, *ibid.* **166**, 1795 (1968), who exhibits the possibility of a q-number ST in the same theory. The BJL definition yields a c-number commutator to order α^2, D. G. Boulware and J. Herbert (unpublished). An even more radical point of view was taken by G. Källén in lectures delivered at winter schools in Karpacz and Schladmig, 1968 (unpublished); he disputed the existence of these commutator anomalies entirely.

[3] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[4] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966).

[5] Further study of the ETC associated with the triangle graph is to be found in R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969); S. L. Adler and D. G. Boulware, Phys. Rev. **184**, 1740 (1969).

[6] Such terms have been found before and commented upon by D. G. Boulware and S. Deser, Phys. Rev. **175**, 1912 (1968).

[7] The metric (1, −1, −1, −1) is used. The Dirac matrices are defined by $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ and $\gamma^5=-\gamma^{5\dagger}=\gamma^0\gamma^1\gamma^2\gamma^3$.

[8] Johnson and Low, Ref. 4, show that this definition is equivalent in position space to the following prescription for calculating the ETC:

$$\langle\alpha|[A(0,\mathbf{x}),B(0)]|\beta\rangle =\lim_{\eta\to0}\langle\alpha|TA(\eta,\mathbf{x})B(0)|\beta\rangle-\langle\alpha|TA(-\eta,\mathbf{x})B(0)|\beta\rangle.$$

If the limit (2.3) diverges, one interprets this as a divergence in the (renormalized) matrix elements of the ETC.

The advantages of this definition of the ETC are the following: (i) To the extent that T^* products are measurable, (2.3) is an operational and measurable definition for the ETC. (ii) When the Low representation for the T product converges, (2.3) gives a plausible expression for the ETC[3]

$$C_{AB}(\mathbf{q})=\int\frac{dq_0'}{(2\pi)}[\rho_{AB}(q_0',\mathbf{q})-\rho_{BA}(q_0',\mathbf{q})], \tag{2.4a}$$

$$\rho_{AB}(q)=\sum_n (2\pi)^4\delta^4(q+p_\alpha-p_n)\times\langle\alpha|A(0)|n\rangle\langle n|B(0)|\beta\rangle, \tag{2.4b}$$

$$\bar{\rho}_{BA}(q)=\sum_n (2\pi)^4\delta^4(q+p_\beta-p_n)\langle\alpha|B(0)|n\rangle\langle n|A(0)|\beta\rangle. \tag{2.4c}$$

[Of course (2.4a) must also converge.] From (2.4), various high-energy sum rules can be derived.[9]

It has been discovered that the ETC calculated by this definition need not coincide with its canonical value.[4,10] In Sec. III we shall also show that the BJL definition need not coincide with the point-spreading technique. When the ETC differs from the expected canonical value, we shall refer to it as anomalous.

The shortcomings of the BJL definition of an ETC, when it is anomalous, are the following: (i) *Canonical* ETC are necessary in setting up the formalism. For example, the Dyson-Schwinger integral equations for n-point functions are derived with the help of canonical ETC. (ii) Various formal gauge properties of the theory can be verified only with the help of canonical ETC. (iii) The commutators which arise from the BJL prescription are valid only when evaluated between restricted matrix elements. In particular, the Jacobi identity, in general, is not satisfied.

It should be emphasized that these shortcomings are not unique to the BJL prescription; they are related to the fundamental consistency of the field theory. At least in perturbation theory, all known nontrivial field theories (with the exception of the φ^3 theory which does not possess a vacuum) develop divergences which are associated with the breakdown of the canonical commutation relations. As a consequence, the gauge properties must be imposed during the renormalization program. If the amplitudes which are calculated by means of the Feynman rules are taken to *define* the covariant time-ordered products and hence the Wightman functions, the BJL prescription then calculates the limit given in Ref. 8.

We have not reconciled these two approaches to ETC. It is known, however,[11] that when the only anomaly is a ST then it is sometimes (but not always) possible to maintain gauge properties by introducing (anomalous) seagull terms in the T^* product, and gauge invariance is effected by a cancelation between ST and divergences of seagull terms. We are not aware of any method to handle the first of the above-mentioned difficulties. What should be the structure of the canonical formalism when the ETC's as calculated by (2.3) are anomalous? We do not pursue here the problems associated with this conflict any further, but confine ourselves to the BJL definition (2.3).

