variables do not appear as group elements in $O(4)$, the space of Bargmann-Fock.

Nambu's realization of $O(4,2)$ is not given by (5.19) with its Casimir invariant $\frac{3}{2}(\bar{\psi}\psi)^2$. Instead, to form the group he utilizes $S_{\mu\nu}$ of (19a) together with the two spacelike vectors that complete the Dirac "vierbein,"

$$W_\mu{}^\pm = \bar{\psi}^{c.c.}\gamma_\mu\psi \pm \bar{\psi}\gamma_\mu\psi^{c.c.}. \tag{5.28}$$

In (5.28), $\psi^{c.c.}$ stands for charge-conjugate four-component spinor. The final group element is $\bar{\psi}\psi$. In this classical theory, the three Casimir invariants of $O(4,2)$ are functions of $\bar{\psi}\gamma_5\psi$.

VI. CONCLUSION

We have analyzed the classical analogs of wave theories with infinite-component wave functions. In all cases, the mass spectrum of the quantum theory is infinite, although discrete. The classical theories are distinguished by the fact that special nonholonomic constraints couple the space-time trajectory to the internal variables. In future papers we shall consider theories predicting mass spectra of elementary particles in closer agreement with experiment. We shall also analyze the spectra of higher multipole moments in these theories.

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Simplified Calculation of Lamb Shift Using Algebraic Techniques*

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The Lamb shift in hydrogenlike atoms is treated by algebraic matrix methods using a unitary, infinite-component representation of the group $SO(4,2)$. The result for the Lamb shift itself, with its dependence upon the Bethe logarithm of the average excitation energy, is identical to previous results, but is obtained more simply. The numerical evaluation of the Bethe logarithm is both simpler and more accurate than by earlier treatments, as demonstrated by a detailed evaluation for the ground state. Numerical values for the Bethe logarithm (obtained *without* the use of an electronic computer) are $\gamma(1S)=2.98412\ 85559(3)$, $\gamma(2S)=2.81176\ 98932(5)$, and $\gamma(2P)=-0.03001\ 67089(3)$ for the three lowest states, with the figures in parentheses giving the number of units of estimated error in the last decimal place. A series of appendices presents the needed properties of $SO(p,1)$ and $SO(p,2)$ representations, the $SO(4,2)$ formulation of the hydrogen atom, and an alternative treatment of the Bethe logarithm which may also be applied to other operators such as the Coulomb Green's function.

INTRODUCTION AND SUMMARY

Although recent developments in particle physics (e.g., current algebra, infinite-component field theories) are employing to an increasing degree the algebraic methods associated with the Heisenberg representation, these methods have not yet been used in atomic physics to any appreciable extent. Except for Pauli's[1] early use of $SO(4)$ symmetry in treating the energy levels and Stark effect in hydrogen, most atomic calculations have been based on a more or less explicit analysis of the wave functions of the Schrödinger representation. This has been true even for those treatments which exploit the $SO(4)$ symmetry of the hydrogen atom, such as Fock's[2] four-dimensional momentum transform of the Schrödinger equation, and Lieber's[3] recent nonrelativistic calculation of the Lamb shift. This has led to the widespread conviction that analytic methods are more fruitful for quantum mechanics than are algebraic methods, especially for numerical applications. A further result is the dearth of experience in applying algebraic methods to the well-understood area of atomic physics—experience which would be useful in attempting to apply similar methods to the less-well-understood areas of particle physics or even nuclear physics.

In view of the above situation, the purpose of this paper is threefold: first, to demonstrate that algebraic methods can be superior to predominantly analytic methods in some cases, even for numerical work; second, to provide more experience in applying algebraic methods in atomic physics with the expectation that this experience will prove useful in other areas of physics; and, third, to suggest that, since the calculation of the Lamb shift is so greatly simplified by this approach, some additional atomic calculations which are presently considered impractical may be made feasible.

* Supported in part by the National Science Foundation.
[1] W. Pauli, Jr., Z. Physik **36**, 336 (1926).
[2] V. Fock, Z. Physik **98**, 145 (1935).
[3] Michael Lieber, Phys. Rev. **174**, 2037 (1968).

FIG. 1. Fluctuation and vacuum polarization diagrams, which give the Lamb shift to lowest order in the radiation field. The wavy lines denote virtual photons, while the double lines denote electrons moving in the external potential.

Previous authors[4–6] have obtained operator expressions—of lowest order in both the radiation field and the external Coulomb potential—whose expectation values with respect to an arbitrary bound state yield the Lamb shift for that state for hydrogenlike atoms. Prior to numerical evaluation, these expressions have always been reduced in the past to alternate forms whose expectation values are to be taken with respect to nonrelativistic Coulomb wave functions, with results equivalent to the original in lowest order. Since this reduction is required for the present calculation also, it is described briefly, in an intuitive, rather than rigorous, manner, in Sec. I, in the form most suitable for subsequent translation into the $SO(4,2)$ matrix algebra formalism. In Sec. II, this translation is performed and the resulting expression for the Lamb shift is evaluated. The major portion of this section consists of an algebraic simplification of the Bethe logarithm of the average excitation energy, followed by an explicit evaluation for the ground state plus results for the $2S$ and $2P$ states. A series of appendices makes this paper self-contained. Appendix A is an introduction to the needed properties of $SO(p,1)$ and $SO(p,2)$ representations, including some new results, as well as earlier results not readily available elsewhere. Appendix B, in which the $SO(4,2)$ form of the Schrödinger equation and several relevant operators are obtained, is primarily a summary and extension of the recent results of Fronsdal.[7] Appendix C contains the details of an evaluation of several $SO(4,2)$ matrices. Appendix D treats the Bethe logarithm by an alternative procedure which may also be applied to other operators and matrix elements, such as the Coulomb Green's function.

I. PRELIMINARY CONSIDERATION OF LAMB SHIFT

A perturbation operator is sought, whose expectation value relative to a particular state gives the Lamb shift ΔE for that state, correct to lowest order in the radiation field. To this order, the only two Feynman diagrams which contribute (in the bound interaction picture) are the fluctuation diagram and the vacuum polarization or tadpole diagram, illustrated in Fig. 1. Of several equivalent treatments of the Lamb shift, that of Ref. 5 is most useful for our purposes. The contributions of the fluctuation diagram are separated into two parts according to the energy k of the virtual photon. Thus, we write $\Delta E = \Delta E_< + \Delta E_>$, where $\Delta E_<$ includes those contributions from the fluctuation diagram with $k < \lambda_c$, while $\Delta E_>$ includes those contributions from the fluctuation diagram with $k > \lambda_c$, plus the complete contribution of the vacuum polarization diagram, with λ_c the separation energy.

If $\mu(Z\alpha)^2 \ll \lambda_c \ll \mu$, where the potential energy of the electron is $V = -Z\alpha/r$, and μ is the electron mass, then $\Delta E_<$ may be calculated nonrelativistically by second-order perturbation theory using Coulomb wave functions throughout, with the result

$$\Delta E_< = -\frac{2\alpha}{3\pi\mu^2}\int_0^{\lambda_c} kdk \sum_{N'}{}^c \left(\frac{(N|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N)}{k+E_{N'}-E_N} - \frac{(N|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N)}{k}\right). \quad (1.1)$$

Here, $\mathbf{q}$ denotes the momentum operator, $|N)$ the initial (and final) state, and $|N')$ the intermediate state, with Schrödinger energies E_N and $E_{N'}$. The superscript c on the summation symbol indicates that an integration over the continuum states is to be included in addition to the sum over the bound states. The second term in the large parentheses is the required mass renormalization term. By applying the completeness relation

$$\sum_{N'}{}^c |N')(N'| = 1, \quad (1.2)$$

Eq. (1.1) reduces to

$$\Delta E_< = -\frac{2\alpha}{3\pi\mu^2}\int_0^{\lambda_c} dk \langle N|\mathbf{q}\cdot\left(\frac{k}{k+H-E_N}-1\right)\mathbf{q}|N\rangle, \quad (1.3)$$

where H is the Hamiltonian.

For $\Delta E_>$, relativity becomes important, but the binding decreases in importance. Thus we need keep only the lowest-order terms in an expansion in powers of the potential strength $Z\alpha$ as illustrated in Fig. 2; with the result

$$\Delta E_> = \frac{\alpha}{3\pi\mu^2}\left(\frac{19}{30}-\frac{3}{8}+\ln\frac{\mu}{2\lambda_c}\right)(N|\nabla^2 V|N) - i\frac{\alpha}{4\pi\mu}(Nd|\gamma\cdot\nabla V|Nd). \quad (1.4)$$

Here ∇ is the gradient operator, $|Nd)$ denotes a four-component Dirac wave function, and $|N)$ denotes the corresponding Coulomb wave function as in Eq. (1.1). The occurrence of $|N)$ in the first term of $\Delta E_>$, rather than the expected $|Nd)$, is explained in the following

[4] H. A. Bethe, Phys. Rev. 72, 339 (1947).
[5] H. A. Bethe and E. E. Salpeter, in *Handubch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. XXXV, pp. 175–193.
[6] M. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), pp. 345–361.
[7] C. Fronsdal, Phys. Rev. 156, 1665 (1967).

way. [Note that $(Nd|\nabla^2 V|Nd)$ is not even defined for s states, because then $|Nd)$ diverges at the origin.] The result $\nabla^2 V$ is correct only for the low-momentum components of the potential and is an overestimate for the momenta greater than the electron mass. Thus, $\nabla^2 V=4\pi Z\alpha\delta^3(\mathbf{r})$ should be replaced by an operator which is nonvanishing in a region of the dimensions of λ_e, the electron Compton wavelength. Since the large components of $|Nd)$ differ significantly from $|N)$ only in a much smaller region, and the small components of $|Nd)$ are smaller than $|N)$ by a factor of order $Z\alpha$, we may replace $|Nd)$ by $|N)$. But $|N)$ is itself slowing varying over a region of the dimensions of $\lambda_e/Z\alpha \gg \lambda_e$, so that the extended operator may in turn be replaced by $\nabla^2 V$.

