Unique Set of $I=0$ S-Wave $\pi\pi$ Phase Shifts*

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We have found that the forward-backward asymmetry exhibits a very slight Δ^2 dependence affording a model-independent extrapolation. When the above method is complemented with a factorization-model analysis, a unique set of $I=0$ S-wave pion-pion phase shifts is obtained, modulo π. New tests of the absorption model have been made.

RECENTLY, two papers[1,2] appeared which attempted to determine the $\pi\pi$ phase shifts by extrapolating to $\Delta^2=-\mu^2$, where Δ^2 denotes the four-momentum transfer to the nucleon in the reactions

$$\pi^-+p\rightarrow\pi^-+\pi^0+p\,, \tag{1}$$

$$\pi^-+p\rightarrow\pi^++\pi^-+n\,. \tag{2}$$

Both of the above attempts had the common feature that they relied on the validity of the simple one-pion-exchange (OPE) model. It was introduced in their analysis by requiring that the inelastic cross sections in reactions 1 and 2 are related to $\pi\pi$ scattering by an explicit $\Delta^2/(\Delta^2+\mu^2)^2$ proportionality factor. This choice was made necessary because of insufficient statistics for the method used in the analysis. Another possible source of error is the sharp Δ^2 dependence of the extrapolated amplitude.

Studying the Δ^2 dependence of ratios[3] of quantities entering into the extrapolation, we found that the forward-backward asymmetry (α) exhibits a very small Δ^2 dependence.[4] In addition to the simplicity of determining this quantity, present statistics are sufficient to determine α at $\Delta^2=-\mu^2$ without invoking the simple OPE or any other model.

The data analyzed in this paper consist of 30 693 events of reaction 2, from a compilation of π^-p experiments[5] with beam momenta between 1.89 and 3.2 BeV/c. This reaction is dominated by ρ^0 production. A Δ^2 cut of about 10 μ^2 gives a rather pure sample of

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$\pi^+\pi^-$ events, with 6% as an upper estimate for $N^*(1236)$ contamination. To study the dipion angular distribution, the z and y axes were taken along the direction of the incident pion momentum and the dipion production normal, respectively. The asymmetry parameter α was defined as

$$\alpha(s,\Delta^2)=\frac{\text{Forward}-\text{Backward}}{\text{Forward}+\text{Backward}}, \tag{3}$$

where s denotes the effective mass squared of the dipion system. Our events were divided into intervals of Δ^2 and s in such a way that we obtained over 100 events in each interval. In each mass interval we determined the values of α as a function of Δ^2 in the physical region. Then by least-squares fitting we extrapolated α to $\Delta^2=-\mu^2$, where μ denotes the mass of the π meson. [Both linear and quadratic fits in Δ^2/μ^2 were attempted. However, the $\alpha(s,-\mu^2)$ values obtained by the two parametrizations agreed well within errors.] It can be shown that $\alpha(s,-\mu^2)$ is related to the $\pi\pi$ phase shifts by the equation[4]

$$\tan\delta_1^1=(\sin 2\delta_0^0)/(3\alpha N/D-2\sin^2\delta_0^0), \tag{4a}$$

where

$$N=1+\frac{4\sin^2\delta_0^0+\sin^2\delta_0^2+4\cos(\delta_0^0-\delta_0^2)\sin\delta_0^0\sin\delta_0^2}{27\sin^2\delta_1^1}, \tag{4b}$$

and

$$D=1+\frac{\cos(\delta_0^2-\delta_1^1)\sin\delta_0^2}{2\cos(\delta_0^0-\delta_1^1)\sin\delta_0^0}. \tag{4c}$$

