PHYSICAL REVIEW VOLUME 186, NUMBER 5 25 OCTOBER 1969

# Quantum Theory of a Randomly Modulated Harmonic Oscillator*

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(Received 23 June 1969)

The time evolution of a randomly modulated quantum harmonic oscillator is studied by introducing the master equation for the reduced density operator $s(t)$. The antinormally ordered representation is adopted for $s(t)$, and one is thus confronted with the problem of solving the partial differential equation obeyed by the reduced matrix element $\bar{s}$ of $s(t)$. The two limiting cases of short- and long-range correlations of the frequency fluctuations are treated in detail.

## I. INTRODUCTION

THE random harmonic oscillator plays a fundamental role in many fields of physics. As a matter of fact, it furnishes a dynamical model the formal features of which are common to a large class of problems. Both for this reason and because of its simplicity, it is used to illustrate the sensitivity of the perturbative approach toward inadequacies in the approximation schemes. In this respect, it resembles certain limiting cases of statistical field theory which are of current interest.

In view of the attention recently paid to the applications of the coherent-state formalism in the frame of statistical physics,[1] it is natural to introduce this picture for investigating the behavior of a randomly modulated quantum harmonic oscillator. In this connection, we remember that the coherent states $|\alpha\rangle$ are defined as the eigenkets of the annihilation operator $a$ with complex eigenvalue $\alpha$. Their usefulness lies mainly in the close analogy they allow one to establish between quantum and classical descriptions. The density operator $\rho$ can be in many cases most conveniently expressed as a superposition of projection operators $|\alpha\rangle\langle\alpha|$ with weight function $P(\alpha,\alpha^*)$, called the $P$ representation.[2] As a consequence of its introduction, the equation of motion obeyed by $\rho$ can be often rewritten in a rather simple form.[1]

In this frame, we remember that the harmonic oscillator may be perturbed by a random fluctuation in its frequency, and the corresponding equation of motion is modified in a manner depending upon the coupling of the oscillator to the perturbing influence. We wish to treat the case in which the initial Hamiltonian $H_0=\frac{1}{2}p^2+\frac{1}{2}\Omega^2q^2$ is modified for the presence of the additive term $f(t)q^2$, $f(t)$ being a centered stationary random function of time. This kind of perturbation may be as well expressed by means of the creation and annihilation operators relative to the unperturbed Hamiltonian $H_0$ as by a term proportional to $f(t)(a+a^\dagger)^2$.

One has to note that the modification of the frequency can also be introduced by means of a perturbative term of the form $f(t)aa^\dagger$. This case has been actually worked out by Glauber[3] as the simplest way to give the oscillating mode of an electromagnetic field a finite bandwidth. The main physical difference between our model and this one lies in the fact that the latter does not give rise to amplitude changes. More precisely, the $P$ representation for an initially coherent state $||\alpha|e^{i\theta}\rangle$ undergoes only a diffusional motion in $\theta$ (see Ref. 3, p. 168). This is no longer true in our case owing to the coupling between different energy levels generated by the terms $a^2$ and $a^{\dagger 2}$.[4]

As a consequence, our model accounts for some of the effects of random amplitude fluctuation as well as phase diffusion.

We treat our problem by writing the master equation for the reduced operator $s(t)$, i.e., the equation of motion for $\rho(t)$ averaged over the ensemble of the realizations of $f(t)$. We look for solutions $s(t)$ in the class of operators which admit an antinormally ordered representation ($P$ representation), thus being able to reduce our problem to the solution of a partial differential equation for an ordinary function $\bar{s}(\alpha,\alpha^*;t)$. We observe that the possibility of expressing the density operator by means of the $P$ representation is always preserved in Glauber's case, while this is not *a priori* true for our kind of

* Supported in part by the Italian National Council of Research.
[1] See, for instance, P. Carruthers and M. M. Nieto, Rev. Mod. Phys. 40, 411 (1968).
[2] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[3] R. J. Glauber, in *Quantum Optics and Electronics, Les Houches, 1964*, edited by C. De Witt *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1965).
[4] The difference between the two cases can be clarified by assuming $f(t)$ to be a well-prescribed function of time. The perturbation $f(t)aa^\dagger$ gives rise only to a variation in the phase of a state $|\alpha\rangle$. Conversely, the perturbation $f(t)(a+a^\dagger)^2$ does not preserve coherence, even if there is a sense in which a state $|\alpha\rangle$ remains coherent when extreme adiabatic conditions are satisfied for the rate of change of $f(t)$. This by no means implies that amplitude variation does not occur. In fact, if the frequency is let to change very slowly from an initial value $\Omega$ to a finite constant value $\omega_f$, the final state is an eigenket $|\alpha_f\rangle$ of the annihilation operator relative to the *final* Hamiltonian $H_f$ with $|\alpha_f|=|\alpha|$ (see Ref. 11). This, in turn, implies that a stochastic perturbation $f(t)(a+a^\dagger)^2$ gives rise to amplitude variation for $P(\alpha)$ unless $f(t)$ is an adiabatic *centered* random process.

