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Propagation and Quantization of Rarita-Schwinger Waves in an External Electromagnetic Potential†

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The Rarita-Schwinger equation in an external electromagnetic potential is shown to be equivalent to a hyperbolic system of partial differential equations supplemented by initial conditions. The wave fronts of the classical solutions are calculated and are found to propagate faster than light. Nevertheless, for sufficiently weak external potentials, a consistent quantum mechanics and quantum field theory may be established. These, however, violate the postulates of special relativity.

1. INTRODUCTION

THE problem of finding a suitable wave equation for electrically charged higher-spin particles has been with us for a long time. Since the pioneering work of Fierz and Pauli,[1] the commonly accepted method, which avoids algebraic inconsistencies, is to find Lagrangian equations of motion whose solutions correspond to free particles of unique mass and spin, and then to account for electromagnetic coupling by substituting

$$i\partial_\mu \to i\partial_\mu + eA_\mu$$

into the free Lagrangian. A familiar example of this method is the Rarita-Schwinger (RS) Lagrangian[2] for spin-$\frac{3}{2}$ particles, which does indeed avoid immediate algebraic inconsistencies.[3] However, a more subtle type of inconsistency appears when the RS field with an external potential is quantized.[4]

In the present article, we show that the difficulty is already present in the RS equation interpreted as a classical field equation, because the solutions propagate at velocities exceeding the speed of light for arbitrarily weak external fields. More precisely, we show that the RS equation is equivalent to a system of hyperbolic partial differential equations, supplemented by initial conditions. Elementary methods then allow one to determine the wave fronts and ray velocities of the solutions to the hyperbolic system. One finds that the propagation of RS wave fronts in an external potential resembles the propagation of light in an anisotropic crystal. There is an ordinary ray which travels at the speed of light and an extraordinary ray which always travels in some direction at a speed exceeding that of light. The violation of causality occurs even though the free retarded propagator is causal, so that every finite order of perturbation theory is causal.

Nevertheless, we find that in the weak-field case [see Eq. (2.16)] a positive definite conserved inner product exists in some, but not all, Lorentz frames. In these frames a consistent quantum mechanics and quantum field theory may be formulated. The equal-time anticommutator is not local in other frames.

The main lesson to be drawn from our analysis is that special relativity is not automatically satisfied by writing equations which transform covariantly. In addition, the solutions must not propagate faster than light. There are simple algebraic criteria on the coefficients appearing in the partial differential equations which determine the velocity of propagation of the signals. These criteria must be applied to other higher-spin equations which describe interactions, as we have done here for the RS equation. In this direction, we have verified that the first-order Duffin-Kemmer-Petiau formalism for spin 0 and 1 and the Wentzel formalism[5] for spin 1 are causal, even though constraints are present.

2. EQUATION OF MOTION AND ITS WAVE FRONTS

We begin with the RS Lagrangian density

$$\mathfrak{L} = \bar{\psi}(\Gamma\cdot\pi - B)\psi. \tag{2.1}$$

Here ψ is the RS vector-spinor ψ_μ, with $\bar{\psi}^\mu = \psi^{\mu\dagger}\gamma^0$; and[6]

$$\pi_\mu = i\frac{\partial}{\partial x^\mu} + eA_\mu, \tag{2.2}$$

with A_μ a given classical four-vector potential. The matrices Γ^μ and B are given by

$$\begin{aligned}(\Gamma\cdot\pi)_\kappa{}^\lambda &= \gamma^5\epsilon_{\kappa\nu}{}^{\mu\lambda}\gamma^\nu\pi_\mu \\ &= g_\kappa{}^\lambda\gamma\cdot\pi - (\gamma_\kappa\pi^\lambda + \pi_\kappa\gamma^\lambda) + \gamma_\kappa\gamma\cdot\pi\gamma^\lambda,\end{aligned} \tag{2.3}$$

$$B_\kappa{}^\lambda = -m\sigma_\kappa{}^\lambda = m(g_\kappa{}^\lambda - \gamma_\kappa\gamma^\lambda). \tag{2.4}$$

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[1] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[2] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

[3] For an alternative spin-$\frac{3}{2}$ wave equation see G. S. Guralnik and T. W. B. Kibble, Phys. Rev. **139B**, 712 (1965); **150**, 1406(E) (1966); S. N. Gupta and W. Repko, *ibid.* **165**, 1415 (1967). Other references may be found here.