We denote the $\pi\pi$ phase shift in the isotopic spin state I and orbital angular momentum l by δ_l^I. The values for δ_1^1 were assumed to be given by the a P-wave Breit-Wigner formula.[3] The mass and width used were determined in this experiment by fitting our $N\langle Y_2^0\rangle$, $N\langle \mathrm{Re}Y_2^1\rangle$, and $N\langle \mathrm{Re}Y_2^2\rangle$ moments[6] to a Breit-Wigner amplitude and extrapolating the values to the pole, $\Delta^2=-\mu^2$. We obtained $\sqrt{s_0}=759\pm7$ MeV, $\Gamma=119\pm20$ MeV. These parameters are in good agreement with previous analyses.[1,2] The δ_0^2 phase shifts were obtained from Ref. 1. Since δ_0^0 is the only unknown in Eq. (4), it can be solved for numerically. Two solutions for δ_0^0 are plotted in Fig. 1(a). In Fig. 1(b) we show one of the extrapolating curves. Note that (a) the extrapolating curves have a small slope in contrast to the extrapolation performed in Ref. 2 [See Fig. 2(b) of Ref. 2]; (b) we anticipated this from absorption model calculations.[7] Following partially the same line of reasoning, Kane and Ross[8] have given a detailed analysis concerning the problem of extrapolation. Their results could also predict a smooth extrapolation for α as a function

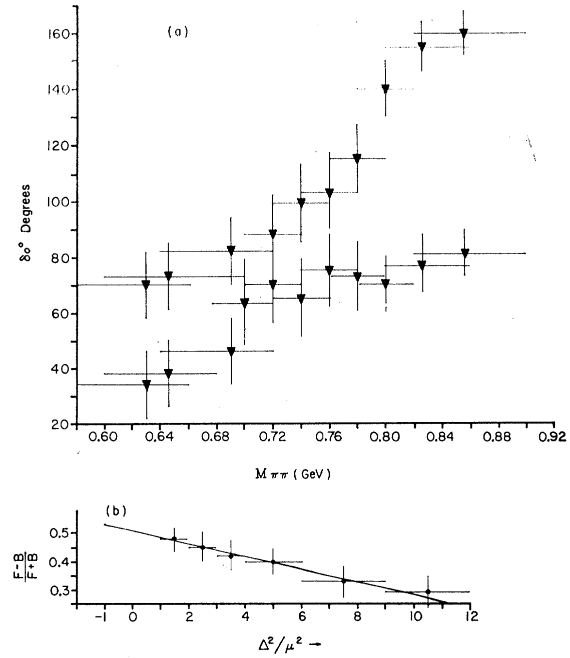

FIG. 1. (a) Phase shifts deduced from the $(F-B)/(F+B)$ ratio. (b) An extrapolation curve for $(F-B)/(F+B)$ versus Δ^2/μ^2, with $0.74\le M_{\pi\pi}\le 0.78$ GeV.

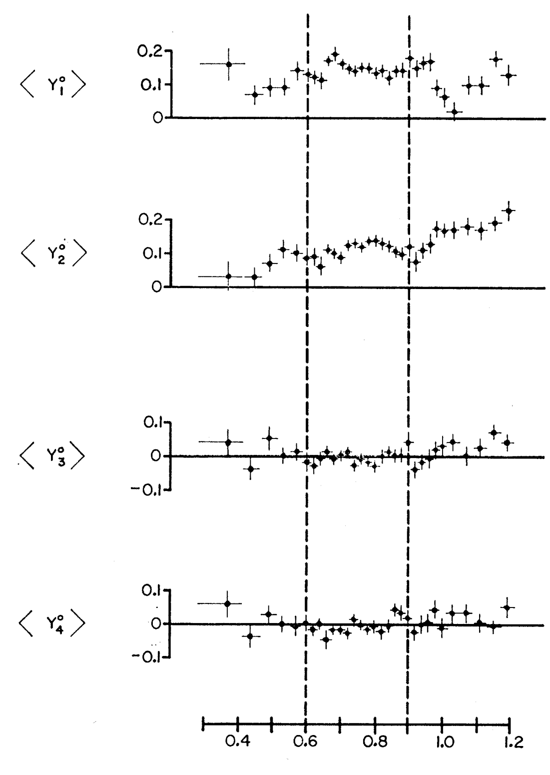

FIG. 2. Expectation values of the spherical harmonics $\langle Y_l^0\rangle$, in the helicity frame, for $\Delta^2<10\,\mu^2$. Dashed curves bracket the mass region considered in our analysis.

[6] P. E. Schlein, Phys. Rev. Letters **19**, 1052 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967).
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[8] G. L. Kane and M. Ross, Phys. Rev. **177**, 2353 (1969).

