Under some suitable hypotheses the general solution of Eq. (42) can be given in terms of Green's functions. More precisely, one has to assume that there exists the inverse Laplace transform of the analytic continuation of $H(x,y;t)$, with respect either to x or y. If this holds true, we can assume $H(x,y;t)$ in the form[18]

$$H(x,y;t)=\int_{-\infty}^{+\infty} d\mu \int_0^{+\infty} d\nu\, \tilde{H}_1(\mu,\nu;t)e^{i\mu x-\nu y} \quad (45a)$$

[where $x\in(-\infty,+\infty)$ and $y\in(a,+\infty)$], or in the form

$$H(x,y;t)=\int_0^{+\infty} d\mu \int_{-\infty}^{+\infty} d\nu\, \tilde{H}_2(\mu,\nu;t)e^{-\mu x+i\nu y}, \quad (45b)$$

where $x\in(a,+\infty)$, $y\in(-\infty,+\infty)$ [and the corresponding relations for x or $y\in(-\infty,a)$]. Substituting Eqs. (45a) or (45b) into Eq. (42), we have

$$\tilde{H}_{1,2}(\mu,\nu;t)=\tilde{H}_{1,2}(\mu,\nu;0)\exp(-\tfrac{1}{4}D\mu^2\nu^2 t), \quad (46)$$

so that, through standard algebra, Eqs. (45a) and (45b) reduce to

$$H(x,y;t)=\int_{-\infty}^{+\infty} dx' \int_{\tau_0-i\infty}^{\tau_0+i\infty} dy'\, H(x',y';0) \times G_1(x,y;t|x',y';0), \quad y>\mathrm{Re}y' \quad (47a)$$

or

$$H(x,y;t)=\int_{\tau_0-i\infty}^{\tau_0+i\infty} dx' \int_{-\infty}^{+\infty} dy'\, H(x',y';0) \times G_2(x,y;t|x',y';0), \quad x>\mathrm{Re}x', \quad (47b)$$

where τ_0 is the abscissa of convergence relative to the analytic continuation of $H(x,y;0)$ with respect to x or y, and

$$G_1(x,y;t|x',y';0)=(2i\pi^{3/2}D^{1/2}t^{1/2})^{-1} \times\int_0^{\infty}\exp[-(x-x')^2/\nu^2Dt-\nu(y-y')]\nu^{-1}d\nu, \quad (48a)$$

$$G_2(x,y;t|x',y';0)=(2i\pi^{3/2}D^{1/2}t^{1/2})^{-1} \times\int_0^{\infty}\exp[-(y-y')^2/\mu^2Dt-\mu(x-x')]\mu^{-1}d\mu. \quad (48b)$$

[18] The following developments are due to J. Peřina and V. Peřinová (private communication).

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Geodesics of Robertson-Walker Universes*

PETER D. NOERDLINGER
New Mexico Institute of Mining and Technology, Socorro, New Mexico 87801
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The complete set of geodesics is obtained for Robertson-Walker universes with arbitrary $R(t)$, from those geodesics whose spatial projections pass through the origin, by inducing a translation of origin through the rotation of a hypersphere whose stereographic projections form the space sections. The results are applied to the calculation, in a Milne universe, of the angular displacement of a particle initially projected at high velocity accross the line of sight. This displacement proves to be bounded, the upper bound being attained reasonably fast, on a cosmic time scale.

I. INTRODUCTION

WITH their present random velocities of $\sim$100 km/sec, galaxies cannot traverse a significant portion of the Universe during its evolution. There are reasons, however, for wanting to know the various trajectories of free particles with large peculiar velocity (i.e., not just the "mean fluid velocity") over a very long period. More and more extreme examples are being found[1] of galaxies with peculiar velocities of the order of thousands of km/sec. Furthermore, it is expected that in the distant past, all random velocities of free objects were much larger, roughly in proportion[2] to R^{-1}, where R is the curvature radius of a space section of a Robert-

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[1] E. M. Burbidge and G. R. Burbidge, Astrophys. J. 134, 244 (1961); P. W. Hodge, *ibid.* 134, 262; W. L. W. Sargent, *ibid.* 153, L135 (1968); and various papers in the Santa Barbara Conference, Astron. J. 66, No. 10 (1961).