[4] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **13**, 126 (1961).

[5] G. Wentzel, *Quantum Theory of Fields* (Wiley-Interscience, Inc., New York, 1969), p. 90.

[6] Our conventions are $\hbar = c = 1$, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $a = a_\mu\gamma^\mu$, and $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, so $(\gamma^5)^2 = -1$, $\epsilon^{0123} = 1$, $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, $\beta = \gamma^0$, and $\boldsymbol{\alpha} = (\gamma^0\gamma^i)$ for $i = 1, 2, 3$.

This Lagrangian is one of a class of possible Lagrangians for spin $\frac{3}{2}$ which differ by the substitution $\psi_\mu \to \psi_\mu + a\gamma_\mu\gamma\cdot\psi$. Variation of the Lagrangian with respect to the 16 components of ψ and $\bar\psi$ independently yields the equations of motion

$$(\Gamma\cdot\pi - B)_\kappa{}^\lambda\psi_\lambda = 0\,,$$
$$\bar\psi^\kappa(\Gamma\cdot\pi - B)_\kappa{}^\lambda = 0\,. \tag{2.5}$$

Let us analyze Eq. (2.5). Because ψ has more components than needed to describe a spin-$\frac{3}{2}$ particle, some of the 16 equations (2.5) will turn out to be constraint equations in the sense that they do not involve time derivatives. In fact, from the form (2.3), we see that when $\kappa=0$, Eq. (2.5) contains no time derivative, but yields, instead, the primary constraint equation

$$(\boldsymbol{\pi} - h\boldsymbol{\alpha})\cdot\boldsymbol{\psi} = 0\,, \tag{2.6}$$

where[6] $\boldsymbol{\pi} = (\pi^i)$, $\boldsymbol{\psi} = (\psi^i)$, $i=1, 2, 3$, and

$$h = \boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta m\,. \tag{2.7}$$

Moreover, from the form (2.3) we see that $(\partial/\partial t)\psi^0$ never appears at all in Eq. (2.5); nor is ψ^0 determined by the primary constraint (2.6), which only involves ψ^1, ψ^2, and ψ^3. To obtain an equation for ψ^0, one must differentiate Eq. (2.5). This may be done covariantly by multiplying Eq. (2.5) successively by γ^κ and π^κ, which yields, respectively,

$$2(\gamma\cdot\pi\gamma - \pi)\cdot\psi + 3m\gamma\cdot\psi = 0\,, \tag{2.8}$$

$$m(\gamma\cdot\pi\gamma - \pi)\cdot\psi - ie\gamma^5\gamma\cdot F^d\cdot\psi = 0\,, \tag{2.9}$$

where

$$F = F_\mu{}^\nu = \partial_\mu A^\nu - \partial^\nu A_\mu\,,$$
$$F^d = F^d{}_\mu{}^\nu = \tfrac{1}{2}\epsilon_\mu{}^{\nu\kappa}{}_\lambda F_\kappa{}^\lambda\,.$$

Comparing Eqs. (2.8) and (2.9), one finds the covariant secondary constraint

$$\gamma\cdot\psi = -\tfrac{2}{3}m^{-2}ie\gamma^5\gamma\cdot F^d\cdot\psi\,, \tag{2.10}$$

which determines ψ^0. Another useful relation is

$$\pi\cdot\psi = -(\gamma\cdot\pi + \tfrac{3}{2}m)\tfrac{2}{3}iem^{-2}\gamma^5\gamma\cdot F^d\cdot\psi\,, \tag{2.11}$$

which follows upon inserting (2.10) into (2.9).