TABLE I. Density matrix elements of dipion systems with orbital angular momenta l and l' and helicity ν. Helicities of the incident and outgoing nucleons are ν_p and ν_n, respectively.

Helicity vectors (l ν ν_n ν_p)	Reduced density matrix elements $R_{\nu\nu'}{}^{ll'}$
$S_\mu = N\langle 0 \quad 0 \quad \frac{1}{2} \vert T_R \vert \mu\rangle$	$R_{00}{}^{00} = S^2$
$T_\mu = N\langle 1 \quad 0 \quad \frac{1}{2} \vert T_R \vert \mu\rangle$	$R_{00}{}^{11} = \frac{1}{3}T^2$
$P_\mu{}^{(\pm1)} = N\langle 1 \quad \pm1 \quad \frac{1}{2} \vert T_R \vert \mu\rangle$	
$D_\mu = \mathbf{P}_\mu{}^{(1)} - \mathbf{P}_\mu{}^{(-1)}$	$R_{11}{}^{11} = \frac{1}{6}(D^2+L)$
$L = 2\mathbf{P}^{(1)}\cdot\mathbf{P}^{(-1)}$	$R_{1-1}{}^{11} = \frac{1}{6}L$
$\cos\lambda = \mathbf{S}\cdot\mathbf{T}/ST$	$R_{00}{}^{10} = ST(\cos\lambda)/\sqrt{3}$
$\cos\gamma = \mathbf{S}\cdot\mathbf{D}/SD$	$R_{10}{}^{10} = SD(\cos\gamma)/2\sqrt{3}$
$\cos(\gamma-\lambda) = \mathbf{D}\cdot\mathbf{T}/DT$	$R_{10}{}^{11} = DT[\cos(\gamma-\lambda)]/6$

of Δ^2. (c) The results for $\delta_0{}^0$ are not sensitive to a possible, less than 6%, $N^*(1238)$ contamination for the 0.6 GeV$<\sqrt{s}<$0.9 GeV region.

We next attempted to resolve the ambiguity in the phase shifts by a maximum-likelihood method.

As suggested by absorption model calculations[9] it was proposed[10] that the single-pion-production amplitude factors into the product of a real function of the kinematic variables and a function of elastic $\pi\pi$ phase shifts.[3,6,11,12] This is a generalization of Watson's theorem[13] to an inelastic process. With this assumption, the single-pion-production cross section can be written as[7]

$$
\begin{aligned}
d^4\sigma/\partial s\partial\Omega\partial\Delta^2
&= (1/4\pi)\{|A_0|^2S^2+\tfrac{1}{3}|A_1|^2(T^2+D^2+L)\\
&\quad+|A_1|^2[\tfrac{1}{3}(T^2-\tfrac{1}{2}(D^2+L))(3\cos^2\theta-1)\\
&\quad-DT\cos(\gamma-\lambda)\sin 2\theta\cos\varphi/\sqrt{2}-\tfrac{1}{2}L\sin^2\theta\cos 2\varphi]\\
&\quad+(\mathrm{Re}A_1A_0{}^*)S[-\sqrt{2}D\cos\gamma\sin\theta\cos\varphi\\
&\qquad+2T\cos\lambda\cos\theta]\}, \qquad (5)
\end{aligned}
$$

where θ and φ denote the polar and azimuthal angles of the final-state π^- in the $\pi^+\pi^-$ rest system. The relationship between this equation and the conventional expression[3] in terms of the density matrix elements is discussed in Ref. 7 and given in Table I. The amplitudes A_0 and A_1 in Eq. (5) are defined as $A_0=\frac{1}{3}(2e^{i\delta_0{}^0}\sin\delta_0{}^0+e^{i\delta_0{}^2}\sin\delta_0{}^2)$ and $A_1=3e^{i\delta_1{}^1}\sin\delta_1{}^1$, where we assume that only $I=1$ P and $I=0$, 2 S waves contribute[14] in the region $\sqrt{s}<0.9$ GeV. This assumption is justified by the fact that $\langle Y_3{}^0\rangle=\langle Y_4{}^0\rangle=0$ in our region of interest. See Fig. 2.