[2] G. B. Van Albada, Astron. J. 66, 590 (1961). If there is drag due to intergalactic matter, the velocities would have been much bigger than given by the $1/R$ law; however, at the relativistic end, the opposite holds. Sturrock has also suggested that *QSO* may be in relativistic motion. See P. A. Sturrock, in *Plasma Astrophysics* (Academic Press Inc., New York, 1967), p. 338 and especially p. 361.

son-Walker universe.[3] Finally, in the Eddington-Lemaitre models,[4] a very long time is available for migration of material across the Universe.

An example is worked out in Sec. III that may be relevant to the assertion that it is "unlikely" for galaxies of large relative velocity to remain in visual alignment, so as to be seen as a group.

The geodesics for Robertson-Walker universe were given by Robertson[3] for the special case of particles moving directly toward or away from the origin, i.e., particles whose geodesics have spatial projection passing through the spatial origin. Since the space sections ($t=$ const) are homogeneous, the restriction to radial motion appears to offer no special problems; if one wants a geodesic corresponding to a particle moving across the line of sight, one merely chooses a new origin through the world line of that particle and applies the old Robertson result. However, this procedure is nontrivial, since the equations for transforming origin are not readily available; in fact, it is the purpose of this paper to derive such equations and apply them to the geodesics. For any practical application, there is no way to avoid this process, since the origin of celestial coordinates is fixed by experimental procedure in the neighborhood of the earth (e.g., the earth or centrum of the solar system). All the cosmological equations are written in this system, and if one wants to introduce a new origin, one must have the explicit transformation.

Clearly, the case of Euclidean space sections is trivial; an ordinary Cartesian translation may be used. Thus, effort will concentrate on the cases of positive ($k=1$) and negative ($k=-1$) curvature. The metric will be written in the form

$$ds^2=c^2dt^2-R^2(t)du^2=c^2dt^2-d\sigma^2\,, \tag{1.1}$$

where the auxiliary metric du^2 assumes one of the standard forms

$$du^2=\frac{dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2}{(1+\frac{1}{4}kr^2)^2} \tag{1.2}$$

or

$$du^2=\frac{dx^2+dy^2+dz^2}{(1+\frac{1}{4}kr^2)^2}\,,\quad r^2=x^2+y^2+z^2\,, \tag{1.3}$$

where

$$x=r\sin\theta\cos\phi\,,\quad y=r\sin\theta\sin\phi\,,\quad z=r\cos\theta\,. \tag{1.4}$$

We refer to geodesics of fixed $\theta=\theta_0$ and $\phi=\phi_0$ as "radial" geodesics or geodesics through the origin (meaning the spatial origin).[5] If we define

$$L_+(\omega)=2\tan\tfrac{1}{2}\omega\,,\quad L_0(\omega)=\omega\,,\quad L_-(\omega)=2\tanh\tfrac{1}{2}\omega\,, \tag{1.5}$$

we have in the cases $k=-1$, 0, and 1, respectively, for the radial geodesics,[3]

$$r=L_k(\omega)\,, \tag{1.6}$$

$$\theta=\theta_0\,,\quad \phi=\phi_0\,,\quad \omega=\pm\int\frac{cdt}{R(t)[1+R^2(t)/Q^2]^{1/2}}\,. \tag{1.7}$$