After these preliminaries, we proceed to the main subject of this section, which is to determine under which conditions a solution to the RS equation exists and to find the velocity of propagation of signals. For this purpose we substitute Eqs. (2.10) and (2.11) for $\gamma\cdot\psi$ and $\pi\cdot\psi$ back into the original RS equation. The resulting equation,

$$(\gamma\cdot\pi - m)\psi_\mu + (\pi_\mu + \tfrac{1}{2}m\gamma_\mu)\tfrac{2}{3}iem^{-2}\gamma^5\gamma\cdot F^d\cdot\psi = 0\,, \tag{2.12}$$

can be put into Hermitian form by again using Eqs. (2.10) and (2.11):

$$\begin{aligned}(\gamma\cdot\pi - m)\psi_\mu &+ (\pi_\mu + \tfrac{1}{2}m\gamma_\mu)\tfrac{2}{3}iem^{-2}\gamma^5\gamma\cdot F^d\cdot\psi \\ &+ \tfrac{2}{3}iem^{-2}F_\mu{}^d\cdot\gamma\gamma^5(\pi + \tfrac{1}{2}m\gamma)\cdot\psi + \tfrac{2}{3}iem^{-2} \\ &\times F_\mu{}^d\cdot\gamma\gamma^5(\gamma\cdot\pi + 2m)\tfrac{2}{3}iem^{-2}\gamma^5\gamma\cdot F^d\cdot\psi = 0\,.\end{aligned} \tag{2.13}$$

So far we have established that every solution of the RS equation satisfies the primary and secondary constraints (2.6) and (2.10) and the new equation of motion (2.12) or (2.13). Conversely, as shown in Appendix A: (a) Equation (2.13) preserves the constraints (2.6) and (2.10) [i.e., every solution of (2.13) which satisfies the constraints (2.6) and (2.10) at a given time satisfies them for all time], and (b) every solution of Eq. (2.13) which satisfies the constraints (2.6) and (2.10) at a given time is a solution of the original RS equation. Thus, Eq. (2.13) contains less information than the original RS equation because it does not imply the constraints. However, it is a true equation of motion because it specifies the time derivative of ψ for any given ψ.

We will now show that, for sufficiently weak fields, Eq. (2.13) is a hyperbolic system of partial differential equations. For this purpose it is sufficient to compute the normals n_μ to the characteristic surfaces,[7] which, for a linear system of the form

$$\left[(\Gamma^\mu)_\kappa{}^\lambda\frac{\partial}{\partial x^\mu} - B_\kappa{}^\lambda\right]\psi_\lambda = 0\,,$$

are determined by[8]

$$D(n) = |(\Gamma^\mu)_{\kappa\lambda}n_\mu| = 0\,. \tag{2.14}$$

This determinant is a polynomial in the components of n_μ, so it is sufficient to evaluate it for n_μ in the future cone, and by Lorentz invariance we may take $n_\mu = (n,0,0,0)$. Taking the coefficient of $\partial/\partial t$ in Eq. (2.13) one has after slight rearrangement

$$\begin{aligned}D(n) = |(g_\mu{}^\rho\gamma^0 + \tfrac{2}{3}iem^{-2}F_\mu{}^d\cdot\gamma\gamma^5 g_0{}^\rho) \\ \times\gamma^0(g_\rho{}^\nu\gamma^0 + \tfrac{2}{3}iem^{-2}g_\rho{}^0\gamma^5\gamma\cdot F^{d\nu})n| \\ = n^{16}[1 - (\tfrac{2}{3}em^{-2})^2\mathbf{B}^2]^4\end{aligned}$$

or, in covariant form,

$$D(n) = (n^2)^4[n^2 + (\tfrac{2}{3}em^{-2})^2(F^d\cdot n)^2]^4 = 0\,. \tag{2.15}$$

This equation determines the normals to the characteristic surfaces passing through each point.

Before analyzing Eq. (2.15), it is convenient to introduce the term "weak-field case" to refer to the situation in which there exists, for each space-time point, a Lorentz frame such that the inequality

$$(\tfrac{2}{3}em^{-2})^2\mathbf{B}^2 < 1 \tag{2.16}$$

is satisfied, and otherwise we refer to the "strong-field case." To avoid inessential complications we will suppose in the weak-field case that there exists a single

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley-Interscience, Inc., New York, 1962), Vol. 2, pp. 590, 596.