[9] K. Gottfried and J. D. Jackson, Nuovo Cimento **34**, 735 (1964); L. Durand and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

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[12] M. Bander, G. L. Shaw, and J. R. Fulco, Phys. Rev. **168**, 1679 (1968); D. Griffiths and R. J. Jabbur, *ibid.* **157**, 1371 (1967); G. L. Kane, *ibid.* **163**, 1544 (1967).

[13] K. M. Watson, Phys. Rev. **88**, 1163 (1952); R. D. Amado, *ibid.* **158**, 1414 (1967); I. J. R. Aitchison and C. Kacser, *ibid.* **173**, 1700 (1968).

[14] The $\langle Y_l{}^m\rangle$ moments for $l\geq 3$ are all consistent with zero in our $\pi\pi$ mass range 0.6–0.9 GeV.

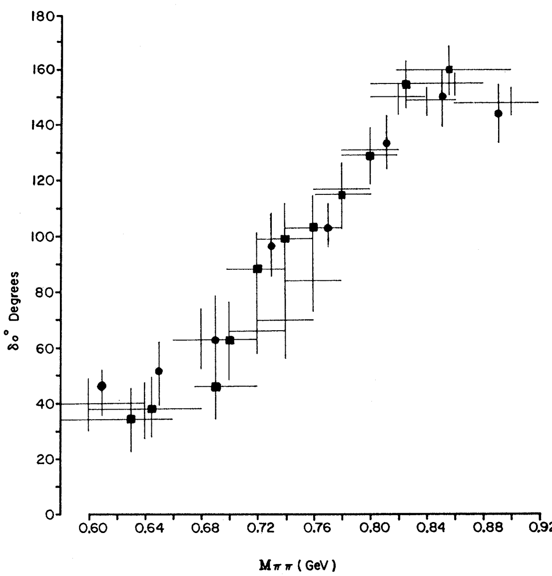

FIG. 3. Comparison of the phase shifts from the factorization model (●) and from the $(F-B)/(F+B)$ ratio (■) with those from Ref. 2 (+). The error bars for our results represent 90% confidence levels for the values from the factorization model and one standard deviation errors for the results from the $(F-B)/(F+B)$ ratio.

It is known from the theory of statistics[15] that if there is a sufficient estimator, the maximum-likelihood method is one and if there is an efficient estimator, maximum likelihood is one, so that a more accurate estimator cannot be found. We therefore constructed the maximum-likelihood function where the parameters

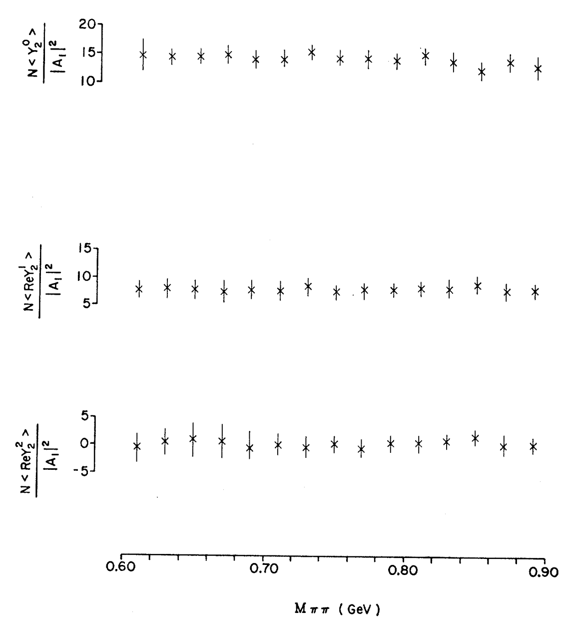

FIG. 4. $N\langle Y_2{}^0\rangle$, $N\langle\mathrm{Re}Y_2{}^1\rangle$, and $N\langle\mathrm{Re}Y_2{}^2\rangle$ moments, for $\Delta^2<10\,\mu^2$, divided by the absolute value of the P-wave scattering amplitude $|A_1|^2$, plotted as a function of the $\pi^+\pi^-$ mass.

[15] R. A. Fisher, *Statistical Methods for Research Workers* (Hafner Publishing Co., New York, 1954).

