Quantization Conditions for Linear and Nonlinear Trajectories*

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It is shown that the relation between linear trajectories of opposite normality found by Ademollo, Veneziano, and Weinberg also holds for nonlinear trajectories which occur in a generalization of the Veneziano form. It is also shown that half-integer spacing of trajectories of opposite normality which are connected by pion emission is possible only if the lowest spin particle on the upper trajectory is missing. Experimentally, there is a distinct absence of such particles (e.g., low-lying $\frac{1}{2}^-$ baryons and baryon resonances).

SOME arguments of Lovelace[1] have been extended by Ademollo, Veneziano, and Weinberg[2] in order to derive "quantization conditions" for linear Regge trajectories. Their relation between trajectories of opposite normality which can be connected by pion emission is obtained by invoking the Adler self-consistency condition[3] in the Veneziano model.[4]

Recently, a generalization[5] of the Veneziano form was found which possesses meromorphy in s and t, polynomial residues, Regge behavior, nonlinear trajectories, and the Veneziano form as a limiting case in which the trajectories become linear. The generalization is an infinite product in which the zeros are displayed so that the arguments of Lovelace and of Ademollo, Veneziano, and Weinberg apply straightforwardly. We find that the relation[2] between trajectories of opposite normality still holds even when the trajectories are nonlinear.

We also find that the assumptions of Ademollo, Veneziano, and Weinberg imply that *the lowest-spin particle on the higher-lying trajectory must be missing in order for their predicted half-integer spacing*[2] *actually to occur.* This prediction is in agreement with current experimental knowledge. In particular, it supplies a dynamical explanation for the absence of certain particles such as low-lying baryon resonances. It should be emphasized that for the case of linear trajectories this result can be proved using the Veneziano representation. We choose to use the more general representation because there is no extra work, the argument is the same, and the result is then obtained under more general conditions.

The generalization of the Veneziano form found in Ref. 5 can be written

$$F(s,t)=\prod_{m=0}^{\infty}\frac{\{1-\exp[K(\alpha_X(s)+\alpha_Y(t)-m)]\}R(m)}{\{1-\exp[K(\alpha_X(s)-m)]\}\{1-\exp[K(\alpha_Y(t)-m)]\}},\tag{1}$$

where the trajectory functions are given by

$$\alpha_X(s)=(1/K)\ln(\Lambda_X s+\Gamma_X)=(1/K)\ln(1+K(\Lambda_X's+\Gamma_X')),\tag{2}$$

$$\alpha_Y(t)=(1/K)\ln(\Lambda_Y t+\Gamma_Y)=(1/K)\ln(1+K(\Lambda_Y't+\Gamma_Y')),\tag{3}$$

and the Λ's, Γ's, and K are real constants with $K\geq 0$. For $K>0$, we can choose $R(m)=1$. In order to have convergence of the infinite product in the Veneziano limit $K\to 0$, we choose $R(0)=K$ and $R(m)=1-e^{-Km}$ for $m\neq 0$. As $K\to 0$, the trajectories become linear.

As in Ref. 2, we imagine that the amplitude for $\pi+A\to B+C$ is some linear combination of terms of the form

$$\prod_{m=0}^{\infty}\frac{\{1-\exp[K(\alpha_X(s)+\alpha_Y(t)-n-J_A-J_B-m)]\}R(m)}{\{1-\exp[K(\alpha_X(s)-k-J_A-m)]\}\{1-\exp[K(\alpha_Y(t)-l-J_B-m)]\}},\tag{4}$$

where k, l, and n are integers, and J_A and J_B are the spins of A and B.

The Adler self-consistency condition states that this amplitude has a zero at $P_\pi{}^\mu=0$, i.e., $s=m_A{}^2$, $t=m_B{}^2$. Assuming that the zero does not arise because of cancellation between terms of the type (4) with different values of k, l, and m, the form (4) must vanish at $s=m_A{}^2$, $t=m_B{}^2$. This can occur only if the numerator of (4) vanishes, which implies that

$$\alpha_X(m_A{}^2)+\alpha_Y(m_B{}^2)=J_A+J_B+N_{AB},\tag{5a}$$

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[2] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).
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where N_{AB} takes on only integer values and

$$N_{AB} \geq n. \tag{5b}$$

Parity conservation implies that only trajectories with normality opposite to that of $A(B)$ can contribute in the s channel (t channel) at $P_\pi{}^\mu = 0$.