perturbation.[5] Anyway, we actually work out its expression, adopting as vector basis the one pertaining to the unperturbed Hamiltonian.

We are able to find $\bar{s}(\alpha,\alpha^*;t)$ in both cases of short- and long-range correlations, i.e., when the coherence time $t_c$ of the perturbation $f(t)$ verifies the extreme relations $t_c \ll \Omega^{-1}$, $t_c \gg \Omega^{-1}$. The assumption that the perturbation is such as to allow one to neglect the variation of $\rho(t)$ over an interval $t_c$ underlies all developments of this paper. The reason for this assumption is that in such a case the evolution of $\rho(t)$ becomes a Markovian process.

The problem of a classical harmonic oscillator perturbed by random fluctuations in its frequency has received some attention as suitable approach to dynamics of nonlinear stochastic systems.[6] In particular, attention has been paid to the case in which the perturbation appears as an additive correction to the square of the oscillator frequency.[7]

In spite of its simplicity, to the best of our knowledge no detailed study of randomly driven quantum oscillator has been presented.[8] However, we remember that examples of harmonic oscillators whose Hamiltonian undergoes a deterministic perturbation have been worked out by many authors. We quote the driven harmonic oscillator whose perturbation has the form $f(t)(a+a^\dagger)$,[9] and the problem of determining the most general Hamiltonian for which an initially coherent state remains coherent.[10] Furthermore, the case in which the frequency undergoes a prescribed temporal variation has recently received some attention in the frame of quantum optics and adiabatic invariant theory.[11]

The main purpose of this paper is to investigate the explicit form of the master equation obeyed by the $P$ representation pertaining to the stochastic perturbation $f(t)(a+a^\dagger)^2$. We emphasize in particular the role played by the correlation time of the perturbation. We shall show that drift and broadening of the oscillation amplitude distribution are present only for short-range correlations.

[5] As an example, the $P$ representation does not exist for an initially coherent state, if $f(t)$ is a prescribed function.

[6] R. H. Kraichnan, J. Math. Phys. **2**, 124 (1961); R. Kubo, *ibid.* **4**, 174 (1963); U. Frisch, in *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Academic Press Inc., New York, 1968).

[7] R. C. Bourret, Can. J. Phys. **43**, 619 (1965).

[8] In principle, one could solve the equation of motion for $\rho$ by resorting to the Kubo's method (see Ref. 6) based on the use of the Liouville operator $L$. However, in our case the statistical properties of $L$ cannot be simply related to the ones pertaining to $f(t)$. On the contrary, Kubo's method could have a direct application to the case of the perturbation $f(t)aa^\dagger$.

[9] W. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Co., New York, 1965).

[10] L. Mišta, Phys. Letters **25A**, 646 (1967); C. L. Mehta, P. Chand, E. C. Sudarshan, and R. Vedam, Phys. Rev. **157**, 1198 (1967).

[11] B. Crosignani, P. Di Porto, and S. Solimeno, Phys. Letters **28A**, 271 (1968); J. Math. Phys. (to be published); H. R. Lewis, Jr., and W. B. Riesenfeld, *ibid.* **10**, 1958 (1969).

## II. EQUATION OF MOTION FOR THE REDUCED DENSITY OPERATOR

Our physical system consists of a harmonic oscillator the frequency of which suffers random fluctuations. We consider the case in which the perturbation appears as an additive correction to the square of the oscillator frequency. Thus, the corresponding Hamiltonian reads

$$H(t)=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2=H_0+f(t)q^2\,, \qquad (1)$$

where $H_0=\frac{1}{2}p^2+\frac{1}{2}\Omega^2q^2$ is the unperturbed Hamiltonian and $2f(t)=\omega^2(t)-\Omega^2$. The equation of motion for the density operator $\rho(t)$ in the Schrödinger picture reads

$$i\hbar(\partial/\partial t)\rho=[H,\rho]. \qquad (2)$$

It is customary to transform Eq. (2) into the interaction picture by resorting to the time translation operator