In (1.7), Q is a constant specifying the velocity of the particle at some chosen time; for photons we have $Q\rightarrow\infty$, while for slow particles Q is small.[6] Taking into account Q and the constant of integration, say, $r(t_0)$, we have a four-parameter family of geodesics. The space should admit a six-parameter family. There are various ways to introduce the two remaining parameters, which must describe geodesics not passing through the origin. For example, one could use the coordinates x and y at which the geodesic meets the $z=0$ plane, although this would fail for a small family of geodesics that do not intercept this plane. It is a little more convenient to set up the additional parameters as follows:

In practice, one is most likely to want to treat particles that start at a common origin and diverge, as in studying the expansion of an unbound galaxy cluster. Thus, it is more convenient to pick some new origin, say, at (x_0,y_0,z_0), with new coordinates $(\bar{r},\bar{\theta},\bar{\phi})$ or $(\bar{x},\bar{y},\bar{z})$. Equation (1.7) gives the geodesics through the old origin, which may be suitably located in the barred system by proper choice of (x_0,y_0,z_0). This procedure results in one redundancy among the parameters, since we are left with x_0, y_0, z_0 $r(t_0)$, Q, θ_0, and ϕ_0. This sevenfold set of parameters describes the sixfold set of geodesics. In applications, this produces no difficulty, and only makes it easier to fit boundary conditions. If one wants, for example, the set of geodesics for a swarm of particles released from a point at various directions (θ_0,ϕ_0) at a common time t_0 with a common initial speed, one simply fixes all parameters save θ_0 and ϕ_0, setting $r(t_0)=0$.

In Sec. II the translation of coordinates is explicitely constructed. In Sec. III, the results are applied to the determination of the angular displacement of a particle ejected from a fixed source.

II. DISPLACEMENT OF SPATIAL ORIGIN

Although the spatial homogeneity of the Robertson-Walker space times is well known, it is certainly not manifest from the metrics (1.1) or (1.2), except for the trivial case $k=0$. In that case, which we discuss no further, a translation of origin is accomplished by the transformation $\bar{x}^\mu=x^\mu-x_0{}^\mu$, which obviously leaves the form of (1.1) or (1.2) invariant. A transformation of this type is obviously not useful for $k=\pm1$, because the new metric would not be of type (1.1) or (1.2). Thus (1.7) could not be applied in the barred coordinates. In this section we construct coordinates $\bar{x}^\mu(x^\alpha,x_0{}^\alpha)$ such that the *form* of the metric is invariant to the transformation.

[3] H. P. Robertson, Rev. Mod. Phys. **5**, 62 (1933).
[4] J. D. North, *The Measure of the Universe: A History of Modern Cosmology* (Clarendon Press, Oxford, England, 1965), pp. 117–131.
[5] This is to include geodesics whose spatial projection, prolonged, intercept the origin; initial conditions may prevent physical interception.

[6] If we define V as the physical velocity $d\sigma/dt$ for the particle, where $d\sigma=Rdu$ is the *spatial* line element, then at all times the constant Q is given by $Q=R(t)\beta\gamma$, where $\beta=V/c$ and $\gamma=(1-\beta^2)^{-1/2}$. See Sec. III.

In other works, we must have

$$du^2=\frac{d\bar{x}^2+d\bar{y}^2+d\bar{z}^2}{(1+\frac{1}{4}k\bar{r}^2)^2}=\frac{d\bar{r}^2+\bar{r}^2d\bar{\theta}^2+\bar{r}^2\sin^2\bar{\theta}\,d\bar{\phi}^2}{(1+\frac{1}{4}k\bar{r}^2)^2}. \quad (2.1)$$