A and B are taken to be currents; thus, we are interested in

$$T^{\mu\nu}(q)=\int d^4x\, e^{iqx}\langle\alpha|Tj^\mu(0)j^\nu(0)|\beta\rangle, \tag{2.5a}$$

$$C^{\mu\nu}(\mathbf{q})=\lim_{q_0\to\infty}-iq_0T^{\mu\nu}(q)\equiv\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\alpha|[j^\mu(0,\mathbf{x}),j^\nu(0)]|\beta\rangle. \tag{2.5b}$$

Internal symmetry will be ignored for the most part. Hence, j^μ is given by $\bar{\psi}\gamma^\mu\psi$, or by $i\bar{\psi}\gamma^5\gamma^\mu\psi$, in which case it will be designated by j_5^μ. The fermion fields are assumed to have only renormalizable interactions with bosons. Thus, we are interested in Yukawa-type scalar, pseudoscalar, and vector couplings, exclusively.

The above matrix elements when α is the vacuum and β a single-meson state have been studied exhaustively in lowest-order perturbation theory.[4,5] Thus we will not concern ourselves with this case here.

When both α and β are the vacuum state, the T^* product is ill defined in lowest-order perturbation theory.[12] However, all the ambiguities and divergences associated with this graph can be collected into a polynomial in q. Dropping such contact terms, one is left with

$$T^{\mu\nu}(q)=\frac{i}{2\pi^2}\int_0^1 dx\,\ln\left(1-\frac{q^2}{m^2}x(1-x)\right)\times[(q^\mu q^\nu-q^{\mu\nu}q^2)x(1-x)+\tfrac{1}{2}m^2g^{\mu\nu}(1\mp1)]. \tag{2.6a}$$

The sign in the term proportional to $m^2g^{\mu\nu}$ is $-$ ($+$) when the current is vector (axial vector). Note that if the currents also carry a discrete internal symmetry—with matrices λ_a—then the generalization of (2.6a)

[9] A recent example of the derivation of sum rules with the help of the BJL limit is due to C. Callan and D. Gross, Phys. Rev. Letters **22**, 156 (1969).

[10] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969); S. L. Adler and Wu-Ki Tung, *ibid.* **22**, 978 (1969); R. Jackiw and G. Preparata, Phys. Rev. **185**, 1748 (1969). The relevant paper by Jackiw and Preparata is "*T* products at high energy and commutators."

[11] For a discussion see, e.g., R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969).

[12] Some discussion of the BJL approach and the vacuum polarization tensor is to be found in R. Jackiw and G. Preparata, Phys. Rev. **185**, 1748 (1969).

which we call $T_{ab}{}^{\mu\nu}$ is related to $T^{\mu\nu}$ by

$$T_{ab}{}^{\mu\nu} = (\mathrm{Tr}\lambda_a\lambda_b)T^{\mu\nu}. \qquad (2.6\mathrm{b})$$

It is clear that the vector–axial-vector T product vanishes. Of interest for the ST is T^{0i}. This is the same both for the vector and axial-vector currents. We have from (2.6a)

$$-iq_0T^{0k}(q) = q^k\frac{q_0^2}{2\pi^2}\int_0^1 dx\, x(1-x)$$
$$\times\ln\left(1-\frac{\mathbf{q}^2}{m^2}x(1-x)\right) \approx \frac{-q^k}{2\pi^2}\int_0^1 dx$$
$$\times\left[-q_0^2x(1-x)\ln\left(\frac{-q_0^2}{m^2}x(1-x)\right)+m^2\right]+\frac{q^k\mathbf{q}^2}{12\pi^2}. \quad (2.7)$$

Terms that go to zero as $q_0\to\infty$ have been dropped in the second equation in (2.7). The coefficient of $-q^k/2\pi^2$ diverges quadratically (up to logarithmic factors) as $q_0\to\infty$; this is interpreted as the usual, quadratically divergent single-derivative ST, already found by Schwinger.[1] However in addition to this term, we find a finite expression proportional to $q^k\mathbf{q}^2$. This is a triple derivative of a δ function. We have therefore from (2.3)

$$\langle\Omega|[j^0(0,\mathbf{x}),j^k(0)]|\Omega\rangle = \langle\Omega|[j_5{}^0(0,\mathbf{x}),j_5{}^k(0)]|\Omega\rangle$$
$$= iS\partial^k\delta(\mathbf{x})+(i/12\pi^2)\partial^k\nabla^2\delta(\mathbf{x}), \qquad (2.8)$$

where S is quadratically divergent.