$$\exp[-i\hbar^{-1}H_0(t-t_0)], \qquad (3)$$

where $t_0$ is the time at which the perturbation is turned on. In such a way, Eq. (2) can be rewritten as

$$i\hbar(\partial/\partial t)\rho_I(t)=f(t)[q_I^2(t),\rho_I(t)], \qquad (4)$$

where

$$\rho_I(t)=e^{i\hbar^{-1}H_0t}\rho(t)e^{-i\hbar^{-1}H_0t}\,,$$
$$q_I^2(t)=e^{i\hbar^{-1}H_0t}q^2e^{-i\hbar^{-1}H_0t}\,, \qquad (5)$$

and we have set $t_0=0$ for notational convenience.

Introducing now the usual annihilation operator $a$ and its adjoint $a^\dagger$ pertaining to the unperturbed oscillator, one gets

$$q_I^2(t)=\tfrac{1}{2}\Omega^{-1}\hbar[a_I^2(t)+a_I^{\dagger 2}(t)+2a_I(t)a_I^\dagger(t)-1]. \qquad (6)$$

Since (see, e.g., Ref. 9, p. 75)

$$a_I(t)\equiv e^{i\hbar^{-1}H_0t}ae^{-i\hbar^{-1}H_0t}=e^{-i\Omega t}a\,, \qquad (7)$$

then

$$q_I^2(t)=\tfrac{1}{2}\Omega^{-1}\hbar(e^{-2i\Omega t}a^2+e^{2i\Omega t}a^{\dagger 2}+2aa^\dagger-1). \qquad (8)$$

Integration of both sides of Eq. (4) over the interval $(0,t)$ followed by an iteration immediately gives

$$\rho_I(t)=\rho(0)-i\hbar^{-1}\int_0^t f(t')[q_I^2(t'),\rho(0)]dt'$$
$$-\hbar^{-2}\int_0^t dt'\int_0^{t'} dt''f(t')f(t'')$$
$$\times[q_I^2(t'),[q_I^2(t''),\rho_I(t'')]], \qquad (9)$$

where use has been made of the relation $\rho_I(0)=\rho(0)$.

We average both sides of Eq. (9) over the ensemble of random functions $f(t)$ (operation hereafter indicated with the symbol $\langle\cdots\rangle$), thus obtaining the following equation for the reduced density operator $s(t)\equiv\langle\rho_I(t)\rangle$:

$$s(t)=\rho(0)-i\hbar^{-1}\int_0^t\langle f(t')\rangle[q_I^2(t'),\rho(0)]dt' -\hbar^{-2}\int_0^t dt'\int_0^{t'}dt''\langle f(t')f(t'') \times[q_I^2(t'),[q_I^2(t''),\rho_I(t'')]]\rangle. \quad (10)$$

If we suppose $f(t)$ to represent a centered stationary process, one has $\langle f(t')\rangle=0$ and $\langle f(t')f(t'')\rangle=g(t'-t'')$.[12] The third term on the right-hand side of Eq. (10) involves the expression $\langle f(t')f(t'')\rho_I(t'')\rangle$, which can be conveniently factorized provided $\rho_I(t)$ does not change appreciably in a time interval $t_c$ [this condition is verified provided the relation $\Omega^{-1}\langle f^2\rangle^{1/2}t_c\ll F(\rho(0))$ is fulfilled, $F$ being a suitable functional of $\rho(0)$ independent on the perturbation]. Indeed, since this assumption amounts to saying that $f(t)$ fluctuates more rapidly than $\rho_I$, we can average $\rho_I$ and $f(t')f(t'')$ separately, thus getting

$$\langle f(t')f(t'')\rho_I(t'')\rangle=g(t'-t'')\langle\rho_I(t'')\rangle. \quad (11)$$

Notice that Eq. (11) does not necessarily imply the weakness of the perturbation $f(t)$.

One then finally obtains, after averaging and differentiating both sides of the resulting equation with respect to $t$,

$$\frac{\partial}{\partial t}s=-\hbar^{-2}\int_0^t dt'g(t-t')[q_I^2(t),[q_I^2(t'),s(t')]], \quad (12)$$

which, in the same order of approximation adopted before, can be written in the Markovian form

$$\frac{\partial}{\partial t}s=-\hbar^{-2}\int_0^t dt'g(t-t')[q_I^2(t),[q_I^2(t'),s(t)]]. \quad (13)$$

Let us now assume $s(t)$ to be expressed by means of a convergent ordered series in $a$ and $a^\dagger$ as

$$s(t)=\sum_{rs}s_{rs}(t)a^ra^{\dagger s}. \quad (14)$$

The possibility of expanding $s(t)$ as in Eq. (14) is equivalent to the existence of the $P$ representation [see Eqs. (20) and (21)]. Therefore, the $P$ representation exists whenever the solution of Eq. (13) can be given in the form of Eq. (14).