The key to finding the desired transformation is to embed the three-space ($t=\text{const}$) as a sphere or pseudosphere in an auxiliary four-space. I follow the embedding method of Adler, Bazin, and Schiffer.[7,8] The four-dimensional coordinates will be denoted X^i (before translation of the origin) or $\bar{X}^i$ (after translation); the fourth coordinate X^0 has nothing to do with time, but is an artificial coordinate introduced for symmetry. As shown by Adler *et al.*,[8] if one sets

$$\sum(X^i)^2=\tilde{R}^2,\quad \tilde{R}^2\equiv kR^2(t) \quad (2.2)$$

then the metric (1.2) may be written[7]

$$du^2=\sum(dX^i)^2, \quad (2.3)$$

where restriction (2.2) is imposed on the differentials and where we identify r with [9]

$$r=2k^{1/2}\rho^{-1}[\tilde{R}-(\tilde{R}^2-\rho^2)^{1/2}],\quad \rho^2\equiv\tilde{R}^2-(X^0)^2. \quad (2.4)$$

If $\tilde{R}$ is real, so are all the X^i and ρ. In that case X^0 and ρ must be less than $\tilde{R}$. If $\tilde{R}$ is imaginary, ρ must still be taken real and it takes the range $(0,\infty)$. Then X^0 is imaginary, but the X^μ are real. In both cases, we have

$$X^1=\rho\sin\theta\cos\phi,\quad X^2=\rho\sin\theta\sin\phi,\quad X^3=\rho\cos\theta. \quad (2.5)$$

Thus, we obtain

$$x^\mu/X^\mu=r/\rho=2k^{1/2}[\tilde{R}-(\tilde{R}^2-\rho^2)^{1/2}]/\rho^2$$
$$=(1+\tfrac{1}{4}kr^2)/\tilde{R}k^{1/2}. \quad (2.6)$$

We shall also need the results derivable from (2.2) and (2.4):

$$X^0=\tilde{R}\frac{1-\frac{1}{4}kr^2}{1+\frac{1}{4}kr^2},\quad r=\frac{2}{\sqrt{k}}\left[\frac{\tilde{R}-X^0}{\tilde{R}+X^0}\right]^{1/2}. \quad (2.7)$$

It is simple to introduce motions in the (X^i) system that preserve the quadratic forms (2.2) and (2.3), and which therefore preserve the form of the metric. Any rotation will do. Clearly, rotations leaving X^0 fixed only amount to rotations about O, but any other rotation will displace O, where O is the origin of the x^μ. It is simplest to choose one convenient rotation in X^i and to introduce any further generality by compounding this with rotations in the x^μ system. We choose the rotation

$$\bar{X}^0=X^0\cos\alpha-X^3\sin\alpha,$$
$$\bar{X}^3=X^0\sin\alpha+X^3\cos\alpha,\ \text{all other } X^i \text{ unchanged}. \quad (2.8)$$

The "angle" α must be imaginary if $k<0$. Schematically, the full transformation from x^μ to $\bar{x}^\mu$ is obtained by the successive transformations:

$$x^\mu \xrightarrow[(2.6),\,(2.7)]{} X^i \xrightarrow[(2.8)]{} \bar{X}^i \xrightarrow[(\bar{2}.\bar{6}),\,(\bar{2}.\bar{7})]{} \bar{x}^\mu, \quad (2.9)$$

where the relevant transformation equations are indicated beneath the arrows.[10]

The new spatial origin ($\bar{x}^\mu=0$) is at $\bar{X}^0=\tilde{R}$, $\bar{X}^\mu=0$. In the old system one finds from Eqs. (2.6)–(2.8) that the new origin $\bar{O}$ has coordinates (see Fig. 1):

$$x=y=0,\quad z=-2k^{-1/2}\tan\tfrac{1}{2}\alpha\quad \text{(new origin)}. \quad (2.10)$$

The transformation seems to be of sufficient generality for all practical cases.