S, T, D, L, γ, λ, $\delta_0{}^0$, Γ, and $\sqrt{s_0}$ were varied and the negative logarithm of the likelihood function was minimized. It was assumed that S, T, D, L, γ, and λ do not vary as a function of energy in the interval 0.6–0.9 GeV, and that $\delta_1{}^1$ is given by a P-wave Breit-Wigner formula.[3] The angles θ and φ in Eq. (5) were measured in the helicity frame. After a very extensive search we found a local minimum and an absolute minimum for the negative logarithm of the likelihood function. At these minima all parameters except $\delta_0{}^0$ were only slightly different. The $\delta_0{}^0$ values obtained for the two minima were related by the equation $\delta_0{}^{0\prime}=\frac{1}{2}\pi-(\delta_0{}^0-\delta_1{}^1)$. [In the absence of the first term, Eq. (5) remains invariant under this transformation.] We found that the likelihood ratios for the two minima obtained were e^{42}, e^9, e^3 (for the 0.5–2.5, 2.5–5, and 5–10 Δ^2/μ^2 intervals), in favor of the solution which is plotted in Fig. 3 together with one of the sets obtained from our extrapolation procedure. This means that we were able to measure the S-wave cross section by separating the isotropic term in the angular distribution into a true S wave and into a term arising from absorption effects (depolarization of ρ^0).

To test the consistency of our assumptions with our results, we divided five of our moments ($N\langle Y_1{}^0\rangle$, $N\langle Y_2{}^0\rangle$, $N\langle \mathrm{Re}Y_1{}^1\rangle$, $N\langle \mathrm{Re}Y_2{}^1\rangle$, and $N\langle \mathrm{Re}Y_2{}^2\rangle$) by either $|A_1|^2$ or $\mathrm{Re}(A_1A_0{}^*)$ and found that the quotient did not vary as a function of s. (See Figs. 4 and 5.) Thus we have a consistent procedure.

Another, yet unexplored, application of the factorization model is to test the absorption model. After the two dominant energy-dependent factors (A_1,A_0) were separated out from the density matrix elements, the ratio of the reduced density matrix elements[7] were plotted in Fig. 6.

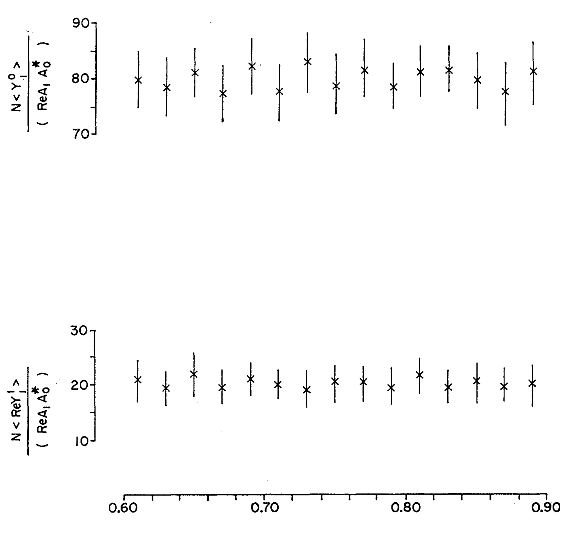

FIG. 5. $N\langle Y_1{}^0\rangle$ and $N\langle \mathrm{Re}Y_1{}^1\rangle$ moments divided by our $S-P$ wave interference term $\mathrm{Re}A_1A_0{}^*$, as a function of the $\pi^+\pi^-$ mass.

From the figures we can conclude that the theory gives predictions which are in rather good agreement with the experimental results. Thus, within the framework of the factorization model, we show the validity of the absorption model for wide resonances when both S and P waves contribute.