Equation (13) is then rewritten, with the aid of Eq. (8), as

$$\frac{\partial}{\partial t}s=-\frac{1}{4\Omega^2}\sum_{rs}\int_0^t dt'g(t-t')s_{rs}(t) \times\{4[aa^\dagger,[aa^\dagger,a^ra^{\dagger s}]]+e^{-2i\Omega(t-t')}[a^2,[a^{\dagger 2},a^ra^{\dagger s}]] +e^{2i\Omega(t-t')}[a^{\dagger 2},[a^2,a^ra^{\dagger s}]]+e^{-2i\Omega(t+t')}[a^2,[a^2,a^ra^{\dagger s}]] +e^{2i\Omega(t+t')}[a^{\dagger 2},[a^{\dagger 2},a^ra^{\dagger s}]]+2e^{-2i\Omega t'}[aa^\dagger,[a^2,a^ra^{\dagger s}]] +2e^{2i\Omega t'}[aa^\dagger,[a^{\dagger 2},a^ra^{\dagger s}]]+2e^{-2i\Omega t}[a^2,[aa^\dagger,a^ra^{\dagger s}]] +2e^{2i\Omega t}[a^{\dagger 2},[aa^\dagger,a^ra^{\dagger s}]]\}. \quad (15)$$

The commutators appearing in Eq. (15) can be evaluated in terms of the expression $[a^m,a^{\dagger n}]$, which in turn is expressed via some algebra as

$$[a^m,a^{\dagger n}]=\sum_{K=1}^{q}(-1)^{K+1}K!\binom{m}{K}\binom{n}{K}a^{m-K}a^{\dagger n-K}, \quad (16)$$

where $q=\min(m,n)$. Therefore, Eq. (15) can be rewritten as

$$(\partial/\partial t)s(t)=Ms(t), \quad (17)$$

$M$ playing the role of a generalized master operator. It thus appears that the evolution of the system can be described within the framework of the master equation. Even if this definition is arbitrary when referred to Eq. (17), where a quasiprobability distribution is dealt with, nevertheless it is adopted in order to emphasize that all probability assumptions are contained in the existence and character of the master operator.[13]

## III. SOLUTION OF MASTER EQUATION FOR SOME TYPICAL CASES

Even if Eq. (15) is in general rather involved, it can be suitably simplified in particular cases, according to whether the correlation time $t_c$ of $g(t)$ satisfies the relations $t_c\gg\Omega^{-1}$ or $t_c\ll\Omega^{-1}$. In particular, when $t_c\ll\Omega^{-1}$, two limiting subcases are possible. The first one, which is discussed in Sec. III B, is characterized by a small variation of $s(t)$ during a time interval $\Omega^{-1}$; in the second one, discussed in Sec. III C, $s(t)$ undergoes a substantial variation in a time which is very short as compared to $\Omega^{-1}$. While the former is representative of a weak frequency perturbation, the latter is related to a strong perturbation. In both subcases, however, it is correct to assimilate $g(t'-t'')$ to a $\delta$ function.

### A. Long-Range Frequency Correlation

As already remarked, we limit ourselves to consider times such that $t>t_c$. This relation, together with the

[12] If $t_c$ is a characteristic correlation time of $g(t)$, stationariness can hold only for $t'>t_c$, $t''>t_c$. In effect, as we shall see, we confine ourselves to such a case.

[13] See, for example, M. Dresden, in *Studies in Statistical Mechanics*, edited by J. de Boer and G. E. Uhlenbeck (North-Holland Publishing Co., Amsterdam, 1962), Vol. I.

disequality $t_c \gg \Omega^{-1}$, implies that

$$\int_0^t g(t-t')e^{\pm 2i\Omega t'}dt' \approx 0\,, \tag{18}$$

so that we can neglect in Eq. (15) the six terms containing the above factor. The terms containing $e^{\pm 2i\Omega t}$ can also be dropped in the rotating-wave approximation,[14] so that Eq. (15) reduces to

$$\frac{\partial}{\partial t}s(t) = -D\sum_{rs} s_{rs}(t)[aa^\dagger,[aa^\dagger,a^r a^{\dagger s}]]\,, \tag{19}$$

where $D=\Omega^{-2}\int_0^t g(t-t')dt'$ is with good approximatino time-independent because of the relation $t>t_c$.