[8] If this criterion were applied directly to the original RS equation, we would find that every surface is a characteristic surface, corresponding to the fact, which we know already, that there are constraints.

common Lorentz frame such that (2.16) holds at every space-time point. It is easy to verify in the weak-field case that Eq. (2.15) has eight positive and eight negative roots n^0, for any given $\mathbf{n}=(n^i)$. This establishes hyperbolicity of Eq. (2.13) in the weak-field case, and allows the definition of "spacelike surfaces" and "future and past cones" with respect to Eq. (2.13).[7] These differ, however, from the familiar spacelike surfaces and light cones of special relativity. In the strong-field case, Eq. (2.13) ceases to be hyperbolic and is not suitable for the description of wave phenomena. Hence, we restrict our considerations to the weak-field case.

We remind the reader that, for hyperbolic equations, the maximum velocity of propagation of signals is the slope of the characteristic surfaces. Significantly, the characteristic surfaces determined by Eq. (2.15) are not all tangent to the light cone and, catastrophically, spacelike characteristic surfaces pass through every point where $F_{\mu\nu}$ is nonvanishing. Consequently, signals are propagated at velocities greater than the speed of light. To see this, we show that there are timelike normals n_μ satisfying (2.15). In fact, choosing $n_\mu=(1,0,0,0)$, the second factor of Eq. (2.15) becomes

$$1-(\tfrac{2}{3}em^{-2})^2\mathbf{B}^2=0\,, \tag{2.17}$$

and whenever $F_{\mu\nu}\neq0$, there exists a Lorentz frame where (2.17) holds.

We remark parenthetically that the propagation of RS waves, according to (2.15), resembles light propagation in an anisotropic medium. There are ordinary rays corresponding to the first factor of Eq. (2.15), and extraordinary rays with wave-front velocity exceeding c corresponding to the second factor. It might be hoped that the constraints (2.6) and (2.10) eliminate the extraordinary ray, but we prove the opposite in Appendix B, so it is true that wave fronts do indeed travel at the extraordinary-ray velocity which exceeds c.

As a final remark, we observe that the hyperbolic system (2.13) possesses "spacelike surfaces" Σ on which initial values $\psi(\Sigma)$ may be specified arbitrarily. Although these spacelike surfaces differ from those of special relativity, nevertheless, in Lorentz frames where Eq. (2.16) holds everywhere, the surfaces $t=t_0$ are "spacelike" and the Cauchy initial-value problem may be posed for any $\psi(t_0)$. We will see in the next section that, in such frames, a conventional quantum-mechanical interpretation of the RS equation can be formulated.

3. CONSERVED POSITIVE DEFINITE INNER PRODUCT

An important feature of the RS equation is the existence of a conserved current or bilinear form. Let ψ_1 and ψ_2 be solutions of (2.5); then the four-vector density

$$\mathcal{J}_\mu(x)=\bar{\psi}_1(x)\Gamma_\mu\psi_2(x)=\bar{\psi}_1{}^\kappa(x)(\Gamma_\mu)_\kappa{}^\lambda\psi_{2\lambda}(x) \tag{3.1}$$

satisfies the equation of continuity

$$\frac{\partial}{\partial x^\mu}\mathcal{J}^\mu(x)=0\,. \tag{3.2}$$

Hence, for solutions ψ that vanish sufficiently rapidly at infinity, one may define the scalar product

$$\langle\psi_1|\psi_2\rangle=\int_\Sigma\bar{\psi}_1\Gamma_\mu\psi_2 d\sigma^\mu\,, \tag{3.3}$$

which is conserved and independent of the surface of integration Σ. For a surface $t=\text{const}$, the inner product becomes[6]

$$\langle\psi_1|\psi_2\rangle=\int\psi_1{}^\dagger\gamma^0\Gamma_0\psi_2 d\mathbf{x}=\int\psi_1{}^\dagger A^0\psi_2 d\mathbf{x}$$
$$=\int[\boldsymbol{\psi}_1{}^\dagger\cdot\boldsymbol{\psi}_2-(\boldsymbol{\alpha}\cdot\boldsymbol{\psi}_1)^\dagger\cdot(\boldsymbol{\alpha}\cdot\boldsymbol{\psi}_2)]d\mathbf{x}, \tag{3.4}$$