The triple derivative term is not usually found in the literature.[13] Indeed a "proof" is given frequently with the result that

$$\langle\Omega|[j^0(0,\mathbf{x}),j^k(0)]|\Omega\rangle = i\partial^k\delta(\mathbf{x})\int_0^\infty da^2\,\rho(a^2), \quad (2.9)$$

where ρ is the spectral function. However such a "proof" is valid only when the integral $\int_0^\infty da^2\,\rho(a^2)$ converges. It is seen that in the present instance this quantity is divergent, and examination of (2.7) shows that it is precisely the presence of a quadratic divergence $q^kq_0^2$ that is responsible for the occurrence of the finite triple-derivative term. Thus, the result (2.9) may be circumvented in the presence of divergences.[14]

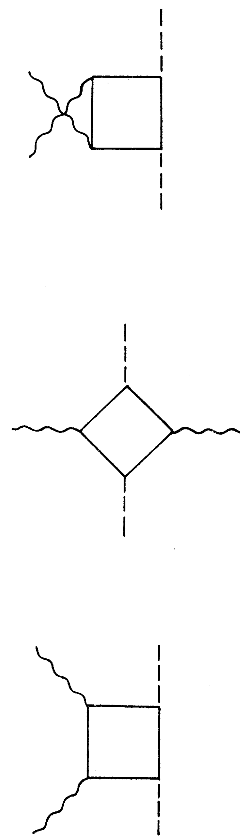

FIG. 1. Diagrams for the two-current–two-boson matrix elements.

We conclude this section with a consideration of the case when the states α and β are each single-meson states. (By crossing this is equivalent to the case when one state is the vacuum, and the other is a two-meson state.) The T^* produce (2.5a) is now represented by the fermion boxes shown in Fig. 1, two legs belonging to the two currents, and the other two to the two bosons. Thus, we are led to consider for vector currents

$$T^{*\mu\nu}(q) = \int d^4x\, e^{iqx}\langle p_1|T^*j^\mu(x)j^\nu(0)|p_2\rangle$$
$$= -2g^2\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\{\gamma^\mu S(r+q)\gamma^\nu S(r+p_1-p_2)O^1S(r-p_2)O^2S(r)$$
$$+\gamma^\nu S(r-q+p_1-p_2)\gamma^\mu S(r+p_1-p_2)O^1S(r-p_1)O^2S(r)+\gamma^\mu S(r+q)O^1S(r+q-p_1)\gamma^\nu S(r-p_2)O^2S(r)\}. \quad (2.10)$$

In the above, the O^i's represent the interaction coupled with strength g. They are each unity for scalar mesons γ^5 for pseudoscalar mesons. For vector mesons O^1 is $\gamma^\alpha\epsilon_\alpha(p_1)$, and O^2 is $\gamma^\beta\epsilon_\beta(p_2)$, where ϵ_α is the vector polarization. The factor 2 in (2.10) comes from collapsing the six topologically distinct boxes to three by charge conjugation.

[13] Its presence has been noted by R. A. Brandt, Ref. 2.

[14] An operator proof has also been presented by D. J. Gross and R. Jackiw [Phys. Rev. 163, 1688 (1967)] to the end that the ST contains at most one derivative of a δ function. However, this proof also requires the existence of the commutator, a circum-stance which does not obtain in the present context.