Equation (5) is invariant under the transformation $\delta_0{}^{0\prime}=\delta_0{}^0+n\pi$; thus we can give information on $\delta_0{}^0$ only modulo π. For example, if $\delta_0{}^0$, set I in Fig. 7, is a

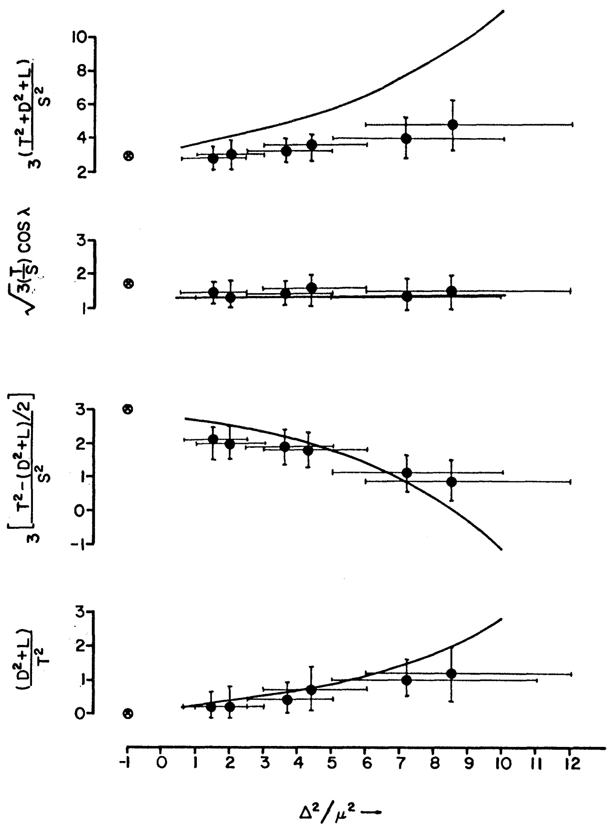

FIG. 6. Ratios of the reduced density matrix elements

$$3(T^2+D^2+L)/S^2=9|A_0|^2(\rho_{00}{}^{11}+2\rho_{11}{}^{11})/|A_1|^2\rho_{00}{}^{00},$$
$$\sqrt{3}(T/S)\cos\lambda=3|A_0|^2\,\mathrm{Re}(\rho_{00}{}^{10})/(\mathrm{Re}A_1A_0{}^*)\rho_{00}{}^{00},$$
$$3[T^2-(D^2+L)/2]/S^2=9|A_0|^2(\rho_{00}{}^{11}-\rho_{11}{}^{11})/|A_1|^2\rho_{00}{}^{00},$$

and $(D^2+L)/T^2=2\rho_{11}{}^{11}/\rho_{00}{}^{11}$ compared with absorption-model predictions. The circles at $\Delta^2/\mu^2=-1$ are the values for on the mass-shell $\pi\pi$ scattering.

solution, set I′, $\delta_0{}^{0\prime}=\delta_0{}^0-\pi$ is equally acceptable. Note, however, that the sign of $\delta_0{}^{0\prime}$ is negative. Analysis of $K_{34}(e^+)$ decay with sufficient statistics could remove this ambiguity.[16]

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[16] R. P. Ely, Jr., G. Gidal, V. Hagopian, G. E. Kalmus, K. Billing, F. W. Bullock, M. J. Esten, M. Govan, C. Henderson, W. L. Knight, F. R. Stannard, O. Treutler, U. Camerini, D. Cline, W. F. Fry, H. Haggerty, R. H. March, and W. J. Singleton, University of California Radiation Laboratory Report No. UCRL-18626, 1968 (unpublished).