where $\psi^\mu=(\psi^0,\boldsymbol{\psi})$ and $A^0=\gamma^0\Gamma_0$. Since the zeroth component of ψ^μ does not appear in the inner product, we retain only the space components and use nonrelativistic notation from now on. Of course, ψ^0 may always be retrieved from the constraint (2.10) when the weak-field assumption (2.16) holds:

$$(1+\tfrac{2}{3}em^{-2}\boldsymbol{\sigma}\cdot\mathbf{B})\psi^0=\boldsymbol{\alpha}\cdot\boldsymbol{\psi}+\tfrac{2}{3}em^{-2}(i\gamma^5\mathbf{B}-\boldsymbol{\sigma}\times\mathbf{E})\cdot\boldsymbol{\psi}. \tag{3.5}$$

The inner product (3.4) allows a quantum-mechanical interpretation in terms of positive probabilities only if it is proved to be positive definite. Because $\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}=3$, this form is in fact indefinite. However, only those functions $\boldsymbol{\psi}$ are allowed that satisfy the constraint (2.6), implied by the RS equation

$$\mathbf{v}\cdot\boldsymbol{\psi}=0\,, \tag{3.6}$$

with

$$\mathbf{v}=\boldsymbol{\pi}-h\boldsymbol{\alpha}. \tag{3.7}$$

Thus, positivity need be established only on the subspace defined by the projector

$$Q_{ij}=\delta_{ij}-v_i{}^\dagger(\mathbf{v}\cdot\mathbf{v}^\dagger)^{-1}v_j\,. \tag{3.8}$$

[Note that $\mathbf{v}\cdot\mathbf{v}^\dagger=(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2+\boldsymbol{\pi}^2+3m^2$ is a positive invertible operator.] With the help of $\mathbf{Q}$, the inner product (3.4) becomes

$$\langle\psi_1|\psi_2\rangle=\int\boldsymbol{\psi}_1{}^\dagger\cdot\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}\cdot\boldsymbol{\psi}_2 d\mathbf{x}\,, \tag{3.9}$$

with

$$A^0{}_{ij}=\delta_{ij}-\alpha_i\alpha_j\,. \tag{3.10}$$

In Appendix C, the positivity of $\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}$ is investigated, and it is found that $\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}$ is in fact positive in those Lorentz frames where inequality (2.16) holds. It is indefinite, however, in other frames.[4]

We now have all the elements for a quantum-mechanical interpretation of the RS equation. The inner product is defined by Eq. (3.4) and the equation of motion is provided by (2.13), with wave function ψ restricted by the conserved subsidiary conditions (2.6) and (2.10). However, we recall that this holds only when the weak condition is satisfied and only for surfaces $t=t_0$ which are "spacelike" for Eq. (2.13).

4. QUANTIZED FIELD FORMULATION

Because the RS equation is linear—corresponding to the fact that charged particles interact only with a given classical vertex potential, and not with each other—the quantum mechanics established in the last sections may be easily formulated in terms of a quantum field. Let t_0 be an early time before the onset of the potential, and let

$$u_i(t_0,\mathbf{x}),\quad u_i^c(t_0,\mathbf{x}), \tag{4.1}$$

satisfying

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)u_i(t_0)=(\mathbf{p}^2+m^2)^{1/2}u_i(t_0)$$

and

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)u_i^c(t_0)=-(\mathbf{p}^2+m^2)^{1/2}u_i^c(t_0),$$

be a set of functions, subject to the constraints (2.6) and (2.10), which is complete and orthonormal on the surface $t=t_0$ with respect to the scalar product (3.4). For the initial values $u_i(t_0,\mathbf{x})$ and $u_i^c(t_0,\mathbf{x})$, the equation of motion (2.13) determines corresponding solutions $u_i(t,\mathbf{x})$, $u_i^c(t,\mathbf{x})$ at all space-time points. Let a_i^{in} and b_i^{in} be Fermi annihilation operators satisfying the usual anticommutation relations. Then the fields defined by

$$\psi(t,\mathbf{x})=\sum_i[a_i^{\text{in}}u_i(t,\mathbf{x})+b_i^{\text{in}\dagger}u_i(t,\mathbf{x})] \tag{4.2a}$$

and

$$\psi^\dagger(t,\mathbf{x})=\sum_i[a_i^{\text{in}\dagger}u_i^\dagger(t,\mathbf{x})+b_i^{\text{in}}u_i^\dagger(t,\mathbf{x})] \tag{4.2b}$$

satisfy the RS equation everywhere, and, for early times, coincide with the canonical free in-fields.