Since the $SO(4,2)$ calculations are to be applied only to the Schrödinger states, and not to the Dirac states, we must reexpress the second term of $\Delta E_>$. For this purpose we denote the large and small components of $|Nd)$ by $|N+)$ and $|N-)$, respectively, and note that

$$\boldsymbol{\sigma}|N-)=(2\mu+Z\alpha/r)^{-1}[\mathbf{q}+i\mathbf{q}\times\boldsymbol{\sigma}]|N+). \quad (1.5)$$

With

$$\gamma=\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$

we obtain

$$-i(Nd|\boldsymbol{\gamma}\cdot\nabla V|Nd=(N+|Q'|N+), \quad (1.6)$$

where

$$Q'=i(2\mu+Z\alpha/r)^{-1}(\nabla V)\cdot(\mathbf{q}+i\mathbf{q}\times\boldsymbol{\sigma}) +i(\mathbf{q}-i\mathbf{q}\times\boldsymbol{\sigma})\cdot(2\mu+Z\alpha/r)^{-1}(\nabla V) \quad (1.7)$$

$$=\left(\nabla\cdot\frac{1}{2\mu+Z\alpha/r}\nabla V\right)+\frac{2}{2\mu+Z\alpha/r}\frac{dV}{dr}(\hat{r}\times\mathbf{q}\cdot\boldsymbol{\sigma}) \quad (1.8)$$

$$=\left(\frac{Z\alpha}{2\mu}\right)^2\frac{1}{r^2(r+Z\alpha/2\mu)^2} +2\left(\frac{Z\alpha}{2\mu}\right)\frac{1}{r^2(r+Z\alpha/2\mu)}\mathbf{L}\cdot\boldsymbol{\sigma}. \quad (1.9)$$

The first term of Q in Eq. (1.9) differs from $\nabla^2 V/2\mu$ in being significantly different from zero in a region of the dimensions of $Z\alpha\lambda_e$, while $|N+)$ differs significantly from $|N)$ only in a much smaller region. Therefore this term may be replaced by $\nabla^2 V/2m$ provided $|N+)$ is first replaced by $|N)$, by the same argument as used for the first term of $\Delta E_>$. The r dependence of the second term of Q in Eq. (1.9) differs from r^{-3} in a similar way, and, except for s states, may be approximated by r^{-3} after replacing $|N+)$ by $|N)$. For s states, neither $(N+|r^{-3}|N+)$ nor $(N|r^{-3}|N)$ exists, but the second term of Q gives zero contribution because of the factor $\mathbf{L}\cdot\boldsymbol{\sigma}$. Subject to the somewhat unesthetic prescription that this term be set equal to zero when $\mathbf{L}\cdot\boldsymbol{\sigma}=0$, the

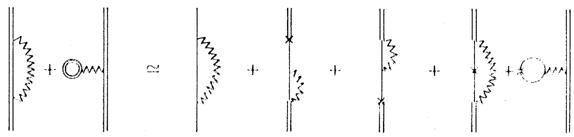

FIG. 2. Expansion of the fluctuation and vacuum polarization diagrams to lowest order in the external potential strength $Z\alpha$. The first term contributes only to mass renormalization, while the remaining four terms yield $\Delta E_>$. The wavy lines denote virtual photons, the double lines denote electrons moving in the external potential, and the single lines denote free electrons.

r^{-3} replacement may also be made for s states, and Eq. (1.4) reduces to

$$\Delta E_>=\frac{Z\alpha^2}{3\pi\mu^2}\langle N|\left[\left(\frac{19}{30}+\ln\frac{\mu}{2\lambda_c}\right)4\pi\delta^3(\mathbf{r})+\frac{3C}{4r^3}\right]|N\rangle, \quad (1.10)$$

where

$$C\equiv J(J+1)-L(L+1)-\tfrac{3}{4}=\mathbf{L}\cdot\boldsymbol{\sigma} \quad (1.11)$$

and J and L are the total and orbital angular momenta.

Although Eqs. (1.3) and (1.10) are the most convenient expressions for $SO(4,2)$ calculations, a further treatment of $\Delta E_<$ will facilitate comparison with previous results. Integrating in Eq. (1.1) yields

$$\Delta E_<=-\frac{2\alpha}{3\pi\mu^2}\sum_{N'}{}^c(N|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N)(E_N-E_{N'}) \times\ln\frac{\lambda_c+E_{N'}-E_N}{|E_{N'}-E_N|}. \quad (1.12)$$

Since the important values of $|E_{N'}-E_N|$ are of order $\mu(Z\alpha)^2$, the numerator of the logarithm may be replaced by λ_c. An equivalent approximation, which we shall adopt, is to take the limit $\lambda_c\rightarrow\infty$ in $\Delta E_<+\Delta E_>$. We define a parameter $\gamma(N,L)$, the Bethe logarithm, which is the same as the quantity $\ln(2K_0/\mu Z^2\alpha^2)$ used in Ref. 5:

$$\gamma(N,L)\sum_{N'}{}^c(N,0|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N,0)(E_{N'}-E_N) \equiv\sum_{N'}{}^c(N,L|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N,L)(E_{N'}-E_N) \times\ln\left|\frac{E_{N'}-E_N}{\frac{1}{2}\mu(Z\alpha)^2}\right|. \quad (1.13)$$

In Eq. (1.13) we have explicitly designated the angular momentum of the initial state, with $|N,0)$ denoting the s state with principal quantum number N. Note from Eq. (1.13) that $\gamma(N,L)$ is independent of Z. Using the relation

$$2\sum_{N'}{}^c(N|\mathbf{q}|N')\cdot(N'|\mathbf{q}|N)(E_{N'}-E_N) =-(N|[\mathbf{q}\cdot[\mathbf{q},H]]|N)=(N|4\pi Z\alpha\delta^3(\mathbf{r})|N) \quad (1.14)$$

with Eqs. (1.12) and (1.13), we obtain, as the final expression to be evaluated,

$$\Delta E(N,L)=(Z\alpha^2/3\pi\mu^2)\times\{(N,L|[(19/30-2\ln Z\alpha)4\pi\delta^3(\mathbf{r})+3C/4r^3]|N,L)-\gamma(N,L)(N,0|4\pi\delta^3(\mathbf{r})|N,0)\},\quad(1.15)$$

with γ given by

$$\gamma(N,L)(N,0|4\pi\delta^3(\mathbf{r})|N,0)=\lim_{\lambda_c\to\infty}\left[\ln\frac{2\lambda_c}{\mu(Z\alpha)^2}\times(N,L|4\pi\delta^3(\mathbf{r})|N,L)-\frac{3\pi\mu^2}{Z\alpha^2}\Delta E_<\right],\quad(1.16)$$

and $\Delta E_<$ given by Eq. (1.3).

II. EVALUATION OF LAMB SHIFT

The desired $SO(4,2)$ expression for the Lamb shift is found by inserting Eqs. (B7), (B11), and (B27) into Eq. (1.15) to obtain

$$\Delta E(N,L)=(Z\alpha^2/3\pi\mu^2)[(19/30-2\ln Z\alpha)Q_1(N,L)+Q_2(N,L)-\gamma(N,L)Q_1(N,0)],\quad(2.1)$$

where

$$Q_1(N,L)\equiv a^3\langle N,L|D|N,L\rangle\quad(2.2)$$

and

$$Q_2(N,L)\equiv\tfrac{3}{4}a^3\langle N,L|C(\Gamma_0-\Gamma_4)^{-2}|N,L\rangle,\quad(2.3)$$

with the matrix D defined by Eq. (B28). The states $|N,L\rangle$ are the physical states, with principal quantum number N, and with normalization given by Eq. (B19); while the states $|N,L\rangle\rangle$ used below are the eigenstates of Γ_0, with eigenvalue N, and with normalization given by Eq. (B17). Applying (B20), (A20), and (C15) in succession yields

$$\begin{aligned}Q_1(N,L)&=\mathfrak{N}_N{}^2a^3\langle\langle N,L|R^{-1}(\theta_N)DR(\theta_N)|N,L\rangle\rangle\\&=\mathfrak{N}_N{}^2a^3e^{2\theta_N}\langle\langle N,L|D|N,L\rangle\rangle\\&=4(\mu Z\alpha/N)^3\delta_{L,0}.\end{aligned}\quad(2.4)$$

Similarly, for $L\neq0$,

$$Q_2(N,L)=3(\mu Z\alpha/N)^3/2L(L+1)(2L1),\quad(2.5)$$

while $Q_2(N,L)$ vanishes for $L=0$, because then $C=0$. Thus, Eq. (2.1) reduces to the familiar[5,6] Lamb-shift result

$$\Delta E(N,L)=\frac{4\mu(Z\alpha)^4\alpha}{3\pi N^3}\times\begin{cases}[19/30-2\ln Z\alpha-\gamma(N,0)], & \text{for } L=0\\ \left[\dfrac{3C}{8L(L+1)(2L+1)}-\gamma(N,L)\right], & \text{for } L\neq0.\end{cases}\quad(2.6)$$

Note that Lieber[3] did not obtain this result because he treated only the $\Delta E_<$ contribution and ignored completely the $\Delta E_>$ contribution.

