# Back Reflection of Scalar and Vector Waves in Gravitational Fields*

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The back reflection of mass-zero scalar or vector waves passing through a gravitational field is calculated. In the Newtonian (equivalence-principle) approximation to Einstein's or Brans and Dicke's gravitational theory, no back reflection occurs. The non-Lorentz part of the spatial metric components $(g_{xx},g_{yy},g_{zz})$ produce the back reflection of the waves. For optimum wavelengths, the reflection coefficient of a wave passing a mass $M$ at distance $d$ is of order $GM/c^2d$.

## I. INTRODUCTION

SOON after Einstein predicted that light would be deflected in a gravitational field, qualitative experimental confirmation of his prediction was made during the solar eclipse of 1919.

Recently, Shapiro[1] has detected the change in the velocity of light when passing through a gravitational field by measuring the round trip time for radar traveling from Earth to Venus and return when the Earth-Venus line of sight passed close by the Sun.

One should be alerted at this point to also expect that light, considered as a wave phenomenon, will be partially back reflected when it passes through a strong gravitational field, for generally when a wave passes through an inhomogeneous region, back reflection occurs.

On the other hand, the equivalence principle states that local experiments do not distinguish between a gravitational field and an accelerated coordinate frame. An accelerated coordinate frame does not lead to back reflection of light, so we might expect that a gravitational field (in lowest order at least) would not back reflect light.

The equivalence principle, however, is known to only predict part of the deflection and velocity change of light, so the answer to the question of the back reflection of light in a gravitational field is not obvious.

The purpose of this paper is to calculate the back reflection of light waves in gravitational fields. It will be found that there is no back reflection in the Newtonian (or equivalence principle) approximation of gravitational theories, but that the complete linearized theories of gravitation of Einstein or Brans and Dicke[2] do predict back reflection.

## II. BACK-REFLECTION CALCULATION

We consider geometrical theories of gravity where the space-time geometry is "curved" by the proximity of matter. Letting $\psi(\mathbf{r})$ be the Newtonian potential produced by quasistatic matter sources,

$$\psi(\mathbf{r})=G\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{1}$$

then the space-time Riemannian metric $g_{\mu\nu}$ which gives the invariant interval

$$ds^2=g_{\mu\nu}dx^\mu\, dx^\nu \tag{2}$$

takes the form (in isotropic coordinates) to linear order;

$$\begin{aligned} g_{00}&=c^2[1-(2/c^2)\psi(\mathbf{r})]\,,\\ g_{xx}&=g_{yy}=g_{zz}=-[1+(2\gamma/c^2)\psi(\mathbf{r})]\,,\\ g_{\mu\neq\nu}&=0\,. \end{aligned} \tag{3}$$

The linear $g_{00}$ metric component gives the Newtonian gravitational theory. But for the propagation of light, the linear part of the spatial metric components $(g_{xx},g_{yy},g_{zz})$ contribute equally to the deflection and velocity change of the light. We have included the dimensionless parameter $\gamma$ in the spatial metric so that we can keep track of how the spatial-metric terms separately influence the light-wave equation. Also, the $\gamma$ parameter facilitates applying our final result to both the Einstein and Brans-Dicke gravitational theories. $\gamma$ (Einstein) $=1$, and $\gamma$ (Brans-Dicke) $=(1+w)/(2+w)^2$, where $w$ is a dimensionless coupling constant in the Brans-Dicke theory.

In terms of the $\gamma$ parameter, the deflection of light passing a mass at distance $d$ is

$$\Theta=(1+\gamma)2GM/c^2d$$

and the position-dependent velocity of light is

$$c(\mathbf{r})=c[1-(1+\gamma)\psi(\mathbf{r})/c^2]\,.$$

In matter-free but curved space, we need the covariant generalization of the wave equation for the electromagnetic vector potential:

$$\frac{1}{c^2}\frac{d^2\mathbf{A}}{dt^2}-\nabla^2\mathbf{A}=0\,.$$

The proper generalization is the four-dimensional covariant Laplacian

$$g^{\mu\nu}A^\lambda{}_{|\mu|\nu}=0\,, \tag{4}$$

with the vector potential subject to the covariant Lorentz gauge condition

$$A^\lambda{}_{|\lambda}=0\,.$$

In Appendix A, the curved-space wave equation (4) is used to obtain the modifications to the wave differ-

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[1] I. I. Shapiro, Phys. Rev. Letters **20**, 1265 (1968).