Having defined the fields by an asymptotic condition, we will now compute the equal-time anticommutator. We begin by considering the anticommutator for arbitrary times,

$$\Delta(x,x')=\{\psi(x),\psi^\dagger(x')\}, \tag{4.3}$$

which by virtue of the expansion (4.2) takes the form

$$\Delta(x,x')=\sum_i[u_i(x)u_i^\dagger(x')+u_i^c(x)u_i^{c\dagger}(x')]. \tag{4.4}$$

As usual, for any solution $f(t,\mathbf{x})$ of the RS equation, we have the identity

$$f(t,\mathbf{x})=\int\Delta(t,\mathbf{x};t',\mathbf{x}')A^0f(t',\mathbf{x}')d\mathbf{x}', \tag{4.5}$$

which holds for any t', with A^0 given by Eq. (3.4). Consequently, for any solutions f and g, the following equality holds:

$$\int f^\dagger(t,\mathbf{x})A^0g(t,\mathbf{x})d\mathbf{x} =\int f^\dagger(t,\mathbf{x})A^0\Delta(t,\mathbf{x};t,\mathbf{x}')A^0g(t,\mathbf{x})d\mathbf{x}d\mathbf{x}', \tag{4.6}$$

where we have set $t'=t$ in (4.5). Because of the form of A^0, [Eqs. (3.4) and (3.10)], only space components appear in Eq. (4.6) and, if we define the space-space equal-time anticommutator

$$C_{ij}(t,\mathbf{x},\mathbf{x}')=[\Delta(t,\mathbf{x};t,\mathbf{x}')]_{ij}, \tag{4.7}$$

then we can write, in nonrelativistic notation,

$$\int \mathbf{f}^\dagger(t,\mathbf{x})\cdot\mathbf{A}^0\cdot\mathbf{g}(t,\mathbf{x})d\mathbf{x} =\int \mathbf{f}^\dagger(t,\mathbf{x})\cdot\mathbf{A}^0\cdot\mathbf{C}(t,\mathbf{x},\mathbf{x}')\cdot\mathbf{A}^0\cdot\mathbf{g}(t,\mathbf{x}')d\mathbf{x}d\mathbf{x}'. \tag{4.8}$$

Because $\mathbf{f}$ and $\mathbf{g}$ are arbitrary in the subspace defined by Q [Eq. (3.8)], and because, by virtue of the expansion (4.4),

$$\mathbf{Q}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbf{Q}=\mathbf{C}, \tag{4.9}$$

we have

$$\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}=(\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q})\mathbf{C}\cdot(\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}). \tag{4.10}$$

In Appendix C, the inverse of $\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q}$ is calculated, and, by Eq. (4.10), we conclude that the equal-time anticommutator coincides with this inverse. Therefore, we have

$$\mathbf{C}=(\mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{Q})^{-1}, \tag{4.11}$$

or explicitly

$$C_{ij}=\delta_{ij}-\tfrac{1}{2}\alpha_i\alpha_j+\tfrac{1}{6}m^{-2}(2\pi_i+\alpha_i\beta m) \times(1+\tfrac{2}{3}em^{-2}\boldsymbol{\sigma}\cdot\mathbf{B})^{-1}(2\pi_j+m\beta\alpha_j). \tag{4.12}$$

This expression coincides with that of Johnson and Sudarshan,[4] who obtained it using Schwinger's canonical quantization principle. They observed that it is an indefinite operator in some Lorentz frames, although it should be positive by virtue of its form (4.3) and (4.4). However, our derivation and our analysis of Sec. 2 show that expression (4.12) represents the equal-time anticommutator only on surfaces which are "spacelike" with respect to Eq. (2.13). For other surfaces the equal-time anticommutator can only be obtained by solving the equations of motion and will be nonlocal because of the faster-than-light propagation of disturbances. The foregoing holds in the weak-field situation, namely, when there are Lorentz frames in which (2.16) holds. In the strong-field situation, Eq. (2.13) ceases to be hyperbolic and no quantum theory can be constructed at all.