We shall need the limit as $q_0 \to \infty$; we expect the result to be independent of all other momenta and masses.[15] Thus, we set $m=0$ and $p_1=p_2=0$. It is now seen that the scalar and pseudoscalar interactions give the same result. We consider these first $(O^i = I, \gamma^5)$

$$T^{*\mu\nu}(q)\Big|_{p_1=p_2=0,m=0} = -2g^2\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\{\gamma^\mu S(r+q)\gamma^\nu S(r)S(r)S(r) + \gamma^\nu S(r-q)\gamma^\mu S(r)S(r)S(r) + \gamma^\mu S(r+q)S(r+q)\gamma^\nu S(r)S(r)\}, \quad (2.11)$$

$$S(r) \equiv i/r_\mu\gamma^\mu .$$

Ignoring polynomials in q, we obtain for the T product

$$T^{\mu\nu}(q) = (g^2/i(g^2/\pi^2)(q^\mu q^\nu/q^2). \quad (2.12)$$

The ST, which is obtained from the $0k$ component, is therefore

$$-iq_0T^{0k} = (q_0^2g^2/q^2)(q^k/\pi^2), \quad C^{0k}(\mathbf{q}) = (g^2/\pi^2)q^k, \quad (2.13a)$$

$$\langle p_1|[j^0(0,\mathbf{x}),j^k(0)]|p_2\rangle = (-ig^2/\pi^2)\partial^k\delta(\mathbf{x}). \quad (2.13b)$$

This suggests that the commutator may be effectively written in the form

$$[j^0(0,\mathbf{x}),j^k(0)] = -\tfrac{1}{2}i\partial^k[(g\varphi/\pi)^2\delta(\mathbf{x})], \quad (2.13c)$$

where φ is the scalar or pseudoscalar boson field.

For vector mesons, the analog of (2.11) is

$$T^{*\mu\nu}(q)\Big|_{p_1=p_2=0,m=0} = \epsilon_\alpha\epsilon_\beta T^{\alpha\beta\mu\nu}(q), \quad (2.14a)$$

$$T^{\alpha\beta\mu\nu}(q) = -2g^2\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\{\gamma^\mu S(r+q)\gamma^\nu S(r)\gamma^\alpha S(r)\gamma^\beta S(r) + \gamma^\nu S(r-q)\gamma^\mu S(r)\gamma^\alpha S(r)\gamma^\beta S(r) + \gamma^\mu S(r+q)\gamma^\alpha S(r+q)\gamma^\nu S(r)\gamma^\beta S(r)\} \quad (2.14b)$$

$$= -2g^2\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\{\gamma^\mu S(r+q)\gamma^\nu S(r)\gamma^\alpha S(r)\gamma^\beta S(r) + \gamma^\mu S(r+q)\gamma^\nu S(r)\gamma^\beta S(r)\gamma^\alpha S(r) + \gamma^\mu S(r+q)\gamma^\alpha S(r+q)\gamma^\nu S(r)\gamma^\beta S(r)\} \quad (2.14c)$$

$$= 2ig^2\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\frac{\partial}{\partial r_\alpha}\{\gamma^\mu S(r+q)\gamma^\nu S(r)\gamma^\beta S(r)\}. \quad (2.14d)$$

Equation (2.14c) follows from (2.14b) by charge conjugation, while (2.14d) is established with the aid of $\partial^\alpha S = iS\gamma^\alpha S$. The representation (2.14d) shows that the T^* product at this point is just a surface term; hence it can depend on q only polynomially [in fact, according to (2.14d), it is independent of q]. Therefore the T product is identically zero and the commutator vanishes.[16] There is no q-number ST in this case.

When the two currents are both axial-vector, the result can be verified to be the same. For the mixed case, vector–axial-vector ETC, the matrix element of the commutator vanishes by charge conjugation.

Thus, we have established the remarkable result: The ST in the vector-vector or axial-vector–axial-vector ETC is a q number for scalar or pseudoscalar interactions. With vector interactions, specifically for quantum electrodynamics, there is no evidence for a q-number ST.

III. POINT-SPLITTING TECHNIQUE

In addition to the BJL method, there exists another technique, with historic priority, that can exhibit the ST—the method of point splitting used by Schwinger.[1] In the present section, we use this approach to study the same ST that we have presented in Sec. II.

The space components of the currents are defined by

$$j_\epsilon^k(y) \equiv \bar\psi(y+\tfrac{1}{2}\epsilon)\gamma^k\psi(y-\tfrac{1}{2}\epsilon), \quad (3.1a)$$

$$j_{5\epsilon}^k(y) \equiv i\bar\psi(y+\tfrac{1}{2}\epsilon)\gamma_5\gamma^k\psi(y-\tfrac{1}{2}\epsilon). \quad (3.1b)$$

Here ϵ is spacelike. The ETC between time and space components are obtained with the help of the canonical relations[17]

$$[j^0(0,\mathbf{x}),\psi(0)] = -\psi\delta(\mathbf{x}), \quad (3.2a)$$

$$[j_5^0(0,\mathbf{x}),\psi(0)] = i\gamma^5\psi\delta(\mathbf{x}). \quad (3.2b)$$

[15] See also Sec. III. This expectation has been verified by explicit calculation in the cases of vector and scalar mesons.