Proceeding now to the evaluation of $\gamma(N,L)$, we introduce the dimensionless quantities

$$\hat{k}\equiv2k/\mu(Z\alpha)^2,\quad(2.7a)$$

$$\lambda\equiv2\lambda_c/\mu(Z\alpha)^2,\quad(2.7b)$$

$$\hat{E}_N\equiv2E_N/\mu(Z\alpha)^2=-1/N^2,\quad(2.7c)$$

$$\hat{a}\equiv a/\mu Z\alpha,\quad(2.7d)$$

and

$$\hat{H}\equiv2H/\mu(Z\alpha)^2=(\Gamma_0-\Gamma_4)^{-1}[\hat{a}^2(\Gamma_0+\Gamma_4)-2\hat{a}].\quad(2.7e)$$

Equations (1.3) and (1.16) now become, with Eqs. (B8a) and (B20),

$$\gamma(N,L)=\lim_{\lambda\to\infty}\left[\delta_{L,0}\ln\lambda+\int_0^\lambda d\hat{k}\,Q_3(N,L)\right],\quad(2.8)$$

where

$$Q_3(N,L)\equiv\langle\langle N,L|R^{-1}(\theta_N)\tilde{Q}R(\theta_N)|N,L\rangle\rangle\quad(2.9)$$

and

$$\tilde{Q}\equiv\tfrac{1}{4}\hat{a}N\Gamma_i[\hat{k}(\hat{H}+\hat{k}-\hat{E}_N)^{-1}-1](\Gamma_0-\Gamma_4)^{-1}\Gamma_i.\quad(2.10)$$

Rotating through an angle $\theta_\nu=\ln(1/\hat{a}\nu)$, with $\nu\equiv(\hat{k}-\hat{E}_N)^{-1/2}$, then yields

$$\begin{aligned}Q(N)&\equiv R^{-1}(\theta_\nu)\tilde{Q}R(\theta_\nu)\\&=(N/4\nu)\Gamma_i[\tfrac{1}{2}\hat{k}\nu^2(\Gamma_0-\nu)^{-1}-(\Gamma_0-\Gamma_4)^{-1}]\Gamma_i.\end{aligned}\quad(2.11)$$

The Γ_i dependence is eliminated by applying Eqs. (A17) and (A19), while the Γ_4 dependence is eliminated next by using the Schrödinger equation [cf. Eq. (B15)]

$$[\nu^2\hat{E}_N(\Gamma_0-\Gamma_4)-(\Gamma_0+\Gamma_4)+2\nu]R^{-1}(\theta_\nu)|N,L\rangle=0,\quad(2.12)$$

with the result

$$Q(N)=\tfrac{1}{16}N\nu^3\hat{k}\Delta_\nu(\Gamma_0-\nu)^{-1},\quad(2.13)$$

where Δ_ν is the second-difference operator defined by

$$\Delta_\nu\equiv f(\nu)\equiv f(\nu+1)-2f(\nu)+f(\nu-1).\quad(2.14)$$

Two alternative procedures are available at this point. The first yields $Q_3(N,L)$ either as a Fourier transform of a single diagonal matrix element of an $SO(4,2)$ rotation operator, or as a *finite* sum of hypergeometric functions. This appears to be more useful for formal manipulations than for the actual evaluation of $\gamma(N,L)$, however, and is presented in Appendix D.

The second alternative, which yields $Q_3(N,L)$ as an infinite sum, will be followed in evaluating $\gamma(N,L)$. Since the basis vectors $|n,L\rangle\rangle$ are eigenstates of $Q(N)$, we may apply Eq. (B18) to Eq. (2.13) to obtain

$$\begin{aligned}Q_3(N,L)&=\tfrac{1}{16}N\nu^3\hat{k}\sum_{n=L+1}^{\infty}|\langle\langle n,L|R(\tilde{\theta})|N,L\rangle\rangle|^2\times\Delta_\nu(n-\nu)^{-1}\quad(2.15)\\&=\tfrac{1}{16}N\nu^3\hat{k}\sum_{n=L-1}^{\infty}(n-\nu)^{-1}\times\Delta_n|\langle\langle n,L|R(\tilde{\theta})|N,L\rangle\rangle|^2,\quad(2.16)\end{aligned}$$

where

$$\tilde{\theta}\equiv\theta_N-\theta_\nu=\ln(\nu/N). \tag{2.17}$$

In Eq. (2.16) we have used the fact that the matrix element vanishes for $n\leq L$. A significant advantage of the $SO(4,2)$ approach over the earlier one may be seen at this point. The sum over the intermediate states in Eq. (1.1) consists of an integration over the continuum states in addition to the discrete sum over bound states, while the corresponding sum in Eq. (2.16) consists only of a discrete sum over the basis vectors $|n,L\rangle\rangle$. (Note, incidentally, that the alternative procedure of Appendix D shows that this infinite sum over n could be performed explicitly, to yield either a hypergeometric function having a k dependence in its parameters as well as its argument, or a *finite* sum of such hypergeometric functions.) Changing to the variables

$$x\equiv(N-\nu)/(N+\nu) \tag{2.18a}$$

and

$$m\equiv n-L-1, \tag{2.18b}$$

we find that Eq. (2.8) reduces to

$$\gamma(N,L)=\sum_{m=-1}^{\infty}J_m(N,L), \tag{2.19}$$

where, except for the case $m=-1$, $L=0$,

$$J_m(N,L)\equiv\int_0^1 dx\, I_m(x), \tag{2.20}$$

$$I_m(x)\equiv\frac{x(1+x)}{(m+L+1+N)(1-x^2)^2(x+A)}\times\Delta_m|\langle\langle m+L+1,\,L|R(\tilde{\theta})|N,L\rangle\rangle|^2, \tag{2.21}$$

$$A\equiv(m+L+1-N)/(m+L+1+N), \tag{2.22}$$

and

$$\cosh\tilde{\theta}=1+2x^2/(1-x^2). \tag{2.23}$$

The case $m=-1$, $L=0$ involves the term $\ln\lambda$, and the upper limit $x_0\equiv1-2[(\lambda N^2+1)^{1/2}-1]^{-1}$ on the x integral cannot immediately be taken in the limit $\lambda\rightarrow\infty$ as it has been in Eq. (2.20). From Eqs. (C18) and (2.23), we obtain for this case

$$J_{-1}(N,0)=\lim_{\lambda\to\infty}\left[\ln\lambda-\int_0^{x_0}dx\frac{x^{2N-1}(1+x)}{(1-x)}\right]=2\left(\sum_{i=1}^{2N}\frac{1}{i}-\frac{1}{4N}-\ln\tfrac{1}{2}N\right). \tag{2.24}$$

Although an explicit evaluation in the general case involves quite complicated expressions, the calculation is actually rather simple for the low-lying states. We will demonstrate this simplicity, as well as show the method to be used in the general case, by evaluating γ for the ground state.

From Eq. (C18), we obtain for the $1S$ state

$$I_m(x)=x^{2m-1}(1+x)\times[(1-x^2)^2-(1-A)(1-x^2)]/(x+A), \tag{2.25}$$

with $A=m/(m+2)$. As the first step we evaluate

$$J_0=-\int_a^1 dx(1+x)(1-x^2)=-\frac{11}{12}$$

to yield, with Eq. (2.24),

$$\gamma_0\equiv J_{-1}+J_0=2\ln2+19/12=2.970. \tag{2.26}$$

It will be seen that γ_0 exceeds the final value for γ by only 0.5%, an amount comparable to the errors already introduced by the various approximations of Sec. I. A similar result is true for S states in general, since the corresponding figures are 2.5% for the $2S$ state, and 8% for the limit $N\rightarrow\infty$.[8]

It is only the presence of the denominator $x+A$ which prevents our obtaining a simple, easily-summable expression for the integral of $I_m(x)$. Therefore we apply the relation

$$(1+x)/(x+A)=1+(1-A)/(x+A) \tag{2.27}$$

to Eq. (2.25), thrice to the first term and twice to the second, thereby obtaining a remainder

$$\tilde{I}_{m,1}(x)=-(1-A)^3x^{2m}(1-x)/(x+A) \tag{2.28}$$

after separating the denominator-free part

$$I_{m,1}(x)=x^{2m-1}(1-x^2)^2-(1-A)x^{2m}(1-x^2)-(1-A)^2x^{2m}(1-x), \tag{2.29}$$

whose integral is

$$J_{m,1}=\frac{1}{m(m+1)(m+2)}-\frac{4}{(m+2)(2m+1)(2m+3)}-\frac{2}{(m+1)(m+2)(^22m+1)}. \tag{2.30}$$

The integral of $\tilde{I}_{m,1}$ is seen to behave like $\tilde{J}_{m,1}\rightarrow m^{-5}$ for large m, compared with $J_m\rightarrow m^{-3}$. To further improve the convergence, we apply the relation

$$(x+A)^{-1}=(1+A)^{-1}[1+(1-x)/(x+A)] \tag{2.31}$$

to Eq. (2.28) three times and obtain a remainder

$$\tilde{I}_{m,2}(x)=-[(1-A)/(1+A)]^3x^{2m}(1-x)^4/(x+A) \tag{2.32}$$

plus a denominator-free part

$$I_{m,2}(x)=-(1-A)^3x^{2m}\sum_{i=1}^{3}[(1-x)/(1+A)]^i, \tag{2.33}$$

[8] Reference 5, pp. 404–406.

TABLE I. Sequence of extrapolated partial sums γ_m, defined by Eq. (2.41), which converge to the Bethe logarithm γ for the ground state. The rapidity of the convergence is illustrated by the successive differences in the last column.

m	γ_m	$10^{10}(\gamma_m-\gamma_{m-1})$
1	2.98412 92620	
2	2.98412 85344	−7276
3	2.98412 85528	+184
4	2.98412 85554	26
5	2.98412 85558	4
6	2.98412 85559	1

whose integral is

$$J_{m,2}=-\frac{1}{2(m+1)^2(m+2)(2m+1)}\times\left[\frac{4}{m+2}+\frac{1}{(m+1)(2m+3)}\left(4+\frac{3}{m+1}\right)\right]. \quad (2.34)$$

The sum $\sum_m (J_{m,1}+J_{m,2})$ may be evaluated exactly by resolving the summand into partial fractions, and using the relation

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\ln 2 \quad (2.35)$$

and the Riemann zeta function

$$\zeta(n)\equiv\sum_{k=1}^{\infty}k^{-n},$$

with the result

$$\begin{aligned}\gamma_{00}&\equiv\gamma_0+\sum_{m=1}^{\infty}(J_{m,1}+J_{m,2})\\&=\tfrac{1}{2}[3\zeta(4)+\zeta(3)+15\zeta(2)-12\ln 2-15+\tfrac{1}{6}] \quad (2.36)\\&=2.98596\ 90535.\end{aligned}$$

Finally, the integral of $\tilde{I}_{m,2}$ is given by

$$\tilde{J}_{m,2}=\tilde{J}_{m,1}-J_{m,2}, \quad (2.37)$$

where the integral of $\tilde{I}_{m,1}$ is easily found to be

$$\tilde{J}_{m,1}=\frac{(1-A)^3}{2m+1}-(1+A)(1-A)^3\times\left[\sum_{i=0}^{2m-1}\frac{(-A)^i}{2m-i}+A^{2m}\ln\frac{1+A}{A}\right]. \quad (2.38)$$

Few of the $\tilde{J}_{m,2}$ need be calculated since $\tilde{J}_{m,2}\rightarrow(m+1)^{-8}$ for $m\rightarrow\infty$, but even fewer are required if we use the approximation

$$\sum_{k=n+1}^{\infty}f(k)\simeq\int_{n+1/2}^{\infty}dk\, f(k)+\frac{1}{24}\frac{df}{dk} \quad (2.39)$$

to obtain the extrapolation term

$$\bar{J}_{m,2}\equiv\sum_{k=m+1}^{\infty}\tilde{J}_{k,2}\simeq\tilde{J}_{m,2}\left(\frac{2m+2}{2m+3}\right)^8\left(\frac{2m+3}{7}-\frac{2}{3(2m+3)}\right). \quad (2.40)$$

The sequence of sums

$$\gamma_m\equiv\gamma_{00}+\sum_{k=1}^{m}\tilde{J}_{k,2}+\bar{J}_{m,2} \quad (2.41)$$

thus rapidly converges to the desired value of γ, as shown in Table I. The rate of convergence is seen to be much more rapid than that obtained by Lieber,[3] and results from the application of Eq. (2.31).