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APPENDIX A

We prove that the constraints (2.6),

$$\chi \equiv (\boldsymbol{\pi} - h\boldsymbol{\alpha}) \cdot \boldsymbol{\psi} = 0\,,$$

and (2.10),

$$\Omega \equiv \gamma \cdot \psi + \lambda \gamma^5 \gamma \cdot F^d \cdot \psi = 0\,,$$

with $\lambda = \frac{2}{3} e m^{-2} i$, are preserved by the equations of motion (2.13), and that the RS equation is satisfied. Since Eq. (2.13) is of first order, it is sufficient to show that if χ and Ω vanish at $t=0$, their time derivatives χ' and Ω' vanish at $t=0$. Note first that Eq. (2.13) may be written

$$\varphi_\mu + [m(-\gamma_\mu + \tfrac{1}{2}\lambda F_\mu{}^d \cdot \gamma\gamma^5) + (\pi_\mu + \lambda F_\mu{}^d \cdot \gamma\gamma^5 \gamma \cdot \pi)]\Omega - m^{-1}[\gamma_\mu + \lambda F_\mu{}^d \cdot \gamma\gamma^5]\pi \cdot \varphi = 0\,, \quad \text{(A1)}$$

with

$$\varphi_\mu \equiv (\Gamma \cdot \pi - B)_\mu{}^\nu \psi_\nu\,.$$

Consider the two equations obtained by contracting Eq. (A1) with γ^μ and also by taking the zeroth component of Eq. (A1). Noting that $\chi = -\gamma^0 \varphi_0$, and assuming that χ and Ω and their spatial derivatives vanish at $t=0$, one finds that

$$\Omega' = 0\,, \quad \pi \cdot \varphi = 0 \quad \text{(A2)}$$

at $t=0$. Comparing with Eq. (A1), we see that $\varphi_i = 0$, so the RS equation is satisfied at $t=0$. From $\chi = -\gamma^0 \varphi_0$ and the second of Eqs. (A2), we deduce that $\chi' = 0$ also. Hence, the constraints are preserved in time and our proof has shown that whenever the constraints are satisfied, the RS equation holds.

APPENDIX B

The discontinuities of the derivatives of a solution to a first-order system of hyperbolic equations are known to propagate along characteristic surfaces.[9] We will show that discontinuities which propagate along the extraordinary ray are compatible with the constraints (2.6) and (2.10). For this purpose, we assume that the external potentials and fields are continuously differentiable functions. Let u be a continuous solution of Eq. (2.13) whose derivative has a discontinuity given by

$$(\pi_\kappa u) = n_\kappa w\,, \quad \text{(B1)}$$

where (f) means the discontinuity in f, w is continuous, and n_κ satisfies

$$n^2 + \tfrac{4}{3} e m^{-2} (F^d \cdot n)^2 = 0\,. \quad \text{(B2)}$$

[9] Reference 7, pp. 618 and 619.

Taking the discontinuity of the primary constraint (2.6), we have

$$\mathbf{n} \cdot \mathbf{w} - \mathbf{n} \cdot \boldsymbol{\alpha}\boldsymbol{\alpha} \cdot \mathbf{w} = 0$$

or

$$\boldsymbol{\sigma} \cdot \mathbf{n} \times \mathbf{w} = 0\,. \quad \text{(B3)}$$

Taking the three-gradient of the secondary constraint (2.10), and equating discontinuities, we find that w must satisfy

$$\gamma \cdot w + \tfrac{2}{3} i e m^{-2} \gamma \cdot F^d \cdot w = 0\,. \quad \text{(B4)}$$

The discontinuity of the equation of motion (2.13) yields

$$n w_\mu + n_\mu \tfrac{2}{3} i e m^{-2} \gamma^5 \gamma \cdot F^d \cdot w + \tfrac{2}{3} i e m^{-2} F_\mu{}^d \cdot \gamma\gamma^5 n \cdot w + \tfrac{2}{3} i e m^{-2} F_\mu{}^d \cdot \gamma\gamma^5 \mathbf{n} \tfrac{2}{3} i e m^{-2} \gamma^5 \gamma \cdot F^d \cdot w = 0\,. \quad \text{(B5)}$$

By choosing $w_\mu = n_\mu f$, Eq. (B3) is automatically satisfied, and Eq. (B4) becomes an eigenvalue equation for f which has a nonzero solution when Eq. (B2) is satisfied, which is true by assumption. Then Eq. (B5) is satisfied identically.