The explicit equations resulting from (2.9) are

$$\bar{x}=2x/P,\quad \bar{y}=2y/P,$$
$$\bar{z}=2[z\cos\alpha+k^{-1/2}(1-\tfrac{1}{4}kr^2)\sin\alpha]/P, \quad (2.11\text{a})$$

where

$$P\equiv1+\tfrac{1}{4}kr^2+(1-\tfrac{1}{4}kr^2)\cos\alpha-zk^{-1/2}\sin\alpha. \quad (2.11\text{b})$$

It is worth noting that

$$1+\tfrac{1}{4}k\bar{r}^2=2(1+\tfrac{1}{4}kr^2)/P. \quad (2.12)$$

Combining (1.4), (1.7), and (2.11), one formally obtains all the geodesics through point

$$\bar{x}=\bar{y}=0,\quad \bar{z}=(2/\sqrt{k})\tan\tfrac{1}{2}\alpha\quad \text{(old origin)}. \quad (2.13)$$

These will be written in detail only for one example. One should also remember that, under the elliptic identification (in the case $k=1$), points beyond $r=2$ ($\omega=\frac{1}{2}\pi$) are identified with other points having $r<2$. Thus, for example, (2.13) will be redundant beyond $\alpha=\pi$. It has been asserted[11] that infinitely many different topological identifications of this sort are possible; these will not be considered here.

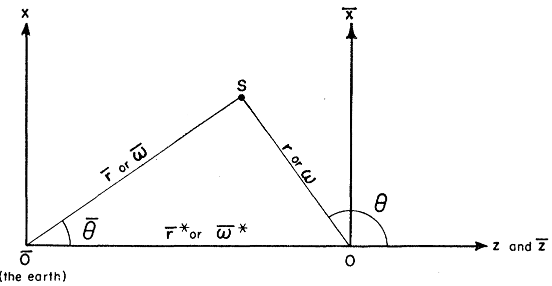

FIG. 1. Schematic of the old $(OXYZ)$ and new $(\bar{O}\bar{X}\bar{Y}\bar{Z})$ coordinate systems, not showing the curvature. $\bar{r}^*$ and $\bar{\omega}^*$ are determined by α, according to Eqs. (1.6) and (2.13). When the motion of a specific particle S is considered, its coordinates may be subscripted S in the text.

III. AN EXAMPLE

The application of the above results is generally cumbersome, especially since to reduce the results to

[7] Greek indices run from 1 to 3, roman, from 0 to 3.

[8] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Co., New York, 1965), pp. 346–349.

[9] Our r is equivalent to u of Adler *et al.*, and our $\tilde{R}$ to their R; the tilde is used to avoid confusion with $R(t)$. ρ cannot be used beyond $r=2$, even if $k=1$.

[10] Bars placed over an equation number means that it is to be written in the barred system.

[11] O. Heckmann and E. Schücking, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 439.

observation one must connect the particle orbits to the (new) origin by light signals. Here we shall give only one rather idealized example, in which the angular displacement of a particle of a certain initial velocity is evaluated in one model universe—the Milne universe—as a function of time. The light propagation time is taken into account, so that only the ejection time and the observation time enter. However, we specialize to the particular angle of ejection (90° to the line of sight), in order to get a simple comparison showing the effects of space curvature. It is possible to interpret the calculation as referring to a galaxy which at a time t^* left a group of galaxies (fixed in the co-moving cosmological reference frame) with velocity V_0. Again, see Fig. 1 for the geometry.

From Equation (1.7), we can derive the result of Ref. 6 for Q, and we shall evaluate Q in terms of V_0 and $R^* \equiv R(t^*)$. For these radial geodesics, clearly, $d\sigma = Rd\omega$, so that

$$V = Rd\omega/dt = c(1+R^2/Q^2)^{-1/2}. \tag{3.1}$$

Solving for Q yields the result of Ref. 6. From (3.1), we see that so long as $Q \gg R$, then $V \ll c$, and the result $V \sim R^{-1}$ is recovered. It is also worth remarking that from (1.7), the geodesics have a translational invariance in ω, which reflects the uniformity of the space sections. A corollary is that two particles projected at the same "universal" time with the same initial velocity along the same ray through the origin will maintain a constant *coordinate* distance $\delta\omega$ from each other. Thus, their *physical* separation $R\ \delta\omega$ will expand in exactly the same way as the neighboring portions of the universe.