TABLE II. Vales of the Bethe logarithm γ for the 1S, 2S, and 2P states, with results of this calculation compared with those of previous authors. The figures in parentheses give the number of units of estimated error in the last decimal place. The last column gives the number of γ_m terms, either evaluated for this calculation, or required to duplicate the *stated* accuracy of previous results.

State	γ	Source	Terms
1S	2.98412 85559(3)	This calculation	6
	2.98412 85(3)	Lieber[a]	3
	2.98414 9(3)	Harriman[b]	2
2Y	2.81176 98932(5)	This calculation	8
	2.81176 9883(28)	Schwartz-Tiemann[c]	6
	2.81176 98(3)	Lieber[a]	5
	2.81179 8(9)	Harriman[b]	4
2P	−0.03001 67089(3)	This calculation	7
	−0.03001 6697(12)	Schwartz-Tiemann[c]	5
	−0.03001 675(6)	Lieber[a]	4
	−0.03001 637(1)	Harriman[b]	5

[a] Michael Lieber, Phys. Rev. **174**, 2037 (1968).
[b] J. M. Harriman, Phys. Rev. **101**, 594 (1956).
[c] C. Schwartz and J. J. Tiemann, Ann. Phys. (N. Y.) **2**, 178 (1959).

The calculated values of $\gamma(N,L)$ for the 1S, 2S, and 2P states are presented in Table II and compared with several previously calculated values. Also listed in Table II are the number of γ_m terms evaluated for the present calculation and the number needed to duplicate the accuracy of the previous calculations. Equation (2.31) was applied a sufficient number of times to yield $\tilde{J}_{m,2}\rightarrow(m+1)^{-9}$ for the 2S state and $\tilde{J}_{m,2}\rightarrow(m+2)^{-10}$ for the 2P state. It should be noted that this evaluation did not employ an electronic computer, nor would it be practical to do so unless far greater accuracy were desired. This is an unlikely desire, since the accuracy of the present calculation already exceeds the current experimental accuracy[9] by over six orders of magnitude. Such numerical accuracy was obtained only to demonstrate the simplicity and capability of algebraic methods in general, and the $SO(4,2)$ formulation in particular, in atomic physics.

[9] W. E. Lamb, Jr., and R. C. Retherford, Phys. Rev. **72**, 241 (1947); **75**, 1325 (1949); **79**, 549 (1950); **81**, 222 (1951); **85**, 259 (1952); **86**, 1014 (1952); S. Triebwasser, E. S. Dayhoff, and W. E. Lamb, Jr., *ibid.* **89**, 98 (1953); **89**, 106 (1953).

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APPENDIX A: REPRESENTATIONS OF $SO(p,1)$ AND $SO(p,2)$

We begin by considering $SO(p,1)$, the group of orthogonal transformations (without reflections) in a $p+1$-dimensional space with a diagonal metric g^{AB} given by $g^{00}=1$ and $g^{11}=g^{22}=\cdots=g^{pp}=-1$.[10] Many calculations involving this group are most conveniently performed by employing the finite-dimensional irreducible representation given by the symmetric and traceless Nth-rank tensor in a $(p+1)$-dimensional space,[11]

$$\psi_{A_1A_2\cdots A_N}\equiv z_{A_1}z_{A_2}\cdots z_{A_N}, \tag{A1}$$

where $z_Az^A\equiv g^{AB}z_Az_B=0$. In this basis, the infinitesimal generators s_{BC} are given by

$$s_{BC}=i\left(z_B\frac{\partial}{\partial z^C}-z_C\frac{\partial}{\partial z^B}\right), \tag{A2}$$

which act on the basis vectors as follows:

$$-is_{BC}\psi_{A_1\cdots A_N} =N\mathcal{S}_A{}^N(g_{CA_1}\psi_{BA_2\cdots A_N}-g_{BA_1}\psi_{CA_2\cdots A_N}). \tag{A3}$$

Here $\mathcal{S}_A{}^N$ denotes symmetrization in the N indices $A_1\cdots A_N$, i.e., $(N!)^{-1}$ times the sum of permutations. The reduction of this representation with respect to $SO(P)$ is given by either of the expressions

$$\psi_{A_1\cdots A_N}=\mathcal{S}_A{}^N\sum_{t=0,1,\ldots}\dot{\psi}_{A_1\cdots A_t}\sum_{n=t,t+2,\ldots}b_p(N,t,n) \times g_{A_{t+1}A_{t+2}}\cdots g_{A_{n-1}A_n}g_{A_{n+1}}{}^0\cdots g_{A_N}{}^0 \tag{A4}$$

$$=\mathcal{S}_A{}^N\sum_{t=0,1,\ldots}\dot{\psi}_{A_1\cdots A_t}\sum_{n=t,t+2,\ldots}c_p(N,t,n) \times \dot{g}_{A_{t+1}A_{t+2}}\cdots\dot{g}_{A_{n-1}A_n}g_{A_{n+1}}{}^0\cdots g_{A_N}{}^0, \tag{A5}$$

where $\dot{g}_{AB}\equiv g_{AB}-g_A{}^0g_{B0}$. The irreducible representations of $SO(p)$ have basis vectors given by the traceless symmetric tth-rank tensors

$$\dot{\psi}_{A_1\cdots A_t}\equiv\tau_A{}^p(\dot{g}_{A_1}{}^{B_1}\cdots\dot{g}_{A_t}{}^{B_t}g_0{}^{B_{t+1}}\cdots g_0{}^{B_N}\psi_{B_1\cdots B_N}), \tag{A6}$$

where $\tau_A{}^p$ denotes the projection operator which picks out that part of its operand which is traceless (relative to $\dot{g}^{AB}$) in the indices $A_1\cdots A_t$. Note that $\dot{\psi}_{A_1\cdots A_t}$ is also traceless relative to g^{AB} since it vanishes if any of its indices equal zero. Requiring the right-hand side of Eq. (A5) to be traceless relative to g^{AB} yields a recursion relation for the coefficients $c_p(N,t,n)$, whose normalization is determined by inserting Eq. (A5) in Eq. (A6), with the result

$$c_p(N,t,n)=a_p(N,t,n)/(t+n+p-2)!!, \tag{A7}$$

where

$$a_p(N,t,n)=(-g_{00})^{(n-t)/2}(2t+p-2)!!N!/ t!(n-t)!!(N-n)!. \tag{A8}$$

A similar procedure yields

$$b_p(N,t,n)=(2N+t-n+p-3)!!a_p(N,t,n)/ (N+t+p-2)!. \tag{A9}$$

From Eqs. (A3), (A5), and (A6), the action of the generators on the basis vectors is found to be

$$-is_{bc}\dot{\psi}_{a_1\cdots a_t}=\mathcal{S}_a{}^t(t\dot{g}_{ca_1}\dot{\psi}_{ba_2\cdots a_t}-t\dot{g}_{ba_1}\dot{\psi}_{ca_2\cdots a_t}), \tag{A10}$$

$$-is_{0c}\dot{\psi}_{a_1\cdots a_t}=-(N-t)\dot{\psi}_{ca_1\cdots a_t}+\frac{t(N+t+p-2)}{(2t+p-2)} \times\left[\mathcal{S}_a{}^t\left(\dot{g}_{ca_1}\dot{\psi}_{a_2\cdots a_t}-\frac{t-1}{2t+p-4}\dot{g}_{a_1a_2}\dot{\psi}_{ca_3\cdots a_t}\right)\right]. \tag{A11}$$

Note that the expression in the brackets in Eq. (A11) is just the traceless projection of $\mathcal{S}_a{}^t\dot{g}_{ca_1}\dot{\psi}_{a_2\cdots a_t}$ (relative to $\dot{g}^{AB}$) in the indices $a_1,\cdots,a_t$. (The traceless projection in the indices $c,a_1,\cdots,a_t$ is zero, of course.) Although Eqs. (A10) and (A11) were derived only for non-negative integral variables of N, they in general provide an irreducible representation of $SO(p,1)$ for any complex N. Further, Eqs. (A3)–(A11) are valid for any traceless symmetric tensor, $\psi_{A_1\cdots A_N}$, and not just for all the polynomials in z_A defined by Eq. (A1). The invariant scalar product is defined by $\langle\phi|\psi\rangle\equiv g^{A_1B_1}\cdots g^{A_NB_N}\phi^*{}_{A_1\cdots A_N}\psi_{B_1\cdots B_N}$ and may be expressed in terms of $SO(p)$ components by using Eqs. (A4) and (A5). The generalization to complex N may be shown to be, after dropping some insignificant t-independent factors,[11a]

$$\langle\phi|\psi\rangle=\sum_{t=0}^{\infty}\frac{(t-N^*-1)!}{(t+N+p-2)!}\frac{(2t+p-2)!!}{t!}(-\dot{g}^{a_1b_1})\cdots \times(-\dot{g}^{a_tb_t})\dot{\phi}^*{}_{a_1\cdots a_t}\dot{\psi}_{b_1\cdots b_t}. \tag{A12}$$

For positive integral N, the same expression holds, except that $(t-N^*-1)!$ is replaced by $(-)^t/(N-t)!$, and the t summation runs from 0 to N. It can be seen that the coefficients in Eq. (A12) are positive and real, and the representation given by Eqs. (A10) and (A11) equivalent to a unitary one, only for the principal series

$$N=-\tfrac{1}{2}(p-1)+i\rho,\quad \rho\text{ real}$$

and the supplementary series

$$-(p-1)<N<0.$$

[10] The treatment, and especially the notation, in this Appendix follows that of Ref. 7.