We recall at this point that an operator admitting a power-series expansion of the type given in Eq. (14) is expressed in the $P$ representation as (see Ref. 15)

$$s(t)=\pi^{-1}\int \bar{s}(\alpha,\alpha^*;t)\,|\alpha\rangle\langle\alpha|\,d^2\alpha\,, \tag{20}$$

with

$$\bar{s}(\alpha,\alpha^*;t)=\sum_{rs} s_{rs}(t)\alpha^r\alpha^{*s}\,. \tag{21}$$

Following now a general method (see, e.g., Ref. 1, p. 436 and Ref. 15), the master equation is expressed in the $P$ representation, thus getting a partial-differential equation for the reduced matrix element $\bar{s}(\alpha,\alpha^*;t)$. Taking into account the relation

$$[aa^\dagger,[aa^\dagger,a^r a^{\dagger s}]]=(r-s)^2 a^r a^{\dagger s}\,, \tag{22}$$

we can derive from Eq. (19) the corresponding equation for $\bar{s}$, which reads[16]

$$\frac{\partial}{\partial t}\bar{s}(\alpha,\alpha^*;t) = -D\left(\alpha\frac{\partial}{\partial\alpha}-\alpha^*\frac{\partial}{\partial\alpha^*}\right)^2\bar{s}(\alpha,\alpha^*;t)\,, \tag{23}$$

or, equivalently,

$$\frac{\partial}{\partial t}\bar{s}(\alpha,\alpha^*;t) = D\frac{\partial}{\partial\theta^2}\bar{s}(\alpha,\alpha^*;t)\,, \tag{24}$$

where the change of variables $\alpha=\rho e^{i\theta}$, $\alpha^*=\rho e^{-i\theta}$ has been performed. Equation (24) is the partial-differential equation for the diffusion of heat on a circular ring, whose Green's function reads

$$G(\theta,t\,|\,\theta_0,0)=\pi^{-1}\sum_m e^{-m^2Dt}\cos m(\theta-\theta_0)\,. \tag{25}$$

We wish to note that Eq. (24) has been obtained in the case in which the perturbation Hamiltonian reduces to $f(t)aa^\dagger$. This is the same Hatmilonian considered by Glauber (see Ref. 3, p. 168), who finds the same diffusionlike equation. However, it is worth stressing that the assumptions of that author are quite different. In effect, he assumes a correlation time $t_c=0$, while we let it to be finite and greater than $\Omega^{-1}$. Our problem differs essentially from Glauber's in the sense that the frequency fluctuation gives rise to a perturbation $f(t)q^2$, which reduces to $f(t)aa^\dagger$ only if $t_c \gg \Omega^{-1}$.

### B. Short-Range Frequency Correlation; Weak Perturbation

In the case $t_c \ll \Omega^{-1}$, we are permitted to put $t=t'$ in the exponential factors appearing in Eq. (15), and then drop the terms with the factors $e^{\pm 2i\Omega t}$, $e^{\pm 4i\Omega t}$ in the rotating-wave approximation. The master equation easily reduces to

$$\frac{\partial}{\partial t}s(t) = -D\sum_{rs} s_{rs}(t)\{[aa^\dagger,[aa^\dagger,a^r a^{\dagger s}]] + 4^{-1}[a^{\dagger 2},[a^2,a^r a^{\dagger s}]] + 4^{-1}[a^2,[a^{\dagger 2},a^r a^{\dagger s}]]\}\,. \tag{26}$$

Taking into account Eq. (22), together with the relations

$$[a^{\dagger 2},[a^2,a^r a^{\dagger s}]] = -4s(r+1)a^r a^{\dagger s} + 2rs(r+s)a^{r-1}a^{\dagger s-1} - rs(r-1)(s-1)a^{r-2}a^{\dagger s-2}\,, \tag{27a}$$

$$[a^2,[a^{\dagger 2},a^r a^{\dagger s}]] = -4r(s+1)a^r a^{\dagger s} + 2rs(r+s)a^{r-1}a^{\dagger s-1} - rs(r-1)(s-1)a^{r-2}a^{\dagger s-2}\,, \tag{27b}$$