We now proceed to the special case of the Milne universe, $q_0 = \sigma_0 = 0$, $k = -1$. This is chosen because it is simple and because it is probably not a very bad model in view of the low density of observed mass in the universe[12] and the uncertainty in the deceleration parameter.[13] For this universe, $R = ct$, and Eq. (1.7) integrates to

$$\omega = \tfrac{1}{2} \ln\left[\frac{(1+c^2t^2/Q^2)-1}{(1+c^2t^2/Q^2)+1}\right] + \text{const}. \tag{3.2}$$

With the use of Eq. (1.6) and much manipulation, Eq. (3.2) becomes

$$r = 2\frac{(1+\xi^2)^{1/2} - \beta_0 - \xi(1-\beta_0^2)^{1/2}}{\beta_0(1+\xi^2)^{1/2} - 1}, \tag{3.3}$$

where

$$\xi \equiv ct/Q, \quad \text{and} \quad \beta_0 = V_0/c, \tag{3.4}$$

and the particle has been chosen to depart from $r = 0$ at time t^* with velocity V_0, so that

$$ct^*/Q = (\beta_0^{-2} - 1)^{1/2}. \tag{3.5}$$

An impressive result may be obtained at once: r is *bounded*. In fact, passing to the limit $t \rightarrow \infty$, we see that $r \rightarrow r_{\max}$, where

$$r_{\max} = 2[1 - (1-\beta_0^2)^{1/2}]/\beta_0 \approx \beta_0, \tag{3.6}$$

the last approximation being for β_0 small. This result does not involve the intended translation of coordinates, but could be substituted therein to obtain an observed maximum angular size. Here we only note two features of the result: First, it does not imply limited maximal *physical* dimensions for an expanding system, because r must be multiplied by a factor porportional to $R(t)$ to get physical dimensions. Second, although this result implies a small final observed angular diameter for a swarm of point masses ejected at small velocity from a distant point, one may verify that the situation is not good for keeping realistic galaxy clusters together this way, because realistic initial velocities seem too high.[14] Thus, gravitational binding seems needed.[1]

To relate the ejection time t^* with the observation time (say, t_1), and so to obtain the apparent rate of growth of the opening angle $\bar{\theta}$, we need the null geodesics from the trajectory to the new origin $\bar{O}$. It is desirable to have some notation for the time t_S at which the light received at t_1 leaves the source S, since this would be helpful in evaluating spectral shifts. To keep the present discussion brief, spectral shifts will not be evaluated, nor will we consider in detail ejection angles other than 90°, even though, because of light delay and non-Euclidean effects, 90° will not generally lead to the maximum apparent separation $\bar{\theta}$ for fixed V_0, t^*, and t_1. For the moment, however, we leave θ arbitrary and seek the null geodesics relating t_1 and t_S with $\bar{\omega}_S$. The t in Eq. (3.4) is then identified with t_S.

For the Milne universe, the required light rays are of the form

$$\bar{\omega}_S = \int_{t_s}^{t_1} t^{-1} dt = \ln(t_1/t_S). \tag{3.7}$$

Returning to $\bar{r}$ instead of $\bar{\omega}$ as a radial coordinate, we find that Eq. (3.7) becomes

$$\mathbf{r}_S = 2(\tau-1)/(\tau+1) \quad \tau \equiv t_1/t_S. \tag{3.8}$$

Note that $(\tau - 1)$ is the red shift of material in the vicinity of S (but not of S).

Now, Eq. (3.3) for the motion of S is parametrized with the time $t \equiv t_S$, which is neither observable nor of much theoretical interest. Actually, we are interested in $\bar{\theta}$ as a function of the distance of the ejection point O and t^*. Although the latter is not observable either, it is significant theoretically, as the time at which a system having S as a member became unbound, or was formed. In an ejection model, such as Arp's,[15] t^* would be the

[12] G. C. McVittie, *General Relativity and Cosmology* (University of Illinois Press, Urbana, Ill., 1965), 2nd ed., pp. 203–204; G. O. Abell, Ann. Rev. Astron. Astrophys. **3**, 1 (1965).