[11] The following notation for indices is used throughout these Appendices: $A, B, \cdots=0, 1, 2, \cdots p$; $a, b, \cdots=1, 2, \cdots p$; $i, j, \cdots=1, 2, \cdots p-1$.

[11a] This complex-type scalar product is invariant only for N real or for the principal series. The corresponding real-type scalar product is invariant for complex N in general, which fact follows almost trivially from the definition of the group.

For the special case $N=-\frac{1}{2}P$, which will be called the Majorana[12] representation, there exists a set of matrices Γ_A which transform among themselves under $SO(p,1)$ in the same way as the z_A, i.e.,

$$[s_{BC},\Gamma_A]=i\Gamma_B g_{AC}-i\Gamma_C g_{AB}, \tag{A13}$$

with Γ_0 satisfying

$$\Gamma_0\dot{\psi}_{a_1\cdots a_t}=(t+\tfrac{1}{2}P-1)\dot{\psi}_{a_1\cdots a_t}. \tag{A14}$$

It then follows from Eqs. (A11) and (A13) that

$$\Gamma_c\dot{\psi}_{a_1\cdots a_t}=(t+\tfrac{1}{2}p)\dot{\psi}_{ca_1\cdots a_t}-\frac{t(2t+p-4)}{2(2t+p-2)}\mathcal{S}_a{}^t\times\left(\dot{g}_{ca_1}\dot{\psi}_{a_2\cdots a_t}-\frac{t-1}{2t+p-4}\dot{g}_{a_1a_2}\dot{\psi}_{ca_3\cdots a_t}\right). \tag{A15}$$

Also, from Eqs. (A11), (A14), and (A15),

$$[\Gamma_A,\Gamma_B]=-is_{AB}, \tag{A16}$$

which shows that this irreducible representation of $SO(p,1)$ is an irreducible representation of $SO(p,2)$ with generators s_{AB} and $s_{-1,A}\equiv\Gamma_A$.

One useful $SO(p)$ invariant is the operator $Q\equiv-\dot{g}^{bc}\Gamma_b\times f(\Gamma_0)\Gamma_c$, where $f(\Gamma_0)$ is any function of the matrix Γ_0. It follows from Eqs. (A14) and (A15) that

$$Q\dot{\psi}_{a_1\cdots a_t}=\tfrac{1}{4}[(2t+p)(t+p-2)f(t+\tfrac{1}{2}p)+t(2t+p-4)f(t+\tfrac{1}{2}p-2)]\dot{\psi}_{a_1\cdots a_t}.$$

Applying Eq. (A14) then yields the operator relation

$$-\dot{g}^{bc}\Gamma_b f(\Gamma_0)\Gamma_c=\tfrac{1}{2}\textstyle\sum_\pm(\Gamma_0\pm1)\times[\Gamma_0\pm\tfrac{1}{2}(p-2)]f(\Gamma_0\pm1). \tag{A17}$$

A second useful relation may be derived by taking $f(\Gamma_0)=(\Gamma_0)^s$, and applying a rotation through an angle $\tanh^{-1}\beta$ in the 0-a plane, i.e., by making the substitutions

$$\Gamma_0\to(1-\beta^2)^{-1/2}(\Gamma_0+\beta\Gamma_a)$$

and

$$\Gamma_a\to(1-\beta^2)^{-1/2}(\Gamma_a+\beta\Gamma_0) \tag{A18}$$

in Eq. (A17), with the remaining Γ_b's remaining unchanged. Finally, by taking the limit $\beta\to\pm1$, we obtain

$$-g^{BC}\Gamma_B(\Gamma_0\pm\Gamma_a)^s\Gamma_C=\tfrac{1}{2}(s+1)(s+p-2)(\Gamma_0\pm\Gamma_a)^s, \tag{A19}$$

provided, of course, that $(\Gamma_0\pm\Gamma_a)^s$ is defined. It can be shown that $(\Gamma_0\pm\Gamma_p)^s$, and therefore $(\Gamma_0\pm\Gamma_a)^s$ in general, exists for integral values $s>-\frac{1}{2}(p-2)$, but not for $s\leq-\frac{1}{2}(p-2)$.

A third useful relation involves the behavior of the frequently occurring combinations $\Gamma_0\pm\Gamma_a$ under rotations in the 0-a plane. The operator $R(\theta)\equiv e^{-i\theta s_{0a}}$, with $\theta=\tanh^{-1}\beta$, generates the transformations of Eq. (A18), and thus gives

$$R^{-1}(\theta)(\Gamma_0\pm\Gamma_a)R(\theta)=(1\pm\beta)(1-\beta^2)^{-1/2}(\Gamma_0\pm\Gamma_a)=e^{\pm\theta}(\Gamma_0\pm\Gamma_a). \tag{A20}$$

The $SO(p,1)$ representation may be further reduced with respect to $SO(p-1)$ with the result

$$\dot{\psi}_{a_1,\cdots,a_t}=\mathcal{S}_a{}^t\sum_{L=0,1,\cdots}\ddot{\psi}_{a_1\cdots a_L}{}^t\sum_{n=L,L+2,\cdots}b_{p-1}(t,L,n)\times\dot{g}_{a_{L+1}a_{L+2}}\cdots\dot{g}_{a_{n-1}a_n}\dot{g}_{a_{n+1}}{}^p\cdots\dot{g}_{a_t}{}^p \tag{A21}$$

$$=\mathcal{S}_a{}^t\sum_{L=0,1,\cdots}\ddot{\psi}_{a_1\cdots a_L}{}^t\sum_{n=L,L+2,\cdots}c_{p-1}(t,L,n)\times\ddot{g}_{a_{L+1}a_{L+2}}\cdots\ddot{g}_{a_{n-1}a_n}\dot{g}_{a_{n+1}}{}^p\cdots\dot{g}_{a_t}{}^p, \tag{A22}$$

where

$$\ddot{g}_{ab}\equiv\dot{g}_{ab}-\dot{g}_a{}^p\dot{g}_{bp},$$

$$\ddot{\psi}_{a_1\cdots a_L}{}^t\equiv\tau_a{}^{p-1}(\ddot{g}_{a_1}{}^{b_1}\cdots\ddot{g}_{a_L}{}^{b_L}\dot{g}_p{}^{b_{L+1}}\cdots\dot{g}_p{}^{b_t}\dot{\psi}_{b_1\cdots b_t}), \tag{A23}$$

and the coefficients are again given by Eqs. (A7)–(A9), with g_{00} replaced by g_{pp} in Eq. (A8). From Eqs. (A21) and (A22), the $SO(p)$-invariant scalar product is found to be

$$(-\dot{g}^{a_1b_1})\cdots(-\dot{g}^{a_tb_t})\phi_{a_1\cdots a_t}{}^{t*}\psi_{b_1\cdots b_t}{}^t=t!(2t+p-4)!!\sum_{L=0}^{t}\frac{(2L+p-3)!!}{(t+L+p-3)!L!(t-L)!}\times(-\ddot{g}^{i_1j_1})\cdots(-\ddot{g}^{i_Lj_L})\ddot{\phi}_{i_1\cdots i_L}{}^{t*}\ddot{\psi}_{j_1\cdots j_L}{}^t. \tag{A24}$$

The L-independent factors cannot be dropped here, because Eq. (A24) is to be combined with Eq. (A12) to yield the $SO(p,1)$-invariant scalar product in terms of $SO(p-1)$ components.

For the Majorana representation, it follows from Eqs. (A15), (A22), and (A23), that the matrix Γ_p acts on the $SO(p-1)$ tensors as

$$\Gamma_p\ddot{\psi}_{i_1\cdots i_L}{}^t=(t+\tfrac{1}{2}p)\ddot{\psi}_{i_1\cdots i_L}{}^{t+1}+\frac{(t-L)(t+L+p-3)}{2(2t+p-2)}\ddot{\psi}_{i_1\cdots i_L}{}^{t-1}. \tag{A25}$$

From Eqs. (A12) and (A24) it is seen that the normalized tensors, with respect to which the generators s_{AB} and Γ_A are Hermitian, are given by

$$|t+\tfrac{1}{2}p-1,L\rangle\rangle\equiv(2t+p-2)!!\times\left(\frac{(2L+p-3)!!}{2(t+L+p-3)!L!(t-L)!}\right)^{1/2}\ddot{\psi}_{i_1\cdots i_L}{}^t. \tag{A26}$$

Inserting these normalized tensors into Eqs. (A14) and

[12] E. Majorana, Nuovo Cimento **9**, 335 (1932).

(A25) gives

$$\Gamma_0|n,L\rangle\rangle = n|n,L\rangle\rangle, \tag{A27}$$

$$\Gamma_p|n,L\rangle\rangle = \tfrac{1}{2}[(n+L+\tfrac{1}{2}p-1)(n-L-\tfrac{1}{2}p+2)]^{1/2} \times|n+1,\, L\rangle\rangle + \tfrac{1}{2}[(n+L+\tfrac{1}{2}p-2) \times(n-L-\tfrac{1}{2}p+1)]^{1/2}|n-1,\, L\rangle\rangle, \tag{A28}$$

where $n=\frac{1}{2}p-1,\, \frac{1}{2}p,\, \ldots,$ are the eigenvalues of Γ_0.