[16] The same conclusion has been arrived at by T. Nagylaki [Phys. Rev. 158, 1534 (1967)] by a method which is related to the present one.

[17] No evidence has been found to cast doubt on these commutators; see Ref. 12.

We find therefore,

$$[j^0(0,\mathbf{x}),j_\epsilon^k(0)]=[j_5^0(0,\mathbf{x}),j_{5\epsilon}^k(0)] =j_\epsilon^k(0)[\delta(\mathbf{x}-\tfrac{1}{2}\epsilon)-\delta(\mathbf{x}+\tfrac{1}{2}\epsilon)], \quad (3.3a)$$

$$[j^0(0,\mathbf{x}),j_{5\epsilon}^k(0)]=[j_5^0(0,\mathbf{x}),j_\epsilon^k(0)] =j_{5\epsilon}^k(0)[\delta(\mathbf{x}-\tfrac{1}{2}\epsilon)-\delta(\mathbf{x}+\tfrac{1}{2}\epsilon)]. \quad (3.3b)$$

In the above formulas, we now expand the arguments of the δ functions and get

$$[j^0(0,\mathbf{x}),j_\epsilon^k(0)]=[j_5^0(0,\mathbf{x}),j_{5\epsilon}^k(0)]=-j_\epsilon^k(0) \times[\epsilon^j\partial_j\delta(\mathbf{x})+(1/24)\epsilon^j\epsilon^m\epsilon^l\partial_j\partial_m\partial_l\delta(\mathbf{x})+O(\epsilon^5)], \quad (3.4a)$$

$$[j^0(0,\mathbf{x}),j_{5\epsilon}^k(0)]=[j_5^0(0,\mathbf{x}),j_\epsilon^k(0)]=-j_{5\epsilon}^k(0) \times[\epsilon^j\partial_j\delta(\mathbf{x})+O(\epsilon^3)]. \quad (3.4b)$$

If the matrix elements of j_ϵ^k and $j_{5\epsilon}^k$ were finite as $\epsilon\to 0$ (or at most logarithmically divergent), one could safely set ϵ to zero in (3.4) and obtain the naive result that the commutators vanish. However, in perturbation theory these matrix elements diverge, and care must be exercized in letting $\epsilon\to 0$. We now enumerate some of the matrix elements that are divergent. (i) The vacuum-expectation value of j_ϵ^k is cubically divergent; hence the first two terms in the brackets of (3.4a) can be expected to survive. For $j_5\epsilon^k$ this matrix element vanishes. (ii) The vacuum single meson (connected) matrix element of j_ϵ^k and of $j_{5\epsilon}^k$ quadratically divergent; hence the first term in the brackets of (3.4a) and (3.4b) can be expected to surfive. (iii) The single meson–single meson (connected) matrix element of j_ϵ^k and $j_{5\epsilon}^k$ is linearly divergent; the first term in the brackets of (3.4a) and (3.4b) can be expected to survive. (iv) All other matrix elements of j_ϵ^k and $j_{5\epsilon}^k$ between states, containing a total of more than two mesons, will be at most logarithmically divergent. Hence, they do not contribute to (3.4).

This analysis indicates that the ST will in general be a q number, linear and quadratic in the boson fields. The presence of such ST was exposed by the BJL approach in Sec. II.

One may continue the split-point investigation to calculate the precise value of the ST. We shall present only two such calculations in order to illustrate the fact that the present method yields results which differ in detail from those of the BJL approach.[18]

The first calculation that we display is that of the triple-derivative c-number ST. The second calculation which we present in detail is the determination of the matrix element of the ST between single vector-meson states.