[13] A. Sandage, Carnegie Institution Year Book **65**, 163 (1965); V. Petrosian, Astrophys. J. **155**, 1029 (1969); R. Wielen, Z. Astrophys. **59**, 129 (1964).

[14] I hope to give fuller discussion of this point at a later date. The situation is worst at very early epochs, and if the cluster can be bound for a time, after which an explosion, mass loss, etc., destroy the binding, the situation is far better.

[15] H. Arp, Astrophys. J. **148**, 321 (1967).

ejection time. In principle, the ultimate results are obtained by reducing r in Eq. (3.3) to $\bar{r}$ [through the use of Eqs. (2.11)], eliminating t_S between the resulting equation and Eq. (3.8), and evaluating $\bar{\theta}$ from Eqs. (2.11). In practice, it was found helpful to retain instead two parameters, namely, r and the parameter

$$T \equiv t_S/t^*. \qquad (3.9)$$

It turns out that by fixing α, β_0, θ, and T, all other quantities are determined. These were worked out numerically for a variety of cases. We shall not write out the equations in detail, since they are only specializations of Eqs. (2.11), (3.3)–(3.5), and (3.8), but we do wish to indicate the fashion in which the given parameters lead to values for the other variables. The significance of ξ becomes clearer if we note from Eqs. (3.4) and (3.5) that

$$\xi = T(\beta_0^{-2} - 1)^{1/2}. \qquad (3.10)$$

Thus, actual evaluation of Q is avoided if T and β_0 are used as variables. Various T values are inserted into Eq. (3.10), yielding values of ξ. These are put into Eq. (3.3) to obtain r, after which $\bar{r}$ is obtained from Eqs. (2.11). When θ is specified (being 90° in the present illustrative example), $\bar{x}$ and $\bar{z}$ are separately determined in like manner. From $\bar{r}$ and Eq. (3.8), one obtains t_1/t_S, and this, in combination with Eq. (3.9), fixes t_1/t^*, setting the time of ejection or unbinding. We deliberately avoid associating a red shift with this ratio, because the event (O,t^*) is assumed not to be presently observed, its light rays long having passed us. The red shift of O, as it is seen now, is generally far less than $t_1/t^* - 1$, and is given by the formula

$$\Delta_O = e^{\alpha} - 1, \quad \Delta_O = \text{red shift of } O. \qquad (3.11)$$

From the results presented in Table 1, it is clear that Δ_O coincides with $t_1/t^* - 1$ at the moment of ejection ($T=1$), and that subsequently, Δ_O does not change, while t_1 increases without limit. The fixity of Δ_O is true only in the Milne cosmology. Of course, the "evolution" described in the table as t_1 changes so much it is not supposed to be observable in a reasonable observation time. Instead, t^* is to be adjusted to fit the presumed or theoretically derived initial conditions. During the evolution, τ changes very little, so that the redshift of cosmic material near S (not partaking of the large peculiar velocity of S) is quite close that of O. This is quite reasonable, since S stays fairly near O, and in this example it was chosen to move initially at right angles to the line of sight. If other directions of motion were chosen, we may expect larger differences in the red shift of material near S and that of O.

The remainder of the discussion will concentrate on Table I. The large values of β_0 chosen may seem unreasonable, but the system is supposed to be observed, on the average, at some time t_1 much later than t^*, so that the velocities have decayed according to Eq. (3.1). Choosing somewhat large β_0 facilitates evaluation of the effects of space curvature, as well. These effects seem appreciable for large β_0, t_1, and α. For example, in a Euclidean model, we would expect $\bar{\theta} \approx \tan^{-1}(r/\alpha)$, while in fact $\bar{\theta}$ tends to be much less in these cases. Much of the increase in τ is probably due to distortion of the track of S as referred to $\bar{O}$, so that although S started out moving perpendicular to the line of sight, it is later moving away.[16]

TABLE I. Angular displacement θ of S from the point of origin, as a function of α, β_0, and (t_1/t^*). The red shift Δ_O of the point of origin and the parameters r and τ are also given. At the head of each group, the values of α, β_0, and Δ_O are given, in that order, in parentheses. Single numbers in parenthesis are powers of ten.