This shows that there are disturbances, compatible with the constraints, which do propagate along the extraordinary ray.

APPENDIX C

We first verify that C, given by Eq. (4.12), satisfies $\mathbf{Q} \cdot \mathbf{C} = \mathbf{C}$ and $\mathbf{C} = (\mathbf{Q} \cdot \mathbf{A}^0 \cdot \mathbf{Q})^{-1}$. The positivity of $\mathbf{Q} \cdot \mathbf{A}^0 \cdot \mathbf{Q}$ will follow from the positivity of $\mathbf{C}$. From the identity

$$(h\boldsymbol{\alpha} - \boldsymbol{\pi}) \cdot (2\boldsymbol{\pi} + \boldsymbol{\alpha}\beta m) = 3m^2 + 2e\boldsymbol{\sigma} \cdot \mathbf{B}\,, \quad \text{(C1)}$$

one easily verifies that $v_i C_{ij} = 0$, with v_i given by Eq. (3.7). Hence, with Q given by Eq. (3.8), one verifies the equality $\mathbf{Q} \cdot \mathbf{C} = \mathbf{C}$. Then we have, with $\mathbf{A}^0$ from Eq. (3.10),

$$\begin{aligned}(\mathbf{Q} \cdot \mathbf{A}^0 \cdot \mathbf{Q}) \cdot \mathbf{C} &= \mathbf{Q} \cdot \mathbf{A}^0 \cdot \mathbf{C} = \mathbf{Q} \cdot (\mathbf{1} - \boldsymbol{\alpha}\boldsymbol{\alpha}) \cdot \mathbf{C} \\ &= [\mathbf{1} - \mathbf{v}^\dagger (\mathbf{v} \cdot \mathbf{v}^\dagger)^{-1} \mathbf{v}] \\ &\quad \cdot [\mathbf{1} - \mathbf{v}^\dagger (3m^2 + e\boldsymbol{\sigma} \cdot \mathbf{B})^{-1} (2\boldsymbol{\pi} + m\beta\boldsymbol{\alpha})] \\ &= [\mathbf{1} - \mathbf{v}^\dagger (\mathbf{v} \cdot \mathbf{v}^\dagger)^{-1} \mathbf{v}] = \mathbf{Q}\,. \quad \text{(C2)}\end{aligned}$$

This shows that $\mathbf{C}$ is the inverse of $\mathbf{Q} \cdot \mathbf{A}^0 \cdot \mathbf{Q}$ in the subspace $\mathbf{Q}$. It is sufficient to show that $\mathbf{C}$ is positive on the subspace $\mathbf{Q}$, and for this purpose we may replace $\boldsymbol{\pi}$ by $\boldsymbol{\alpha}h$ and $h\boldsymbol{\alpha}$ on the left- and right-hand sides, respectively. This gives

$$\mathbf{C} = \mathbf{Q} \cdot \{1 + \tfrac{1}{2}\boldsymbol{\alpha}\{(2\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + 3\beta m)[(3m^2 + 2e\boldsymbol{\sigma} \cdot \mathbf{B})^{-1} - (2\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + 3\beta m)^{-2}](2\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + 3\beta m)\}\boldsymbol{\alpha}\} \cdot \mathbf{Q}\,.$$

We assume that $3m^2 + 2e\boldsymbol{\sigma} \cdot \mathbf{B} > 0$. (weak-field condition), and therefore from the fact that

$$(2\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + 3\beta m)^2 > (3m^2 + 2e\boldsymbol{\sigma} \cdot \mathbf{B})\,,$$

it follows that $\mathbf{C} \geq \mathbf{Q}$. Hence, $\mathbf{C}$ is positive definite on the subspace.