According to (3.4a) and (i) above, the triple-derivative, c-number ST is obtained from

$$-(1/24)\epsilon^j\epsilon^m\epsilon^l\langle\Omega|j_\epsilon^k(0)|\Omega\rangle =-\frac{1}{24}\epsilon^j\epsilon^m\epsilon^l\langle\Omega|\bar\psi(\tfrac{1}{2}\epsilon)\gamma^k\psi(-\tfrac{1}{2}\epsilon)|\Omega\rangle. \quad (3.5a)$$

By letting ϵ take on a positive, timelike component, (3.5a) becomes, in the limit $\epsilon^0\to 0$,

$$(1/24)\epsilon^j\epsilon^m\epsilon^l\,\mathrm{Tr}\gamma^k\langle\Omega|T\psi(-\tfrac{1}{2}\epsilon)\bar\psi(\tfrac{1}{2}\epsilon)|\Omega\rangle =-\frac{1}{24}\epsilon^j\epsilon^m\epsilon^l\,\mathrm{Tr}\gamma^k G(-\epsilon) =-(\epsilon^k\epsilon^j\epsilon^m\epsilon^l/12\pi^2\epsilon^4)i+O(\epsilon). \quad (3.5b)$$

In the second line of (3.5b), G is the fermion propagator. The last equation in (3.5b) is the lowest-order result for the previous expression. We now must set $\epsilon^0\to 0$ and then $\boldsymbol{\epsilon}\to 0$. However, to do this, meaning must be given to the undefined expression $(\epsilon^k\epsilon^j\epsilon^m\epsilon^l)/\epsilon^4$ on the limit as $\epsilon\to 0$. This may be done in two ways. One may first set $\epsilon^0\to 0$ and then average over three dimensions. Alternatively, one may average over four dimensions. The first method has the advantage that it respects the fact that the parameter ϵ was introduced in (3.1) without a time component, so that canonical ETC (3.2) could be used to calculate (3.3). On the other hand, the four-dimensional averaging preserves Lorentz invariance, and is the position-space analog of symmetric integration in momentum space, a practice conventionally adopted to give meaning to ambiguous expressions. The two methods give different limits for (3.5b). These are

$$-(i/12\pi^2)d(g^{mj}g^{kl}+g^{mk}g^{jl}+g^{ml}g^{jk}), \quad (3.5c)$$

$d=1/15$ for three-dimensional averaging,

$d=1/24$ for four-dimensional averaging.

Combining (3.4a) with (3.5c), we obtain

$$\langle\Omega|[j^0(0,\mathbf{x}),j^k(0)]|\Omega\rangle=iS\partial^k\delta(\mathbf{x}) +[i/(12\pi^2)](3d)\partial^k\nabla^2\delta(\mathbf{x}) \quad (3.6)$$

Here, S is the quadratic divergence discussed by Schwinger. It is seen that neither three-dimensional nor four-dimensional averaging allows the result of this calculation to agree with that given by the BJL method, (2.8).

The vector-meson–vector-meson matrix element of the ST is, according to (3.4a), proportional to (in the limit $\epsilon\to 0$)

$$\mathrm{Tr}\tfrac{1}{2}\epsilon^j\int\frac{d^4r}{(2\pi)^4}e^{i\epsilon r}[\gamma^\alpha S(r)\gamma^\beta S(r-p_1)\gamma^k S(r+p_2) +\gamma^\beta S(r)\gamma^\alpha S(r+p_2)\gamma^k S(r-p_1)]\equiv I^{j\alpha\beta k}. \quad (3.7a)$$

Here α, β are the polarization indices and p_1, p_2 the momenta of the external mesons. The linearly divergent $[O(1/\epsilon)]$ portion of the r integral is independent of the

[18] The point-splitting technique has been recently reviewed critically by C. R. Hagen, Informal Meeting on Renormalization Theory, Trieste, 1969 (unpublished).

masses and momenta. Therefore,

$$I^{j\alpha\beta k}=4\epsilon^{j}\frac{1}{i}\int\frac{d^4r}{(2\pi)^4}\frac{e^{i\epsilon r}}{r^6}\times[4r^\alpha r^\beta r^k-r^2(g^{\alpha\beta}r^k+g^{\alpha k}r^\beta+g^{\beta k}r^\alpha)]. \quad (3.7b)$$