Finally, consider the function

$$f(z)\equiv\sum_{t=0}^{\infty}\frac{(t-N^*-1)!}{(t+N+p-2)!}\frac{(2t+p-2)!!}{t!} \times(-)^t f_{a_1\cdots a_t}(z_0)^{N-t}z^{a_1}\cdots z^{a_t}, \tag{A29}$$

where the coefficients $f_{a_1\cdots a_t}$ are symmetric in their indices, traceless relative to $\dot{g}^{ab}$, and transform under $SO(p,1)$ like the $\dot{\psi}_{a_1\cdots a_t}$. Then it follows from Eq. (A12) that $f(z)$ is transformed by an element G of $SO(p,1)$ according to

$$f(z)\to Gf(z)=f(G^{-1}z). \tag{A30}$$

Note that $f(z)$ is a homogeneous function of degree N in the z^A. The $SO(p,1)$-invariant scalar product of two such functions, $f(z)$ and $h(z)$, is given in terms of their coefficients by Eq. (A12), but we also wish to find an integral form for this scalar product. The integral

$$\int\delta(z^c z_c)d^{p+1}z f^*(z)h(z)$$

is an obvious candidate because it is manifestly invariant, but this integral has the unfortunate defect of not existing. However, it can be shown that

$$I^{a_1\cdots a_n}\equiv\int\delta(z^A z_A)d^p z z^{a_1}\cdots z^{a_n} =(-)^{n/2}(z_0)^{n+p-2}V_p[n!/(n+p-2)!!n!!] \times S_a{}^n(\dot{g}^{a_1a_2}\cdots\dot{g}^{a_{n-1}a_n}) \tag{A31}$$

for n even, and $I^{a_1\cdots a_n}=0$ for n odd, where

$$V_p\equiv\int\delta(1+\dot{g}^{ab}x_a x_b)d^p x.$$

It therefore follows that

$$\tau_a{}^p\tau_b{}^p I^{a_1\cdots a_t b_1\cdots b_{t'}}=\delta_{t,t'}(z_0)^{2t+p-2}\frac{t!V_p(-)^t}{(2t+p-2)!!} \times\tau_a{}^p S_a{}^t(\dot{g}^{a_1b_1}\cdots\dot{g}^{a_tb_t}) \tag{A32}$$

and consequently,

$$I\equiv\int\frac{\delta(z^A z_A)d^p z}{V_p(z_0)^{N+N^*+p-2}}f^*(z)h(z) \tag{A33}$$

$$=\sum_{t=0}^{\infty}\frac{(t-N^*-1)!(t-N-1)!(2t+p-2)!!}{(t+N+p-2)!(t+N^*+p-2)!t!} \times(-\dot{g}^{a_1b_1})\cdots(-\dot{g}^{a_tb_t})\dot{f}_{a_1\cdots a_t}{}^*\dot{h}_{b_1\cdots b_t}. \tag{A34}$$

For representations in the principal series, the integral I is seen to reduce to the scalar product of Eq. (A12). But for the Majorana representation, the right-hand side of Eq. (A34) has one more factor $(t+\frac{1}{2}p-1)$ than does the scalar product. A comparison with Eq. (A14) thus shows that the integral

$$I'\equiv\int\delta(z^A z_A)(z_0)^2 d^p z\, f^*(z)\Gamma_0{}^{-1}h(z) \tag{A35}$$

is a suitable $SO(p,2)$-invariant scalar product.

APPENDIX B: $SO(4,2)$ REFORMULATION OF THE NONRELATIVISTIC HYDROGEN ATOM

The familiar Schrödinger theory for hydrogenlike atoms is easily translated to $SO(4,2)$ form by expressing the momentum wave functions $\phi(\mathbf{q})$ in terms of functions $\hat{\phi}(z)$ which transform according to Eq. (A30):

$$\hat{\phi}(z)\equiv\phi(\mathbf{q})(q^2+a^2)^2/2\sqrt{2}(z_0)^2a^{5/2}, \tag{B1}$$

where

$$q_i\equiv a z_i/(z_0-z_4), \tag{B2a}$$

$$q^2\equiv -q^i q_i = a^2(z_0+z_4)/(z_0-z_4), \tag{B2b}$$

or

$$z_i=2z_0 a q_i/(q^2+a^2), \tag{B3a}$$

$$z_4=z_0(q^2-a^2)/(q^2+a^2), \tag{B3b}$$

with a an arbitrary positive parameter. Note that $z^A z_A=0$ and that the factor $(z_0)^{-2}$ in Eq. (B1) causes $\hat{\phi}(z)$ to be a homogeneous function of order $N=-\frac{1}{2}p=-2$ in the z^A. From Eqs. (B2), it follows that

$$d^3q=2z_0a^3\delta(z^A z_A)d^4z/(z_0-z_4)^3, \tag{B4}$$

and the physical scalar product

$$(\phi|\psi)\equiv\int d^3q\phi^*(\mathbf{q})\psi(\mathbf{q}) \tag{B5}$$

thus becomes

$$(\phi|\psi)=\int\delta(z^A z_A)z_0 d^4z\hat{\phi}^*(z)\hat{\psi}(z)\cdot(z_0-z_4). \tag{B6}$$

This differs from the invariant scalar product of Eq. (A35) by a factor of $\Gamma_0(z_0-z_4)/z_0$, which transforms like $(\Gamma_0-\Gamma_4)$ under $SO(4,2)$. Therefore, we adopt the relation

$$(\phi|\psi)=\langle\hat{\phi}|(\Gamma_0-\Gamma_4)|\hat{\psi}\rangle \tag{B7}$$

between the two scalar products. It is necessary that $\Gamma_0-\Gamma_4$ be positive definite, but this can be shown easily from Eqs. (A27) and (A28).

Similarly, in Eqs. (B2) we replace z_A by Γ_A to obtain

$$q_i=a(\Gamma_0-\Gamma_4)^{-1}\Gamma_i, \tag{B8a}$$

$$q^2=a^2(\Gamma_0-\Gamma_4)^{-1}(\Gamma_0+\Gamma_4). \tag{B8b}$$

For the position operator r_i we obtain

$$-ir_i\hat{\phi}(z)=[(q^2+a^2)^2/4a^{5/2}(z_0)^2]\frac{\partial}{\partial q_i}\phi(\mathbf{q})$$

$$=\left(\frac{\partial}{\partial q_i}-\frac{2z_i}{az_0}\right)\hat{\phi}(z)$$

or

$$-iar_i=\frac{2z_i}{z_0}+a\frac{\partial z_a}{\partial q_i}\frac{\partial}{\partial z_a}$$

$$=-\frac{z_i}{z_0}\left(2+z_j\frac{\partial}{\partial z_j}\right)+(z_0-z_4)\left(\frac{\partial}{\partial z_i}+\frac{z_i}{z_0}\frac{\partial}{\partial z_4}\right)$$

$$=-\frac{z_i}{z_0}\left(2+z_A\frac{\partial}{\partial z_A}\right)+(z_0-z_4)\frac{\partial}{\partial z_i}+z_i\left(\frac{\partial}{\partial z_4}+\frac{\partial}{\partial z_0}\right).$$

From Eq. (A2), and since $z_A(\partial/\partial z_A)$ is a Casimir operator of the group with eigenvalue $N=-2$, this reduces to

$$-iar_i=-i(s_{i0}-s_{i4}) \tag{B9}$$

$$=[\Gamma_i,(\Gamma_0-\Gamma_4)]. \tag{B10}$$

Application of Eq. (A19) then shows that

$$a^2r^2\equiv-a^2r^ir_i=(\Gamma_0-\Gamma_4)^2$$

or

$$ar=\Gamma_0-\Gamma_4, \tag{B11}$$

since ar is required to be positive definite on physical grounds.

From Eqs. (B8) and (B11), it follows that the Hamiltonian

$$H=q^2/2\mu-Z\alpha/r \tag{B12}$$

becomes

$$H=(\Gamma_0-\Gamma_4)^{-1}[(a^2/2\mu)(\Gamma_0+\Gamma_4)-Z\alpha a]. \tag{B13}$$

Note that the matrix form of H staisfies the physical requirements of being Hermitian with respect to the physical scalar product of Eq. (B7), as a result of the matrices Γ_A being Hermitian with respect to the invariant scalar product. The Schrödinger equation now assumes the form

$$[E(\Gamma_0-\Gamma_4)-(a^2/2\mu)(\Gamma_0+\Gamma_4)+Z\alpha a]|\hat{\phi}\rangle=0. \tag{B14}$$

From Eq. (A20), application of the rotation operator $R(\theta)\equiv e^{-i\theta s_{04}}$ yields

$$[E(\Gamma_0-\Gamma_4)e^{-\theta}-(a^2/2\mu)e^{\theta}(\Gamma_0+\Gamma_4)+Z\alpha a]\times R^{-1}(\theta)|\hat{\phi}\rangle=0. \tag{B15}$$

For $\theta=\frac{1}{2}\ln(2\mu|E|/a^2)$, and $E<0$, this reduces to

$$[E\Gamma_0+(\tfrac{1}{2}\mu|E|)^{1/2}Z\alpha]R^{-1}(\theta)|\hat{\phi}\rangle=0. \tag{B16}$$

Let $|n,L,L_Z\rangle\rangle$ be an eigenstate of Γ_0, $\mathbf{L}^2\equiv(s_{23})^2+(s_{31})^2+(s_{12})^2$, and s_{12}, with eigenvalues n, $L(L+1)$, and L_Z, respectively, and with normalization given by the invariant scalar product

$$\langle\langle n',L',L_Z'|n,L,L_Z\rangle\rangle=\delta_{nn'}\delta_{LL'}\delta_{L_ZL_Z'}. \tag{B17}$$

These basis vectors form a complete set

$$\sum_{n,L,L_Z}|n,L,L_Z\rangle\rangle\langle\langle n,L,L_Z|=1. \tag{B18}$$

Further, let the solutions $|\hat{\phi}\rangle$ of Eq. (B16) be denoted by $|N,L,L_Z\rangle$, where N is the principal quantum number, and L and L_Z are the angular momentum quantum numbers, with normalization given by the physical scalar product

$$\langle N',L',L_Z'|(\Gamma_0-\Gamma_4)|N,L,L_Z\rangle=\delta_{NN'}\delta_{LL'}\delta_{L_ZL_Z'}. \tag{B19}$$