and following the procedure previously adopted, Eq. (26) yields

$$(\partial/\partial t)\bar{s} = -DL\bar{s}\,, \tag{28}$$

where the master operator $L$ is given by

$$L=\alpha^2\frac{\partial^2}{\partial\alpha^2}+\alpha^{*2}\frac{\partial^2}{\partial\alpha^{*2}}-4\alpha\alpha^*\frac{\partial^2}{\partial\alpha\partial\alpha^*} +2\frac{\partial^2}{\partial\alpha\partial\alpha^*}+\alpha\frac{\partial^3}{\partial\alpha^2\partial\alpha^*}+\alpha^*\frac{\partial^3}{\partial\alpha\partial\alpha^{*2}}-\frac{1}{2}\frac{\partial^4}{\partial\alpha^2\partial\alpha^{*2}}\,, \tag{29}$$

or, in polar coordinates,

$$4L=4\rho^2\frac{\partial^2}{\partial\rho^2}+\left(-6\rho^2+\rho\frac{\partial}{\partial\rho}+2\right)\nabla^2-8^{-1}\nabla^2\nabla^2\,. \tag{30}$$

While the study of Eq. (28) is in general a formidable task, it suitably simplifies if we look for angle-independent solutions (initial random-phase assump-

[14] See, e.g., M. O. Scully and W. E. Lamb, Jr., Phys. Rev. 159, 208 (1967).
[15] W. Louisell, in Proceedings of the International School of Physics "Enrico Fermi," Course 42, 1967 (unpublished).
[16] It is hereafter assumed in performing these calculations that $\alpha^*$ is independent of $\alpha$.

tion). In this case the differential operator $L$ reduces to

$$L=-32^{-1}\frac{\partial^4}{\partial\rho^4}+4^{-1}(\rho-4^{-1}\rho^{-1})\frac{\partial^3}{\partial\rho^3}+4^{-1}(-2\rho^2+3+8^{-1}\rho^{-2})\frac{\partial^2}{\partial\rho^2}+4^{-1}(-6\rho+\rho^{-1}-8^{-1}\rho^{-3})\frac{\partial}{\partial\rho}\equiv L_\rho. \quad (31)$$

A simple calculation shows that the operator $L_\rho$ maintains the normalization of $\bar{s}$, thus preserving its role of quasiprobability density. In this respect, it is useful to recall that the hypothesis of $\bar{s}$ vanishing rapidly enough together with its derivatives (in practice up to third order) as $\rho\rightarrow\infty$ or $\rho\rightarrow 0$, underlies this and other following results.

Explicit solutions of Eq. (28) for $L=L_\rho$ can be given in the extreme case $\rho\rightarrow\infty$, that is, when the initial condition is such that $\bar{s}$ is vanishing when $|\alpha|<c$, $c$ being a quantity of the order of magnitude of unity. This, in turn, implies the same condition for the mean number of energy quanta associated with the oscillator (see Ref. 2, p. 2769). The interest of this case arises from the fact that the quantum-mechanical description in terms of coherent states tends asymptotically to the classical one as $|\alpha|\rightarrow\infty$.

Anyway, we can obtain interesting information directly from Eq. (31), by confining ourselves to studying the evolution of some mean quantities. Indeed, it is a matter of simple algebra to obtain the following mean equations of motion for $\langle\rho\rangle$ and $\langle\rho^2\rangle$:

$$(d/dt)\langle\rho\rangle=D[3.2^{-1}\langle\rho\rangle+4^{-1}\langle\rho^{-1}\rangle+32^{-1}\langle\rho^{-3}\rangle], \quad (32)$$

$$(d/dt)\langle\rho^2\rangle=D[4\langle\rho^2\rangle+2], \quad (33)$$

where

$$\langle\rho\rangle=2\pi\int_0^\infty \rho^2\bar{s}d\rho$$

and

$$\langle\rho^2\rangle=2\pi\int_0^\infty \rho^3\bar{s}d\rho.$$

According to these equations, the mean amplitude of the oscillation and the relative variance undergo an amplification because of the frequency perturbation.