[2] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

ential equation for the vector potential to linear order in the gravitational field. The result is

$$\frac{d^2\phi}{dz^2}+k^2\left(1+2(1+\gamma)\frac{\psi}{c^2}\right)\phi+\frac{(\gamma-1)}{c^2}\frac{d\psi}{dz}\frac{d\phi}{dz}=0. \quad (5)$$

$\phi(z)$ is interpreted as the invariant amplitude of the wave

$$\mathbf{A}(z,t)=[1-\gamma(\psi(z)/c^2)]\phi(z)\hat{\epsilon}e^{ikct}$$

with $|\phi(z)|=1$ being the zeroth-order flat-space approximation for a propagating plane wave. $z$ is the length variable along the wave trajectory.

To first approximation in solving (5), we have

$$\phi(z)=e^{ikL(z)}, \quad (6)$$

with

$$L(z)=\int^z\left(1+(1+\gamma)\frac{\psi(z')}{c^2}\right)dz'. \quad (7)$$

To better approximation, we look for a solution

$$\phi(z)=[1+R(z)]e^{ikL(z)} \quad (8)$$

which leads to the differential equation for $R(z)$

$$\frac{d^2R}{dz^2}+2ik\left(1+(1+\gamma)\frac{\psi}{c^2}\right)\frac{dR}{dz}=-\frac{2i\gamma k}{c^2}\frac{d\psi}{dz}. \quad (9)$$

This differential equation has the inhomogeneous solution

$$R(z)=\gamma e^{-2ikL(z)}\int^z\frac{d\psi}{dz'}\frac{e^{2ikL(z')}}{c^2}dz'. \quad (10)$$

The complete solution to (5) is then

$$\phi(z)=e^{ikL(z)}+e^{-ikL(z)}\left(\frac{\gamma}{c^2}\int_{+\infty}^{z}\frac{d\psi}{dz'}e^{2ikL(z')}dz'\right), \quad (11)$$

where we have invoked the physical boundary condition that the incoming wave is coming from $z=-\infty$, and therefore there is no left-going wave amplitude at $z=+\infty$. Assuming that $z=-\infty$ is a gravity-free region, we can read out the total reflection coefficient of an incident wave of wave number $k$

$$\mathfrak{R}(k)=-\frac{\gamma}{c^2}\int_{-\infty}^{\infty}\frac{d\psi}{dz'}e^{2ikL(z')}dz'. \quad (12)$$

In the Newtonian approximation of gravitational theories the $\gamma$ metric terms are neglected, and we see from (12) that there is no back reflection of the electromagnetic wave, in agreement with our intuition concerning the equivalence principle. But it is also seen that back reflection does occur when the full linearized gravitational theory is taken into account.

Consider a light wave passing a mass $M$ at distance $d$. The reflection amplitude (12) can be evaluated and is given by a Bessel function,

$$\mathfrak{R}(k)=-\frac{2\gamma GM}{c^2}\int_0^{\infty}\frac{z'\sin 2kz'dz'}{(d^2+z'^2)^{3/2}} \quad (13)$$
$$=-8\gamma(GM/c^2)kK_0(2kd).$$

If the wave is of wave number such that $kd\sim1$, then $\mathfrak{R}(k)$ reaches its maximum value of about

$$\mathfrak{R}(kd\sim1)\simeq-\gamma GM/c^2d.$$

For an incident wave of very long wavelength, the reflection coefficient diminishes as

$$\lim_{k\to0}\mathfrak{R}(k)\simeq-8\gamma\frac{GM}{c^2}k\ln\left(\frac{1}{kd}\right),$$

while for short wavelengths, the reflection coefficient diminishes very rapidly:

$$\lim_{kd\gg1}\mathfrak{R}(k)\simeq-4\gamma\frac{GM}{c^2}\left(\frac{\pi k}{d}\right)^{1/2}e^{-2kd}.$$

We show in Appendix A that a scalar wave undergoes the same back reflection as the electromagnetic wave when propagating through a gravitational field.

The practical applications of the back reflection of wave energy by a gravitational field are in situations of very strong gravitational potential, i.e., where

$$GM/c^2d\sim1.$$

Then, a substantial fraction of wave energy can be reflected. Such strong field conditions are fulfilled in neutron stars and other near-gravitational collapse entities in cosmology.

In Appendix B, a physical optics, intuitive interpretation of the reflection formula (12) is given.