For convenience we will suppress the L_Z dependence of these basis vectors and physical states. From Eq. (B16) it is seen that the solutions $R^{-1}(\theta)|\hat{\phi}\rangle$ are eigenstates of Γ_0 with eigenvalues $N=1, 2, \ldots,$ and energies $E_N=-\mu(Z\alpha)^2/2N^2$, i.e.,

$$|N,L\rangle=\mathfrak{N}_NR(\theta_N)|N,L\rangle\rangle \tag{B20}$$

with $\theta_N=\ln(\mu Z\alpha/aN)$. The normalization constant is determined from

$$1=|\mathfrak{N}_N|^2\langle\langle N,L|R^{-1}(\theta_N)(\Gamma_0-\Gamma_4)R(\theta_N)|N,L\rangle\rangle$$
$$=|\mathfrak{N}_N|^2e^{-\theta_N}\langle\langle N,L|(\Gamma_0-\Gamma_4)|N,L\rangle\rangle$$
$$=N|\mathfrak{N}_N|^2e^{-\theta_N},$$

or

$$\mathfrak{N}_N=(\mu Z\alpha/aN^2)^{1/2}. \tag{B21}$$

Although the basis vectors $|N,L\rangle\rangle$ form a complete set, the physical states $|N,L\rangle$ do not, because $R(\theta_N)$ is N-dependent. In order to complete the set, it is necessary to include the positive-energy continuum states, to obtain the matrix expression corresponding to Eq. (1.2),

$$\sum_{N,L}{}^{c}|N,L\rangle\langle N,L|(\Gamma_0-\Gamma_4)=1. \tag{B22}$$

As in Sec. I, the superscript on $\sum^c$ merely indicates that the continuum states are included in the summation. This completeness relation is the only point in the present study which explicitly involves the matrix form of the continuum states, and they need not be considered further at this time.

One remaining operator to be expressed in $SO(4,2)$ form is the Dirac δ function. For this purpose, we introduce the differential operator

$$D'\equiv r(\partial/\partial r)=i\mathbf{r}\cdot\mathbf{q} \tag{B23}$$

and write the Laplacian operator in spherical coordinates,

$$\nabla^2=r^{-2}[D'(1+D')-\mathbf{L}^2].$$

The Laplacian of a spherically symmetric function $U(r)$ can thus be written as

$$(\nabla^2U)=[\nabla^2,U]-(2/r^2)[D',U]D'. \tag{B24}$$

By setting $U=-1/r$, and using Eqs. (B8b) and (B11), we obtain the desired result

$$4\pi\delta^3(\mathbf{r})=a^3(\Gamma_0-\Gamma_4)^{-1}\{[(\Gamma_0+\Gamma_4),(\Gamma_0-\Gamma_4)^{-1}]+2(\Gamma_0-\Gamma_4)^{-1}[D',(\Gamma_0-\Gamma_4)^{-1}]D'\}, \quad \text{(B25)}$$

where, from Eqs. (B8a), (B10), and (A19),

$$D'=-[\Gamma_i,(\Gamma_0-\Gamma_4)](\Gamma_0-\Gamma_4)^{-1}\Gamma_i=[\Gamma_0,\Gamma_4]-1. \quad \text{(B26)}$$

From Eqs. (A13) and (A16), or by direct calculation with Eq. (C6), it follows that $[D'(\Gamma_0-\Gamma_4)^{-1}]=-(\Gamma_0-\Gamma_4)^{-1}$, and Eq. (B25) reduces to

$$4\pi\delta^3(\mathbf{r})=a^3(\Gamma_0-\Gamma_4)^{-1}D, \quad \text{(B27)}$$

where

$$D\equiv[(\Gamma_0+\Gamma_4),(\Gamma_0-\Gamma_4)^{-1}]-2(\Gamma_0-\Gamma_4)^{-1}[(\Gamma_0-\Gamma_4)^{-1}D']. \quad \text{(B28)}$$

Note that the order in which the various factors $(\Gamma_0-\Gamma_4)^{-1}$ are combined is crucial, because $(\Gamma_0-\Gamma_4)^{-2}$ is undefined on the manifold of states with $L=0$. When the multiplications are performed in the order indicated by the square brackets, the matrix D is well defined. The remaining factor of $(\Gamma_0-\Gamma_4)^{-1}$ will be cancelled by the $(\Gamma_0-\Gamma_4)$ factor in the physical scalar product of Eq. (B7). Although D is merely the matrix form of $4\pi r\delta^3(\mathbf{r})/a^2$, it is (contrary to what one might expect) not identically zero. Only the physical matrix elements of D, defined with the physical scalar product of Eq. (B7), are required to vanish. Since the matrix $(\Gamma_0-\Gamma_4)D$ is identically zero, this physical requirement is satisfied.

APPENDIX C: EVALUATION OF SEVERAL SPECIAL MATRICES

For the Majorana representation of $SO(p,2)$, we evaluate the reciprocal matrices, $G_\pm\equiv(\Gamma_0\pm\Gamma_p)^{-1}$. If $|n,L\rangle\rangle$ is represented by the monomial $f_n x^{n-K-1}$, where

$$f_n\equiv[(n+K)!/(n-K-1)!]^{1/2}, \quad \text{(C1)}$$

and

$$K\equiv L+\tfrac{1}{2}p-2,$$

then Eqs. (A27) and (A28) are satisfied if $\Gamma_0\pm\Gamma_p$ is represented by the differential operator

$$D_\pm\equiv\pm\tfrac{1}{2}(1\pm x)^{-2K}(\partial/\partial x)(1\pm x)^{2(K+1)}. \quad \text{(C2)}$$

If, in addition, $G_\pm|n,L\rangle\rangle$ is represented by the polynomial

$$G_n(x)=\sum_m \langle\langle m,L|G_\pm|n,L\rangle\rangle f_m x^{m-K-1}, \quad \text{(C3)}$$

then the equation $(\Gamma_0\pm\Gamma_p)G_\pm=1$ becomes

$$D_\pm G_n(x)=f_n x^{n-K-1}, \quad \text{(C4)}$$

with the solution

$$G_n(x)=(1\pm x)^{-2(K+1)}\left[\pm2f_n\int dx(1\pm x)^{2K}x^{n-K-1}+C_{n,L}'\right]$$
$$=\frac{2f_n}{(2K+1)}\left(\frac{x^{n-K-1}}{1\pm x}\mp(n-K-1)\sum_{m=0}^{n-K-2}(\mp)^m x^{n-K-2-m}\frac{(n-K-2)!(n+K-1-m)!}{(n-K-2-m)!(n+K)!}\right)+C_{n,L}'(1\pm x)^{-2(K+1)}$$
$$=\frac{2}{2K+1}\left(\sum_{m=K+1}^{n-1}(\mp)^{n-m}x^{m-K-1}(f_m)^2/f_n+\sum_{m=n}^{\infty}(\mp)^{n-m}x^{m-K-1}f_n\right)+\frac{C_{n,L}'}{(2K+1)!}\sum_{m=K+1}^{\infty}(\mp x)^{m-K-1}(f_m)^2. \quad \text{(C5)}$$

Thus, $G_\pm=G_\pm'+G_\pm''$, with

$$\langle\langle m,L|G_\pm'|n,L\rangle\rangle=\frac{2(\mp)^{n-m}}{2K+1}\times\begin{cases} f_m/f_n, & K+1\le m\le n<\infty \\ f_n/f_m, & K+1\le n\le m<\infty\end{cases} \quad \text{(C6)}$$

and

$$\langle\langle m,L|G_\pm''|n,L\rangle\rangle=(\mp)^{m-K-1}C_{n,L}'f_m/(2K+1)!. \quad \text{(C7)}$$

Since $(\Gamma_0\pm\Gamma_p)$ is a symmetric matrix, the requirement that $G_\pm$ also be a left inverse reduces to the requirement that the transpose $\tilde{G}_\pm$ also be a right inverse. This restricts the integration constant to be of the form $C_{n,L}'=(\mp)^n f_n C_L$. However, unless $C_L=0$, not only will $G_\pm$ be unbounded, but no higher powers of $G_\pm$ will exist. We therefore take Eq. (C6) as the desired inverse.

Note that for $m\to\infty$, f_m has the asymptotic form

$$(f_m)^{-2}\sim m^{-(2L+p-3)}. \quad \text{(C8)}$$

Then, as a result of the infinite summations involved, integral powers of $G_\pm$ will have the asymptotic form

$$\langle\langle m,L|(G_\pm)^M|n,L\rangle\rangle\sim m^{M-L-(p-3)/2} \quad \text{(C9)}$$

and will exist only for $M<\frac{1}{2}(p-2)$, thus proving the assertion made in conjunction with Eq. (A19). However, when restricted to the manifold of basis vectors having a given value of L, $(G_\pm)^M$ is defined for $M<L+\frac{1}{2}(p-2)$.

The evaluation of $(G_\pm)^2$ is straightforward, and involves the two sums

$$\Sigma_1\equiv\sum_{n=K+1}^{m-1}f_n^2=(2K+1)\sum_{n=K+1}^{m-1}\binom{n+K}{2K+1}$$
$$=(2K+1)!\binom{m+K}{2K+2}$$
$$=(m+K-1)f_m^2/2(K+1) \quad \text{(C10)}$$

and

$$\Sigma_2 \equiv \sum_{n=M+1}^{\infty} 1/f_m^2 = \frac{(N-K)!}{(N+K+1)!} {}_2F_1 \times (1, N-K+1; N+K+2; 1) = (M-K)/2Kf_M^2, \quad K \neq 0. \quad \text{(C11)}$$

Thus, for $K\neq 0$

$$\langle\langle m_1,L|(\Gamma_0 \pm \Gamma_p)^{-1}|m_2,L\rangle\rangle = \frac{2(\mp)^{M-m}}{2K+1}\left(\frac{M}{K}-\frac{m}{K+1}\right) f_m/f_M, \quad \text{(C12)}$$

where $M\equiv \max(m_1,m_2)$ and $m\equiv \min(m,m_2)$. For SO-(4,2), we merely have $K=L$.