A further insight into the behavior of $\bar{s}$ can be gained by specializing Eq. (31) to the case in which $\rho\rightarrow\infty$:

$$L_\rho\underset{\rho\rightarrow\infty}{\longrightarrow}-\frac{1}{32}\frac{\partial^4}{\partial\rho^4}-\frac{1}{4}\rho\frac{\partial^3}{\partial\rho^3}-\frac{1}{2}\rho^2\frac{\partial^2}{\partial\rho^2}-\frac{3}{2}\rho\frac{\partial}{\partial\rho}\equiv L_\infty. \quad (34)$$

By performing the change of variable $\rho=e^\xi$, and saving, as above, only the dominant terms, we obtain

$$L_\infty=-\frac{1}{32}\frac{1}{\rho^4}\frac{\partial^4}{\partial\xi^4}+\frac{1}{4\rho^2}\frac{\partial^3}{\partial\xi^3}-\frac{1}{2}\frac{\partial^2}{\partial\xi^2}-\frac{\partial}{\partial\xi}. \quad (35)$$

The operator on the right-hand side of Eq. (35) is in a suitable form to be simplified if all the derivatives appearing there are assumed to be bounded. In this case, we get

$$L_\infty\underset{\rho\rightarrow\infty}{\longrightarrow}-\frac{1}{2}\frac{\partial^2}{\partial\xi^2}-\frac{\partial}{\partial\xi}, \quad (36)$$

so that Eq. (28) reduces to

$$(\partial/\partial t)\bar{s}=D(\tfrac{1}{2}\partial^2/\partial\xi^2+\partial/\partial\xi)\bar{s}. \quad (37)$$

This is a Fokker-Planck equation,[17] with negative drift coefficient $-D$, the Green's function of which reads

$$G(\xi,t|\xi_0,0)=(2\pi Dt)^{-1/2}\exp[-(\xi-\xi_0+Dt)^2/2Dt]. \quad (38)$$

### C. Short-Range Frequency Correlation; Strong Perturbation

The relations $t_c\ll\Omega^{-1}$ and $t\Omega\ll1$ allow us to set the exponentials appearing in Eq. (15) equal to unity, thus getting

$$(\partial/\partial t)s=-\tfrac{1}{4}D[a^2+a^{\dagger2}+2aa^\dagger,[a^2+a^{\dagger2}+2aa^\dagger,a^ra^{\dagger s}]]. \quad (39)$$

These commutators can be evaluated via tedious algebra and the resulting equation for $\bar{s}$ reads

$$\frac{\partial}{\partial t}\bar{s}(\alpha,\alpha^*;t)=-\frac{1}{4}D\left[\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\alpha^{*2}}\right)^2-4(\alpha+\alpha^*)\left(\frac{\partial^3}{\partial\alpha^3}-\frac{\partial^3}{\partial\alpha^2\partial\alpha^*}-\frac{\partial^3}{\partial\alpha\partial\alpha^{*2}}+\frac{\partial^3}{\partial\alpha^{*3}}\right)+4[(\alpha+\alpha^*)^2-1]\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\alpha^*}\right)^2\right]\bar{s}(\alpha,\alpha^*;t), \quad (40)$$

or, performing the change of variables $\alpha=x+iy$, $\alpha^*=x-iy$,

$$\frac{\partial}{\partial t}\bar{s}(\alpha,\alpha^*;t)=\frac{1}{4}D\frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial x}-4x\right)^2\bar{s}(\alpha,\alpha^*;t); \quad (41)$$

solution $\bar{s}$ can be also rewritten as $e^{2x^2}H(x,y;t)$, $H$ being the solution of the equation

$$\frac{\partial}{\partial t}H(x,y;t)=\frac{1}{4}\frac{\partial^4}{\partial x^2\partial y^2}H(x,y;t). \quad (42)$$

As before, some kind of information about the evolution of $\bar{s}$ can be gained through the mean equations of motion of $\langle x^m\rangle=\int x^m\bar{s}dx\,dy$ and $\langle y^m\rangle$:

$$(d/dt)\langle x^m\rangle=0, \quad (43)$$

$$(d/dt)\langle y^m\rangle=Dm(m-1)(4\langle x^2y^{m-2}\rangle+\langle y^{m-2}\rangle). \quad (44)$$

[17] See, e.g., W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, Inc., New York, 1967), Vol. I, p. 358.