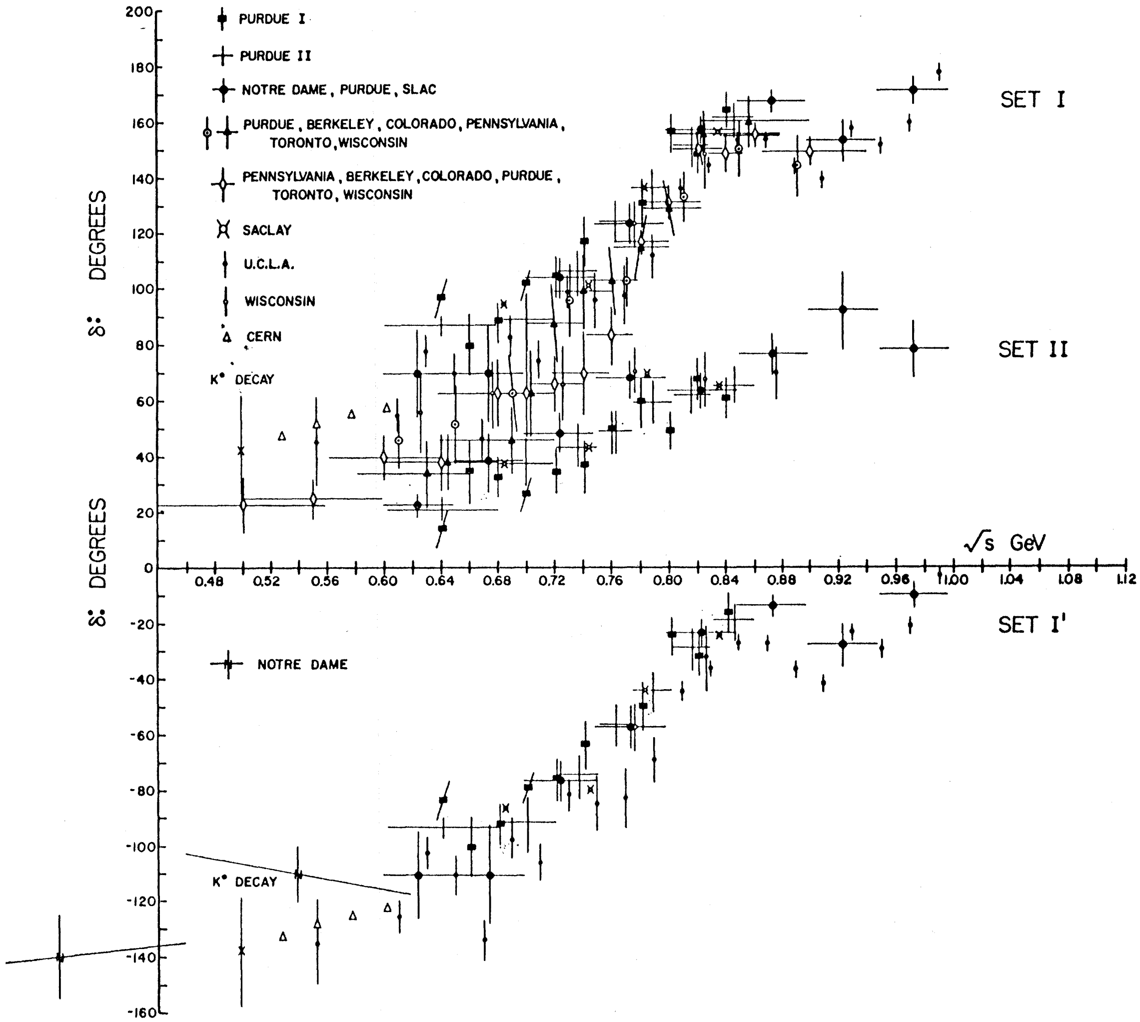

FIG. 7. World data on δ_0^0 phase shifts, taken from Refs. 2–4, 6, 11, 17, and 18.

Various results of earlier investigators[2,3,4,6,11,17,18] are presented in Fig. 7. Comparison of our solution for δ_0^0 with $\pi^0\pi^0$ effective-mass distributions will be done elsewhere. We received several comments concerning Eq. (5). Reference 7 is a review paper where the historical development in both the experimental and theoretical physics leading to Eq. (5) is given. It is a logical synthesis of many earlier theoretical and experimental investigations. The most important use was made of the works of Gottfried and Jackson[9] (absorption model), Csonka and Gutay[10] (factorization model), Malamud,[6] Meiere,[7] and Schlein[6] (helicity vectors). To test these ideas we made use of the efforts of an estimated thirty physicists[5] who obtained the data; but most of whose names do not appear on this paper. We extend our thanks to all of the above-mentioned people.

Note added in proof. After this paper was submitted for publication a paper was published[19] where the ratio of the $I=0$ and $I=2$ scattering length was determined and found to be negative. Since δ_0^2 is negative,[1] the result of Ref. 19 rules out set I'.

We are grateful to the authors of Ref. 5 for supplying us with their data necessary for this analysis. We would like to thank Professor V. Hagopian, Professor W. Selove, and Professor W. D. Walker for critical remarks.

[17] W. D. Walker, J. Carrol, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967).

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[19] L. J. Gutay, F. T. Meiere, and J. H. Scharenguivel, Phys. Rev. Letters **23**, 431 (1969).