This integral is easily done by exponentiating the denominator, and we find

$$I^{j\alpha\beta k}=4\epsilon^{i}\int_0^\infty ds\int\frac{d^4r}{(2\pi)^4}e^{isr^2}e^{i\epsilon r}\times[2s^2r^\alpha r^\beta r^k-is(g^{\alpha\beta}r^k+g^{\alpha k}r^\beta+g^{\beta k}r^\alpha)]$$

$$=\frac{2}{(4\pi)^2}(\epsilon^j\epsilon^\alpha\epsilon^\beta\epsilon^k)\frac{1}{i}\int_0^\infty\frac{ds}{s^3}e^{-i\epsilon^2/4s}$$

$$=i\frac{2}{\pi^2}\frac{\epsilon^j\epsilon^\alpha\epsilon^\beta\epsilon^k}{\epsilon^4}. \quad (3.7c)$$

Then, for the three-dimensional average, α and β can only take on space indices, so (3.7c) becomes

$$i(2/\pi^2)(1/15)(\delta^{j\alpha}\delta^{k\beta}+\delta^{k\alpha}\delta^{j\beta}+\delta^{kj}\delta^{\alpha\beta}). \quad (3.8a)$$

For a four-dimensional average, α and β can take on all values, and we obtain for (3.7c)

$$i(2/\pi^2)(1/24)(\delta^{j\alpha}\delta^{k\beta}+\delta^{k\alpha}\delta^{j\beta}-\delta^{jk}g^{\alpha\beta}). \quad (3.8b)$$

Thus, we find here a q-number ST, in contradistinction to the BJL result which yielded zero. Furthermore, the two averages not only give different numerical values, but different tensor structures. In terms of an effective commutator, we have

$$[j^0(\mathbf{r}),j^k(0)]=(i/\pi^2)(1/15)(2B^kB^j+\delta^{kj}\mathbf{B}\cdot\mathbf{B})\partial_j\delta(\mathbf{r}) \quad (3.9a)$$

for the three-dimensional average and

$$[j^0(\mathbf{r}),j^k(0)]=i(1/\pi^2)(1/24)(2B^kB^j-\delta^{kj}B^\mu B_\mu)\partial_j\delta(\mathbf{r}) \quad (3.9b)$$

for the four-dimensional average. Note that the above ST is independent of momenta and masses.

IV. CONCLUSIONS

(a) We have exhibited q-number ST in ETC between fermionic currents by two methods: the BJL high-energy technique and the point-splitting technique. The ST are seen to involve terms that are linear and quadratic in boson fields, and their precise form depends both on the nature of the interaction and on the method of calculation.

(b) In the context of the point-splitting technique, it is clear that no further bosonic ST are present. In the BJL technique no such definite statement can be made on the basis of the present calculations. However the following comment can shed some light on further bosonic ST arising from the BJL method. The precise mechanism in perturbation theory which leads to noncanonical high-energy behavior of T products, thus to noncanonical ETC, in particular to ST, has been explained elsewhere.[12] It is to be recalled that the canonical results can be regained formally from a Feynman integral representation of the T product by replacing a fermion propagator, which carries an internal integration momentum r and the external momentum q (which is getting large in the BJL limit), by just the the external momentum, i.e., by replacing $S(r+q)$ by $i\gamma^0/q_0$. Noncanonical behavior is encountered only when such replacement is illegitimate because of divergences. Thus, vacuum polarization bubbles, triangle graphs, box diagrams, and pentagons become under this replacement cubically, quadratically, linearly, and logarithmically divergent, respectively. Therefore, one may expect noncanonical behavior here. The ST discussed in this paper and elsewhere are those associated with the first three enumerated cases. The above discussion indicates that a ST may arise from the noncanonical high-energy behavior of the pentagon. Such a ST would be cubic in the boson fields. However, it does not occur; a brief calculation and estimate of the high-energy behavior of the relevant graphs shows that it vanishes. The same considerations indicate that diagrams with hexagonal structure, as well as structures with more corners, should have canonical high-energy behavior. Thus, no q-number ST which are cubic or higher in the boson fields are to be expected in the BJL context.

We have not examined any diagrams which might exhibit q-number ST which involve fermion fields. It is to be recalled that previous investigations have never encountered such terms.[19]

(c) It has been demonstrated that different results for the ST are obtained by the two different methods which were used to calculate them. It is to be expected that still other expressions can be arrived at by introducing yet further prescriptions for calculating the ETC.[20] It is our opinion that none of these approaches is wrong; different definitions are appropriate for different contexts. So far only two calculational procedures for the ETC have been found to have practical significance: the canonical formal commutator and the BJL method. The former is relevant for defining the canonical theory, for verifying gauge and/or conservation properties, and for determining low-energy theorems. The latter method is appropriate for the high-energy sum rules,[21] and for the behavior of the light-cone singularity in configuration space.