The D matrix of Eq. (B28) is easily evaluated once we observe (via direct calculation) that

$$\langle\langle n,L|(\Gamma_0-\Gamma_4)^{-1}D'|m,L\rangle\rangle = -\epsilon_{nm}\langle\langle n,L|(\Gamma_0-\Gamma_4)^{-1}|m,L\rangle\rangle, \quad \text{(C13)}$$

where

$$\begin{aligned} \epsilon_{nm} &= L+1, \quad n<m \\ &= \tfrac{1}{2}, \quad n=m \\ &= -L, \quad n>m. \end{aligned} \quad \text{(C14)}$$

Again using the summations of Eqs. (C10) and (C11), we obtain

$$\langle\langle m_1,L|D|m_2,L\rangle\rangle = 4(m_1m_2)^{1/2}\delta_{L,0}. \quad \text{(C15)}$$

Note that in obtaining this result for $L=0$, the infinite sum of Eq. (C11) does not contribute, because each term of the sum is zero as a result of the vanishing of the coefficients ϵ_{nm}.

In order to evaluate the matrix $R(\theta)=e^{-i\theta s_{04}}$, we note from Eqs. (A13), (A16), (A27), and (A28) that the representation of the operators $-i\Gamma_4$, is_{04}, and Γ_0 in the $|n,L\rangle\rangle$ basis is precisely the same as the familiar representation of the $SO(3)$ generators L_x, L_y, and L_z. Thus, we merely need the analytic continuation (in indices and argument) of the function[13]

$$d_{mm'}{}^L(\phi) \equiv \langle\langle L,m|e^{-i\phi L_y}|L,m'\rangle\rangle = \frac{1}{\Gamma(m-m'+1)}\left[\frac{\Gamma(L+m+1)\Gamma(L-m'+1)}{\Gamma(L+m'+1)\Gamma(L-m+1)} \times\left(\frac{1+\omega}{2}\right)^{m+m'}\left(\frac{1-\omega}{2}\right)^{m-m'}\right]^{1/2} {}_2F_1 \times(-L+m, L+1+m; m-m'+1; \tfrac{1}{2}(1-\omega)), \quad \text{(C16)}$$

where $\omega\equiv\cos\phi$. Upon using the identity

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z) \quad \text{(C17)}$$

for the hypergeometric function, we obtain the desired result

$$\langle\langle n,L|R(\theta)|N,L\rangle\rangle = d_{nN}{}^L(-i\theta) = \frac{1}{(n-N)!}\left[\frac{(n-L-1)!(n+L)!}{(N-L-1)!(N+L)!}\left(\frac{2}{\omega+1}\right)^{n+N}\left(\frac{\omega-1}{2}\right)^{n-N}\right]^{1/2} \times {}_2F_1(L+1-N, -L-N; n-N+1; \tfrac{1}{2}(1-\omega)), \quad \text{(C18)}$$

where $\omega\equiv\cosh\theta$.

APPENDIX D: ALTERNATIVE EXPRESSION FOR $\gamma(N,L)$

Since Γ_0 has an integer spectrum, the factor $(\Gamma_0-\nu)^{-1}$ in Eq. (2.13) may be expressed as

$$(\Gamma_0-\nu)^{-1} = B_\nu \int_0^{2\pi} id\theta\, e^{i\theta(\Gamma_0-\nu)}, \quad \text{(D1)}$$

with

$$B_\nu \equiv 1/(e^{-2\pi i\nu}-1). \quad \text{(D2)}$$

Then Eqs. (2.9) and (2.13) reduce to

$$Q_3(N,L) = \tfrac{1}{16}N\nu^3\hat{k}\Delta_\nu B_\nu \times \int_0^{2\pi} id\theta\, e^{-i\theta\nu}\langle\langle N,L|\mathcal{R}|N,L\rangle\rangle, \quad \text{(D3)}$$

where

$$\mathcal{R} \equiv R^{-1}(\tilde{\theta})e^{i\theta\Gamma_0}R(\tilde{\theta}) \quad \text{(D4)}$$

and $\tilde{\theta}$ is given by Eq. (2.17). Since the $SO(4,2)$ rotation operator may be written in the Euler angle form

$$\mathcal{R} = e^{i\alpha\Gamma_0}R(\beta)e^{i\gamma\Gamma_0}, \quad \text{(D5)}$$

the matrix element in Eq. (D3) becomes

$$\begin{aligned}\langle\langle N,L|\mathcal{R}|N,L\rangle\rangle &= \epsilon^N\langle\langle N,L|R(\beta)|N,L\rangle\rangle \\ &= [2\epsilon/(1+\omega)]^N \times {}_2F_1(L+1-N, -L-N; 1; \tfrac{1}{2}(1-\omega)),\end{aligned} \quad \text{(D6)}$$

where

$$\epsilon \equiv e^{i(\alpha+\gamma)} \quad \text{(D7)}$$

and

$$\omega \equiv \cosh\beta. \quad \text{(D8)}$$

We easily determine the relation of α, β, and γ to $\tilde{\theta}$ and θ by taking the vector representation of Eqs. (D4) and

[13] J. Strathdee, J. F. Boyce, R. Delbourgo, and Abdus Salam, Trieste Report No. IC/67/9, 1967, pp. 54–55 (unpublished). Minor typographical errors in this report have been corrected in Eqs. (C16) and (C18).

(D5), with the results

$$\omega=(1-\cos\theta)\sinh^2\tilde{\theta}+1, \quad \text{(D9)}$$

$$\epsilon=(1+\omega)^{-1}[(\cos\theta+1)+(\cos\theta-1)\cosh^2\tilde{\theta}+2i\sin\theta\cosh\theta]. \quad \text{(D10)}$$

Changing to the variable $z=e^{i\theta}$ and to the variable x given by Eqs. (2.18a) and (2.23), with associated integration upper limit $x_0=1-2[(\lambda N^2+1)^{1/2}-1]^{-1}$, we obtain

$$\gamma(N,L)=\lim_{\lambda\to\infty}\left(\delta_{L,0}\ln\lambda+\int_0^{x_0}dx\,Q_x(N,L)\right), \quad \text{(D11)}$$

where

$$Q_x(N,L)\equiv x(1-x^2)^{-2}B_\nu \times\oint dz\,z^{-2-\nu}(1-z)^2\langle\langle N,L|\mathcal{R}|N,L\rangle\rangle \quad \text{(D12)}$$

and

$$\nu=N(1-x)/(1+x). \quad \text{(D13)}$$

The matrix element in Eq. (D12) is given by Eq. (D6) with

$$2\epsilon/(1+\omega)=z(1-x^2)^2/(1-x^2z)^2 \quad \text{(D14)}$$

and

$$\tfrac{1}{2}(1-\omega)=(1-z)^2x^2/z(1-x^2)^2, \quad \text{(D15)}$$

while the z-integration contour runs around the unit circle from $z=e^0$ to $z=e^{2\pi i}$.

Although Eqs. (D3) and (D12) may well be the most convenient forms for $Q_3(N,L)$ and $Q_x(N,L)$ for some purposes, the integrals can be replaced by a *finite* sum if desired. Since

$${}_2F_1[L+1-N,\,-L-N;\,1;\,\tfrac{1}{2}(1-\omega)]=\sum_{n=0}^{N-L-1}D_n[\tfrac{1}{2}(1-\omega)]^n, \quad \text{(D16)}$$

where

$$D_n\equiv\binom{N-L-1}{n}\binom{N+L}{n}, \quad \text{(D17)}$$

Eq. (D12) may be written as

$$Q_x(N,L)=\sum_{n=0}^{N-L-1}x^{2n+1}(1-x^2)^{2(N-n-1)}D_nK_n(x), \quad \text{(D18)}$$

with

$$K_n(x)\equiv B_\nu\oint dz\,z^{N-n-\nu-2}(1-z)^{2n+2}(1-x^2z)^{-2N}. \quad \text{(D19)}$$

By using the expansion

$$(1-x^2z)^{-2N}=\sum_{m=0}^{\infty}\binom{2N-1+m}{m}z^mx^{2m} \quad \text{(D20)}$$

and integrating by parts $2n+2$ times, we obtain

$$K_n=\sum_{m=0}^{\infty}\binom{2N-1+m}{m}\frac{(2n+2)!(N+m+n-\nu-2)!}{(N+m+n-\nu+1)!}x^{2m}$$
$$=[(M-n-\nu-2)!/(N+n-\nu+1)!]\times{}_2F_1(2N,\,N-n-\nu-1;\,N+n-\nu+2;\,x^2). \quad \text{(D21)}$$

For the ground state, Eqs. (D12), (D18), and (D21) become

$$Q_x(1,0)=xB_\nu\oint dz\,z^{-1-\nu}(1-x^2z)^{-2} \quad \text{(D22)}$$
$$=[\nu(\nu-2)(1+\nu)]^{-1}\times{}_2F_1[2,\,-\nu;\,3-\nu;\,(1-\nu)^2/(1+\nu)^2], \quad \text{(D23)}$$

the latter expression similar to that obtained by Fronsdal[14] for the related process of Compton scattering.

Unfortunately, the x dependence of the parameters as well as the argument of the hypergeometric function makes these results too cumbersome to be of much use for the actual evaluation of $\gamma(N,L)$. However, Eq. (D12), and possibly Eq. (D18), may be useful for purposes of formal manipulation. A similar treatment may be applied to other operators and matrix elements. For example, the Coulomb Green's function operator $G(E)$ and its matrix elements may be written in a form similar to Eq. (D12). This corresponds to the result of Schwinger[15] with respect to the use of the z integration, but the latter result does not possess the formal simplicity which is made possible by using $SO(4,2)$ operators. Further evaluation yields the matrix elements $\langle\langle N|G(E)|M\rangle\rangle$ as a *finite* sum of hypergeometric functions, corresponding to Eqs. (D18) and (D21), and, for bound states, the physical matrix elements $(N|G(E)|M)$ can be written as a *finite* double sum of such hypergeometric functions.

[14] C. Fonsdal, Phys. Rev. **179**, 1513 (1969).
[15] J. Schwinger, J. Math. Phys. **5**, 1606 (1964).