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### APPENDIX A

Consider the covariant wave equation

$$g^{\mu\nu}A^{\lambda}{}_{|\mu|\nu}=0. \quad (A1)$$

The Lorentz gauge condition on the vector potential yields no influence on the wave equation to the order we are solving the problem. Keeping terms in (A1) to linear order in the gravitational field, we get

$$g^{00}\frac{\partial^2A^\lambda}{\partial t^2}+g^{ss}\nabla^2A^\lambda+\left(\eta^{\mu\nu}\frac{\partial\Gamma_{r\nu}{}^{\lambda}}{\partial x^\mu}\right)A^r$$
$$+2(\eta^{\mu\nu}\Gamma_{r\mu}{}^{\lambda})\frac{\partial A^r}{\partial x^\nu}-(\eta^{\mu\nu}\Gamma_{\mu\nu}{}^{r})\frac{\partial A^\lambda}{\partial x^r}=0, \quad (A2)$$

where $\eta^{\mu\nu}$ is the flat-space Lorentz metric. $\Gamma_{\mu\nu}{}^{\tau}$ are the Christoffel symbols of the second kind;

$$\Gamma_{\mu\nu}{}^{\tau} \simeq \tfrac{1}{2}\eta^{\tau r}\left(\frac{\partial g_{\mu r}}{\partial x^{\nu}}+\frac{\partial g_{\nu r}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{r}}\right) \quad \text{(A3)}$$

to linear order in the gravitational field.

We first insert the lowest-order approximation for the electromagnetic vector potential for a light wave traveling in the $z$ direction, polarized in the $x$ direction

$$A^{x}(z,t)=A(z)e^{ikct}. \quad \text{(A4)}$$

The condition on $A(z)$ such that it represent a wave of constant magnitude (plane wave) is

$$g_{xx}(A^{x})^{2}=\text{const},$$

which becomes, upon using the metric (3),

$$|A(z)|=1-\gamma\psi(z)/c^{2}. \quad \text{(A5)}$$

We assume a solution to (A2) then of the form

$$A^{x}(z,t)=(1-\gamma\psi(z)/c^{2})\phi(z)e^{ikct}$$

and obtain the differential equation for $\phi(z)$

$$\frac{d^{2}\phi}{dz^{2}}+k^{2}\left(1+\frac{2}{c^{2}}(1+\gamma)\psi(z)\right)\phi+\frac{(\gamma-1)}{c^{2}}g(z)\frac{d\phi}{dz}=0 \quad \text{(A6)}$$

with $g(z)=d\psi/dz$. We obtain the same differential equation (A6) starting from the covariant scalar field wave equation

$$g^{\mu\nu}\phi_{\mu|\nu}=0, \quad \text{(A7)}$$

with

$$\phi_{\mu}=\partial\phi/\partial x^{\mu}.$$

Therefore, both scalar and vector electromagnetic waves backscatter in the same manner in a gravitational field.

## APPENDIX B

Here we give a simple physical-optics derivation of the solution to (A6) for the reflection coefficient of an incident wave. Let the gravitational potential $\psi(z)$ change in small steps from $\psi(z_i)$ to $\psi(z_{i+1})$.

Integrating (A6) across the boundary, we have

$$\left(\frac{d\phi}{dz}\right)_{+}-\left(\frac{d\phi}{dz}\right)_{-} = -(\gamma-1)\delta\psi\left(\frac{(d\phi/dz)_{+}+(d\phi/dz)_{-}}{2}\right). \quad \text{(B1)}$$

In each region of constant potential $\psi(z_i)$, the wave number of the waves is

$$k_i=\pm k[1+(1+\gamma)\psi(z_i)/c^{2}]. \quad \text{(B2)}$$

At each boundary we assume an incident, transmitted, and reflected wave:

$$e^{ik_i z}+r_i e^{-ik_i z} \quad \text{(B3a)}$$

and

$$t_i e^{ik_{i+1}z}. \quad \text{(B3b)}$$

At the boundary we then have the conditions $\phi_{+}=\phi_{-}$ and the condition (B1). These give the equations

$$1+r_i=t_i,$$

$$\left[1-\left(\frac{\gamma-1}{2}\right)\frac{\delta\psi}{c^{2}}\right]k_i(1-r_i)=\left[1+\left(\frac{\gamma-1}{2}\right)\frac{\delta\psi}{c^{2}}\right]k_{i+1}t_i,$$

which can be solved for $r_i$ to give

$$r_i=-\gamma(\delta\psi_i/c^{2}). \quad \text{(B4)}$$

In lowest order, then, the complete reflection coefficient for a wave incident upon a region of varying gravitational potential is obtained by summing the differential reflections of (B4) with the proper phase lags:

$$\mathcal{R}(k)=\sum_i r_i e^{2ikL_i}. \quad \text{(B5)}$$

$kL_i$ is the phase the incident wave obtains in propagating to $z_i$. Letting

$$\delta\psi_i=(d\psi/dz)dz$$

and replacing the sum in (B5) by an integral yields

$$\mathcal{R}(k)=-\frac{\gamma}{c^{2}}\int_{-\infty}^{\infty}\frac{d\psi}{dz'}e^{2ikL(z')}dz', \quad \text{(B6)}$$

with

$$L(z')=\int^{z'}\left[1+(1+\gamma)\frac{\psi(z'')}{c^{2}}\right]dz'',$$

which is identical to the text's solution (12).