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[19] T. Nagylaki, Ref. 16; S. L. Adler and Wu-Ki Tung, Ref. 10.

[20] Such further techniques are to be found for example in R. A. Brandt, Ref. 2.

[21] This point of view has been stressed by J. S. Bell, Nuovo Cimento 47, 616 (1967).

Other methods, such as the point-splitting technique do not seem to have any practical significance.

It is unfortunate that the point-splitting technique cannot be used to calculate the various anomalies that have been discovered. It is seen that this method simplifies the calculation considerably in that it presents a formal expression for the anomaly which is moderately easy to evaluate; see (3.4). Moreover from the structure of the formulas for the anomalies, in the point-splitting context, one may hope to be able to prove results independent of perturbation theory. Perhaps a useful point of view about the point-splitting device is that it provides a clue to the existence of anomalies. The precise value then must be computed by the method relevant to the application—typically by the BJL method.

(d) Although we have found q-number ST in some of our models, their significance to the usual applications of current algebra seems to be minimal. The only important role that ST have had, to our knowledge, has been in connection with Weinberg's first sum rule.[22] It is true that in derivations of that result, frequently the assumption is employed that the ST are c numbers, which is not valid in our models. However that assumption is in fact too strong—the sum rule requires merely the equality of the vacuum expectation values of the ETC between vector currents and axial-vector currents. This equality is maintained in the present investigation, as is seen from (2.7). Thus, the first Weinberg sum rule holds, even though the ST are q numbers.

[22] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

On the other hand, our considerations indicate that the use of canonical commutation relations to draw conclusions concerning the asymptotic behavior of electroproduction amplitudes or the convergence of radiative corrections in general and mass shifts in particular is highly suspect. Not only can the interactions change the values and tensor structure of the commutators,[10] they also can introduce entirely new forms. The usual "proofs" contain strong implicit assumptions concerning the dynamics.

Electromagnetic mass shifts may be of particular interest; if there is a neutral scalar meson, then the Cottingham formula applied to any shift should yield a quadratic divergence even though there are no charged boson fields. To be sure, this divergence would, in a complete theory, be associated with the electromagnetic mass renormalization of the neutral scalar particle,[6] but the Cottingham formula is indifferent to the source of the divergence.

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Three-Dimensional Covariant Integral Equations for Low-Energy Systems*

FRANZ GROSS
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
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Integral equations suitable for the dynamical treatment of strongly interacting particles are derived. The equations can be described as Bethe-Salpeter equations with one particle restricted to the mass shell, resulting in a three-dimensional covariant equation which can be easily interpreted physically. To restore the dynamical terms omitted in the process of restricting one particle to the mass shell, additional kernels are added to the irreducible kernels from the original Bethe-Salpeter equation. The addition of these extra terms leads to a resulting simplification in the kernels themselves, since the new kernels have the same structure as the original ones, with some partial cancellations. Estimates as to the convergence of the procedure and the sizes of the various potentials are given. The special case of the hydrogen atom is discussed briefly, and comments are made on the application of these equations to the nuclear-force problem. Connections between scattering equations and bound-state equations are discussed, and the relativistic normalization condition for bound-state wave functions is derived.

I. INTRODUCTION AND DISCUSSION

EVERYONE knows that the hydrogen atom can be quite well described by nonrelativistic quantum mechanics and that only the finer details require the application of the ideas of relativistic field theory (in the form of the Bethe-Salpeter equation). On the other hand, even though much progress has been made in the last 20 years toward an understanding of the nuclear force, no one has yet been able to construct a simple reasonably accurate theoretical description of the deuteron. The striking simplicity of the hydrogen atom is sharply contrasted with the complexity of the current models of the nuclear force, which usually have about 10 adjustable parameters and can be said to be as complicated as is theoretically possible.[1] Is it really

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[1] See Rev. Mod. Phys. 39, 495–718 (1967) for a recent review of the status of the nucleon-nucleon interaction.