	(0.02, 0.02, 0.02)				(0.2, 0.1, 0.2214)		
t_1/t^*	r	$\bar{\theta}$	τ	t_1/t^*	r	$\bar{\theta}$	τ
1.1	1.2(−3)	6.0(−2)	1.02	1.23	9.6(−4)	4.8(−3)	1.22
1.2	3.0(−3)	1.5(−1)	1.02	1.25	2.0(−3)	9.6(−3)	1.22
1.3	4.3(−3)	2.1(−1)	1.02	1.27	3.9(−3)	1.9(−2)	1.22
1.5	6.3(−3)	3.1(−1)	1.02	1.31	6.7(−3)	3.3(−2)	1.22
1.8	8.6(−3)	4.1(−1)	1.02	1.4	1.3(−2)	6.4(−2)	1.22
2.2	1.1(−2)	5.0(−1)	1.02	1.6	2.4(−2)	1.2(−1)	1.22
3.0	1.3(−2)	5.8(−1)	1.02	2.0	3.9(−2)	1.9(−1)	1.23
5.0	1.6(−2)	6.7(−1)	1.03	3.0	6.0(−2)	2.9(−1)	1.23
7.0	1.7(−2)	7.0(−1)	1.03	8.0	8.5(−2)	4.0(−1)	1.24
10.0	1.8(−2)	7.3(−1)	1.03	12.0	9.0(−2)	4.2(−1)	1.25
	(1.0, 0.2, 1.718)				(2.0, 0.4, 6.39)		
t_1/t^*	r	$\bar{\theta}$	τ	t_1/t^*	r	$\bar{\theta}$	τ
2.8	5.8(−3)	4.9(−3)	2.72	7.6	1.1(−2)	3.1(−3)	7.39
3.0	2.1(−2)	1.8(−2)	2.72	7.84	2.3(−2)	6.3(−3)	7.39
3.77	5.6(−2)	4.7(−2)	2.72	8.33	4.5(−2)	1.3(−2)	7.40
5.0	9.1(−2)	7.8(−2)	2.73	9.11	7.6(−2)	2.1(−2)	7.41
6.0	1.1(−1)	9.3(−2)	2.74	10.0	1.1(−1)	3.0(−2)	7.43
12.0	1.5(−1)	1.3(−1)	2.76	20.0	2.6(−1)	6.9(−2)	7.64
27.0	1.8(−1)	1.5(−1)	2.78	80.0	3.8(−1)	1.0(−1)	7.95

The most striking results, however, are relatively independent of the space curvature, and are only associated with the behavior of the velocity and displacement, as fixed by Eqs. (3.1) and (3.6). In every case, the particle proceeds relatively quickly to its maximum displacement and then sits there; notice the very uneven time steps. If we envision a cloud of particles all leaving a common event, they will soon settle down to occupy their maximum allowed solid angle, and the probability is large that they would be seen at this stage. A more definitive study of this point would involve allowing θ to vary. Retardation effects would introduce considerable distortion into the picture. The effects of space curvature would probably be more striking, also.

In Table I, one sees that the red shift $(\tau - 1)$ of cosmic material near S starts out matching Δ_O, as it must, and then varies little. This is due both to the relatively small excursion of S, and to the use of the Milne model, with zero deceleration parameter at all times. It would be interesting to investigate the effect of varying the model, especially the deceleration parameter.

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[16] I refer here to the coordinate velocity, neglecting the cosmological velocity.