Under some suitable hypotheses the general solution of Eq. (42) can be given in terms of Green's functions. More precisely, one has to assume that there exists the inverse Laplace transform of the analytic continuation of $H(x,y;t)$, with respect either to $x$ or $y$. If this holds true, we can assume $H(x,y;t)$ in the form[18]

$$H(x,y;t)=\int_{-\infty}^{+\infty}d\mu\int_{0}^{+\infty}d\nu\,\tilde{H}_1(\mu,\nu;t)e^{i\mu x-\nu y} \quad (45a)$$

[where $x\in(-\infty,+\infty)$ and $y\in(a,+\infty)$], or in the form

$$H(x,y;t)=\int_{0}^{+\infty}d\mu\int_{-\infty}^{+\infty}d\nu\,\tilde{H}_2(\mu,\nu;t)e^{-\mu x+i\nu y}, \quad (45b)$$

where $x\in(a,+\infty)$, $y\in(-\infty,+\infty)$ [and the corresponding relations for $x$ or $y\in(-\infty,a)$]. Substituting Eqs. (45a) or (45b) into Eq. (42), we have

$$\tilde{H}_{1,2}(\mu,\nu;t)=\tilde{H}_{1,2}(\mu,\nu;0)\exp(-\tfrac{1}{4}D\mu^2\nu^2t), \quad (46)$$

so that, through standard algebra, Eqs. (45a) and (45b) reduce to

$$H(x,y;t)=\int_{-\infty}^{+\infty}dx'\int_{\tau_0-i\infty}^{\tau_0+i\infty}dy'\,H(x',y';0)\times G_1(x,y;t|x',y';0),\quad y>\mathrm{Re}y' \quad (47a)$$

or

$$H(x,y;t)=\int_{\tau_0-i\infty}^{\tau_0+i\infty}dx'\int_{-\infty}^{+\infty}dy'\,H(x',y';0)\times G_2(x,y;t|x',y';0),\quad x>\mathrm{Re}x', \quad (47b)$$

where $\tau_0$ is the abscissa of convergence relative to the analytic continuation of $H(x,y;0)$ with respect to $x$ or $y$, and

$$G_1(x,y;t|x',y';0)=(2i\pi^{3/2}D^{1/2}t^{1/2})^{-1}\times\int_0^\infty \exp[-(x-x')^2/\nu^2Dt-\nu(y-y')]\nu^{-1}d\nu, \quad (48a)$$

$$G_2(x,y;t|x',y';0)=(2i\pi^{3/2}D^{1/2}t^{1/2})^{-1}\times\int_0^\infty \exp[-(y-y')^2/\mu^2Dt-\mu(x-x')]\mu^{-1}d\mu. \quad (48b)$$

[18] The following developments are due to J. Peřina and V. Peřinová (private communication).

## ACKNOWLEDGMENTS

The authors are indebted to J. Peřina for useful suggestions and to Carlo Crosignani for editorial assistance. Stimulating discussions with Professor D. Sette, Professor G. Franceschetti, and Professor M. Bertolotti were most helpful in the completion of this work.

PHYSICAL REVIEW VOLUME 186, NUMBER 5 25 OCTOBER 1969

# Geodesics of Robertson-Walker Universes*

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(Received 12 June 1969)

The complete set of geodesics is obtained for Robertson-Walker universes with arbitrary $R(t)$, from those geodesics whose spatial projections pass through the origin, by inducing a translation of origin through the rotation of a hypersphere whose stereographic projections form the space sections. The results are applied to the calculation, in a Milne universe, of the angular displacement of a particle initially projected at high velocity accross the line of sight. This displacement proves to be bounded, the upper bound being attained reasonably fast, on a cosmic time scale.

## I. INTRODUCTION

With their present random velocities of $\sim$100 km/sec, galaxies cannot traverse a significant portion of the Universe during its evolution. There are reasons, however, for wanting to know the various trajectories of free particles with large peculiar velocity (i.e., not just the "mean fluid velocity") over a very long period. More and more extreme examples are being found[1] of galaxies with peculiar velocities of the order of thousands of km/sec. Furthermore, it is expected that in the distant past, all random velocities of free objects were much larger, roughly in proportion[2] to $R^{-1}$, where $R$ is the curvature radius of a space section of a Robert-

* Work supported in part by the National Science Foundation, under Grant No. GU 2921.

[1] E. M. Burbidge and G. R. Burbidge, Astrophys. J. 134, 244 (1961); P. W. Hodge, *ibid.* 134, 262; W. L. W. Sargent, *ibid.* 153, L135 (1968); and various papers in the Santa Barbara Conference, Astron. J. 66, No. 10 (1961).

[2] G. B. Van Albada, Astron. J. 66, 590 (1961). If there is drag due to intergalactic matter, the velocities would have been much bigger than given by the $1/R$ law; however, at the relativistic end, the opposite holds. Sturrock has also suggested that *QSO* may be in relativistic motion. See P. A. Sturrock, in *Plasma Astrophysics* (Academic Press Inc., New York, 1967), p. 338 and especially p. 361.