Equation (5a) implies that $2N_{AB} = N_{AA} + N_{BB}$. Thus, N_{AA} is either odd-integer-valued or even-integer-valued.[2]

For trajectory functions of the form (2), the spectrum of external masses is given in terms of spin by

$$m_A{}^2 = (e^{K_A J_A} - \Gamma_A)/\Lambda_A. \tag{6}$$

Letting $X = Y$ and $A = B$ in Eq. (5a) and substituting Eq. (6), we find that

$$N_{AA} = \frac{2}{K} \ln\left(\frac{\Lambda_X}{\Lambda_A}\right) + 2\left(\frac{K_A}{K} - 1\right) J_A + \frac{2}{K} \ln\left[1 + \left(\frac{\Lambda_A}{\Lambda_X}\Gamma_X - \Gamma_A\right) e^{-K_A J_A}\right]. \tag{7}$$

For N_{AA} to be even-integer-valued or odd-integer-valued for all spins J_A, we must have

$$\Lambda_A \Gamma_X = \Lambda_X \Gamma_A, \tag{8}$$

$$\frac{2}{K} \ln\left(\frac{\Lambda_X}{\Lambda_A}\right) = Q, \quad K_A = PK, \tag{9}$$

where P and Q are integers with $P \geq 0$. These equations together with Eq. (2) imply that

$$\alpha_X(s) = P\alpha_A(s) + \tfrac{1}{2}Q. \tag{10}$$

Considering amplitudes with X and A interchanged[2] leads to the result that $1/P$ is also an integer. Therefore, $P = 1$. This fact, together with Eqs. (7)–(9), implies that N_{AA} is independent of J_A and that $Q = N_{AA}$. Thus Eq. (10) becomes

$$\alpha_X(s) = \alpha_A(s) + \tfrac{1}{2}N_{AA}. \tag{11}$$

This result states that two trajectories of opposite normality which are connected by pion emission have the same slope at each value of s and have intercepts which differ by half-integers. For linear trajectories, this is just the result of Ademollo, Veneziano, and Weinberg. Their mass formulas which depend on trajectories being linear will follow from Eq. (11) only if $K = 0$, although they follow approximately for small K.

In order to derive a further result, we now consider the integers k, l, and n in (4). Since $l + J_B$ is the minimum t-channel angular momentum, we have the inequalities

$$l + J_B \geq 0 \text{ for } Y \text{ and } B \text{ bosons}, \tag{12}$$

$$l + J_B \geq \tfrac{1}{2} \text{ for } Y \text{ and } B \text{ fermions}. \tag{13}$$

The order of the polynomial in t of the residue of the pole in (4) at $\alpha_X(s) = j$ can be shown to be $j + l - n - J_A$. The requirement that the order of the polynomial be less than or equal to the angular momentum j (no ancestors) implies that

$$n \geq -J_A + l \text{ for } X \text{ and } A \text{ bosons}, \tag{14}$$

$$n \geq -J_A + l + \tfrac{1}{2} \text{ for } X \text{ and } A \text{ fermions}. \tag{15}$$

We now let $X = Y$ and $A = B$. From Eq. (5b) we have $N_{AA} \geq n$. Combining this with Eqs. (12)–(15) gives $N_{AA} \geq -2J_A$ for bosons and $N_{AA} \geq -2J_A + 1$ for fermions. We have already established that N_{AA} is independent of J_A, and therefore we have

$$N_{AA} \geq -2J_{A\,\min} \text{ for bosons}, \tag{16}$$

$$N_{AA} \geq -2J_{A\,\min} + 1 \text{ for fermions}, \tag{17}$$

where $J_{A\,\min}$ is the lowest spin of all physical particles on the A trajectory. If we repeat these considerations for amplitudes with X and A interchanged, we find

$$N_{AA} \leq 2J_{X\,\min} \text{ for bosons}, \tag{18}$$

$$N_{AA} \leq 2J_{X\,\min} - 1 \text{ for fermions}. \tag{19}$$

If $J_{A\,\min} = J_{X\,\min} = 0$ or $\frac{1}{2}$, then, by Eqs. (16)–(19), $N_{AA} = 0$, and therefore $\alpha_X(s) = \alpha_A(s)$. If a spin-0 (spin-$\frac{1}{2}$) particle is missing on a boson (fermion) X trajectory, then $J_{X\,\min} = 1$ ($J_{X\,\min} = \frac{3}{2}$) and $N_{AA} \leq 2$. This would permit the X trajectory to lie $\frac{1}{2}$ or 1 unit above the A trajectory.

Thus, we see that for the half-integer spacing of Ademollo, Veneziano, and Weinberg to occur, the lowest-spin particle on the upper trajectory must be missing. For a difference of $\frac{3}{2}$ between the intercepts of two leading trajectories, the two lowest-spin particles on the upper trajectory must be absent.[6]

For the trajectories with half-integer-spaced partners given in Ref. 2, we will mention the missing particles. On the Δ and $Y_1{}^*$ trajectories, there are no $\frac{1}{2}^-$ particles, and on the ρ and K^* trajectories there are no scalar particles. In the case of $\frac{1}{2}^-$ baryons, there has been considerable interest in finding the dynamical reason for their absence. Our result could be of use in baryon spectroscopy as a means of correlating the less-well-established trajectories.

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[6] This result helps us understand the observation of Ref. 2 that observed intercept differences are all $\frac{1}{2}$ and not $\frac{3}{2}$ or $\frac{5}{2}$.